A General Ritz Algorithm for Static Analysis of Arbitrary Laminated Composite Plates using First Order Shear Deformation Theory

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Abstract: This paper is concerned with the bending of laminated composite plates with arbitrary lay-up and general boundary conditions. The analysis is based on the small deflection, first-order shear deformation theory of composite plates, which utilizes the Reissner-Mindlin plate theory. In solving the aforementioned plate problems, a general algorithm based on the Ritz method and the use of beam orthogonal polynomials as coordinate functions is derived. This general algorithm provides an analytical approximate solution that can be applied to the static analysis of moderately thick laminated composite plates with any lamination scheme and combination of edge conditions. The convergence, accuracy, and flexibility of the obtained general algorithm are shown by computing several numerical examples and comparing some of them with results given in the literature. Some results, including general laminates and nonsymmetrical boundary conditions with free edges, are also presented.

Keywords: Plate bending, General laminated composite plates, First order shear deformation theory

خواريزم رتز العام للتحليل الإستاتيكي للوح الرقائق لمنضد مركب باستخدام الدرجة الأولى لنظرية الانحراف

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الملخص: يتعلق هذا البحث بدراسة الإنحناء للوحات منضد مركب ضمن عشوائية وضع المتابعة والحدود العامة. ويستند هذا التحليل إلى الانحراف الصغير من الدرجة الأولى لنظرية الإنحراف للوحات المركب الذي يستخدم نظرية لوحات رايسنر ليندلن لحل مشاكل اللوحات المذكورة سابقا. تم اشتقاق خواريزم عام يعتمد على طريقة رتز مع إستخدام شعاع متعامد متعدد الحدود. يوفر هذا البرنامج حلا نظريا تقريبيا من الممكن تطبيقه لتحليل لوحات منضد مركب معتدل السمك وفي وضع الثبات لأي منضد مع أي نظام ومزيج من ظروف الحافة. ويظهر البرنامج التقارب والدقة والمرونة التي تم الحصول عليها من خلال حساب العديد من الأمثلة، ومقارنة بعضهما مع النتائج الواردة في الأبحاث المنشورة.

الكلمات المفتاحية: الدرجة الأولى لنظرية الانحراف، رقائق الواح منضد عام، انحناء لوح

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1. Introduction

The advantages of composite materials over conventional ones have been well-established because of their high strength-to-density and stiffness-to-density ratios. These advantages are especially desirable in aerospace, aircraft, automotive, and other industrial applications. Among the different types of composite materials, fiber-reinforced laminates are very important as structural plate components. The classical laminate plate theory (CLPT) (Lekhnitskii 1968), which neglects the effects of out-of-plane strains, has been widely used for obtaining the mechanical response of several composite plates. However, composite laminated plates have relatively low transverse shear stiffness making, in consequence, the transverse shear strains more noticeable for laminated plates than for isotropic plates. CLPT predicts the response of thin isotropic plates with reasonable accuracy, but it usually fails to yield similar accuracy for composite plates of a similar configuration. Because of this, CLPT often under-predicts deflections and over-predicts natural frequencies and buckling loads, even in thin laminated plates.

There are numerous plate theories that include the transverse shear strains in the analysis *eg*. (Noor and Burton 1989; Maiti and Sinha 1996; Liew *et al*. 1996; Reddy 2003). The first-order shear-deformation theory (FSDT) proposed by (Reissner 1945 and Mindlin 1951) assumes that planes normal to the midplane remain straight, but not necessarily normal to it after deformation. Since FSDT accounts for a constant state of transverse shear stresses, shear correction coefficients are needed to rectify the unrealistic variation of the shear strain/stress through the thickness and which ultimately define the shear strain energy. Some other plate theories, such as the higher order shear deformation theories (HSDT), include the effect of transverse shear strains (Khdeir and Reddy 1989; Reddy 2003; Xiao *et al*. 2008; Oktem and Chaudhuri 2008) and do not need the use of shear correction coefficients in computing the transverse shear stresses.

Notwithstanding the limitations of FSDT, from an engineering point of view it is still a very attractive approach due to its simplicity and low computational cost. It is well recognized that while FSDT is adequate for global structural behavior (*eg*. transverse deflections, fundamental vibration frequencies, critical buckling loads, force and moment resultants), it is not adequate for accurate prediction of local response parameters, such as the interlaminar stress distributions (Qi and Knight 1996).

For problems involving different common and refined plate theories, several numerical and analytical approximate solutions have been proposed (Tessler 1993; Nguyen *et al*. 2005; Daghia *et al*. 2008; Fares and Elmarghany 2008; Xiang 2009; Bodaghi and Saidi 2010). Most papers are limited to certain edges support and only to cross-ply or angle-ply laminated plates. Closed-form solutions for laminated composite plates using FSDT have been provided for some simple boundary conditions and some particular lamination schemes (Whitney 1987; Reddy 2003). Other analytical solutions obtained by the Ritz method employ trigonometric and hyperbolic beam functions to form the approximate shape functions. However, that approach for plates with general anisotropy leads to resultant moments that appear to be oscillating about a relative constant value, as has been shown by (Nallim and Grossi 2003).

In this study, FSDT was employed in conjunction with the Ritz method and sets of beam characteristic orthogonal polynomials for obtaining a generalized approximate analytical solution to study the static behavior of arbitrary composite laminated plates with all possible combinations of boundary conditions. All coupling effects including, stretching-bending, stretching-shear, and bending-twisting were considered in the formulation. The accuracy of the present method, is shown through the convergence tests for selected plate problems and selected results are compared with those published by other authors. Few selected cases are solved and the corresponding deflections and resultant moments are presented by means of plots and in tabular form. Results include laminated plates with general lamination sequences, non-symmetric boundary conditions and free edges.

The algorithm developed offers an interesting alternate from an engineering view point to perform design analysis. It can be applied to the analysis of a wide range of rectangular laminated plates with different boundary conditions, number of layers, stacking sequences, and fiber orientation.

2. Formulation

2.1 Energy Functional Components

A rectangular fiber reinforced composite laminated plate of uniform thickness h is shown in Fig. 1. The plane x, y coincides with the middle surface along the plate thickness, while z remains normal to it. In each layer of the laminate, θ denotes the angle of fiber orientation and the major and minor principal material axes are denoted by 1 and 2, respectively.

In the first order shear deformation theory, the plate kinematics is governed by the midplane displacements u^o, v^o, w^o and rotations ϕ_x, ϕ_y given by:

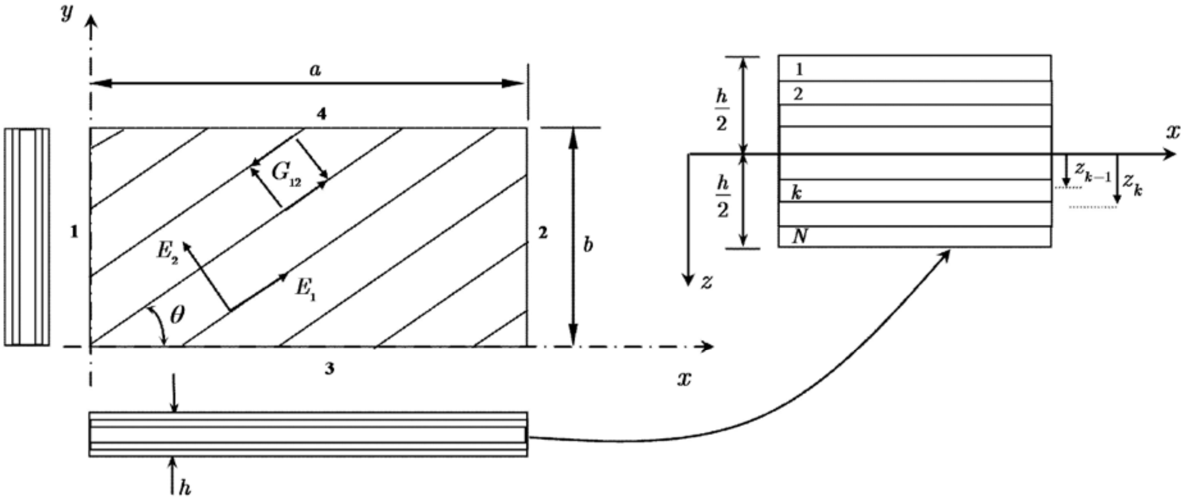

Figure 1. General description of the composite plate model

$$u(x,y,z) = u^0(x,y) + z\phi_x(x,y);\; v(x,y,z) = v^0(x,y) + z\phi_y(x,y);\; w(x,y,z) = w^0(x,y) \tag{1}$$

$$\begin{aligned}
\bar{U} = \frac{1}{2}\iint_R & (A_{11}\left(\frac{\partial u^0}{\partial x}\right)^2 + 2A_{12}\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y} + A_{22}\left(\frac{\partial v^0}{\partial y}\right)^2 + 2A_{16}\frac{\partial u^0}{\partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) \\
& + 2A_{26}\frac{\partial v^0}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right) + A_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)^2 + KA_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)^2 \\
& + 2KA_{45}\left(\phi_x + \frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y} + \phi_y\right) + KA_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right)^2 + 2B_{11}\frac{\partial u^0}{\partial x}\frac{\partial \phi_x}{\partial x} \\
& + 2B_{12}\left(\frac{\partial u^0}{\partial x}\frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_x}{\partial x}\frac{\partial v^0}{\partial y}\right) + 2B_{22}\frac{\partial v^0}{\partial y}\frac{\partial \phi_y}{\partial y} + 2B_{16}\left(\frac{\partial u^0}{\partial x}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + \frac{\partial \phi_x}{\partial x}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right) \\
& + 2B_{26}\left(\frac{\partial v^0}{\partial y}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + \frac{\partial \phi_y}{\partial y}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\right) + 2B_{66}\left(\frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x}\right)\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \\
& + D_{11}\left(\frac{\partial \phi_x}{\partial x}\right)^2 + 2D_{12}\frac{\partial \phi_x}{\partial x}\frac{\partial \phi_y}{\partial y} + D_{22}\left(\frac{\partial \phi_y}{\partial y}\right)^2 + 2D_{16}\frac{\partial \phi_x}{\partial x}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + 2D_{26}\frac{\partial \phi_y}{\partial y}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \\
& + D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)^2)dx\,dy
\end{aligned} \tag{2}$$

where z is the thickness coordinate as measured from the reference plane shown in Fig. 1.

It is necessary that the two in-plane mid-surface translational displacements u^o and v^o are included in the analysis due to the coupling between in-plane and out-of-plane behavior in laminates with unsymmetrical lay-up.

Considering the established kinematics and basic assumptions of the first order theory, the strain energy, $\bar{U}$, is given by (2), where R is the mid-surface area and K is the shear correction factor which can be estimated by using special methods (Whitney 1987).

In Eq. (2), the extensional stiffness, bending-extensional coupling stiffness, bending stiffness, and transverse shearing stiffness are respectively given by:

$$\begin{aligned}
(A_{ij}, B_{ij}, D_{ij}) &= \int_{-h/2}^{h/2} (1, z, z^2)\bar{Q}_{ij}\,dz \quad i,j = 1,2,6 \\
A_{ij} &= \int_{-h/2}^{h/2} \bar{Q}_{ij}\,dz \quad i,j = 4,5
\end{aligned} \tag{3}$$

where $\bar{Q}_{ij}$ represents the elastic mechanical constants referred to in the x, y axes (Reddy, 2003). The potential energy of a transversal load $q(x,y)$ distributed over the plate surface is given by:

Table 1. Notations for various combinations of boundary conditions, in which *n* and *s* indicate the normal and tangential directions to the respective plate edges

Transverse boundary supports	In plane constraints			
	$u_n = 0$ $u_s = 0$	$N_n = 0,$ $u_s = 0$	$u_n = 0,$ $N_{ns} = 0$	$N_n = 0,$ $N_{ns} = 0$
Clamped: $w = 0;\ \phi_n = 0$	C_1	C_2	C_3	C_4
Simply supported: $w = 0;\ M_n = 0; \phi_s = 0$	S_1	S_2	S_3	S_4
Free: $M_n = 0;\ M_{ns} = 0;\ Q_n = 0$	F_1	F_2	F_3	F_4

$$\bar{V} = -\iint_R q(x,y)\, w\, dxdy \tag{4}$$

2.2 Boundary Conditions and Approximating Functions

The situation for asymmetrically laminated plates is more complex than for symmetrical ones because the transverse and in-plane displacements are coupled. Actually, there are four types of boundary conditions that can be called simply supported (S), clamped (C), or free edges (F), and the combinations of these constraints are summarized in Table 1.

In the application of the Ritz method only the essential boundary conditions are required to be satisfied by the assumed functions. The use of beam orthogonal polynomials to study anisotropic plates is very satisfactory, as has been demonstrated by Nallim et al. (2003, 2005, 2008), since the convergence of the solution is rapid and practically without oscillations. In the present work, the five displacement components are expressed by sets of beam characteristic orthogonal polynomials as follows:

$$\begin{aligned}
u^0(x,y) &\approx \sum_{i=1}^{N}\sum_{j=1}^{M} c_{ij}^{(u)} p_i^{(u)}(x)\, q_j^{(u)}(y),\\
v^0(x,y) &\approx \sum_{i=1}^{N}\sum_{j=1}^{M} c_{ij}^{(v)} p_i^{(v)}(x) q_j^{(v)}(y),\\
w^0(x,y) &\approx \sum_{i=1}^{N}\sum_{j=1}^{M} c_{ij}^{(w)} p_i^{(w)}(x) q_j^{(w)}(y),\\
\phi_x(x,y) &\approx \sum_{i=1}^{N}\sum_{j=1}^{M} c_{ij}^{(\phi_x)} p_i^{(\phi_x)}(x)\, q_j^{(\phi_x)}(y),\\
\phi_y(x,y) &\approx \sum_{i=1}^{N}\sum_{j=1}^{M} c_{ij}^{(\phi_y)} p_i^{(\phi_y)}(x)\, q_j^{(\phi_y)}(y)
\end{aligned} \tag{5}$$

where $\{p_i^{(\bullet)}(x)\}$, $\{q_j^{(\bullet)}(y)\}$, $(\bullet) = u^0, v^0, w^0, \phi_x, \phi_y$ are sets of orthogonal polynomials generated by the Gram-Schmidt recurrence formulas (Bhat 1985; Nallim and Oller 2008), $c_{ij}^{(u)}$, $c_{ij}^{(v)}$, $c_{ij}^{(w)}, c_{ij}^{(\phi_x)}$, $c_{ij}^{(\phi_y)}$ are the unknown coefficients, and M, N are the numbers of polynomials in each coordinate.

For the construction of the orthogonal polynomials the following procedure is followed. The first members of the sets are given by $p_1(x) = \sum_{i=1}^{5} a_i x^{i-1}$ y and $q_1(y) = \sum_{i=1}^{5} b_i y^{i-1}$ where a_i, y b_i coefficients are obtained from the geometrical boundary conditions in each direction. The rest of the sets are constructed by employing the Gram–Schmidt orthogonalization procedure as follows:

$$p_2(x) = \left(x - B_2\right) p_1(x)$$

$$p_k(\bar{x}) = \left(\bar{x} - B_k\right) p_{k\text{-}1}\left(\bar{x}\right) - C_k p_{k\text{-}2}\left(\bar{x}\right)$$

where B_k, C_k are obtained using the orthogonality property of the $p_k(x)$ polynomials:

$$B_k = \frac{\int_0^a x\left(p_{k-1}(x)\right)^2 dx}{\int_0^a \left(p_{k-1}(\bar{x})\right)^2 d\bar{x}} \quad C_k = \frac{\int_0^b x p_{k-1}(x)\, p_{k-2}(x)\, dx}{\int_0^b \left(p_{k-2}(x)\right)^2 dx}$$

The same procedure is used for y variable.

3. Application of the Ritz Method

The total energy functional for the static bending analysis of laminated plates is given by:

$$\Pi = \bar{U} + \bar{V} \tag{6}$$

where $\bar{U}$ is the strain energy given by Eq. (2) and $\bar{V}$ is the potential energy given by Eq. (4).

The Ritz procedure requires the minimization of the energy functional (6) with respect to each of the unknown coefficients:

$$\frac{\partial \Pi}{\partial c_{ij}^{(u)}} = 0, \frac{\partial \Pi}{\partial c_{ij}^{(v)}} = 0, \frac{\partial \Pi}{\partial c_{ij}^{(w)}} = 0, \frac{\partial \Pi}{c_{ij}^{(\phi_x)}} = 0,$$

$$\frac{\partial \Pi}{c_{ij}^{(\phi_y)}} = 0 \quad (i, j = 1, \ldots, M, N)$$

Substituting Eq. (5) into the expression of the functional Eq. (6), and subsequently applying Eq. (7) results in the following governing equation:

$$[K]\ \{C\} = \{f\} \tag{8}$$

where {f} is the load vector and [K] is the syammetric stiffness matrix given by:

$$[\mathbf{K}] = \begin{bmatrix} \left[K^{uu}_{ijkh}\right] & \left[K^{uv}_{ijkh}\right] & \left[K^{uw}_{ijkh}\right] & \left[K^{u\phi_x}_{ijkh}\right] & \left[K^{u\phi_y}_{ijkh}\right] \\ & \left[K^{vv}_{ijkh}\right] & \left[K^{vw}_{ijkh}\right] & \left[K^{v\phi_x}_{ijkh}\right] & \left[K^{v\phi_y}_{ijkh}\right] \\ & & \left[K^{ww}_{ijkh}\right] & \left[K^{w\phi_x}_{ijkh}\right] & \left[K^{w\phi_y}_{ijkh}\right] \\ & & & \left[K^{\phi_x\phi_x}_{ijkh}\right] & \left[K^{\phi_x\phi_y}_{ijkh}\right] \\ \mathrm{Sym} & & & & \left[K^{\phi_y\phi_y}_{ijkh}\right] \end{bmatrix} \tag{9}$$

and the unknown coefficient vector is:

$$\{\mathbf{C}\} = \{\{c^{(u)}_{kh}\}\{c^{(v)}_{kh}\}\{c^{(w)}_{kh}\}\{c^{(\phi_x)}_{kh}\}\{c^{(\phi_y)}_{kh}\}\}^T \tag{10}$$

The elements of the stiffness matrix are given in Appendix A.

4. Numerical Results

A number of numerical examples are presented in this section to demonstrate the performance of the general algorithm developed for bending analysis of laminated plates with various boundary conditions, span-to-thickness ratios, fiber orientations, etc. The following terminology is used for describing the plate boundary conditions. For instance, $S_iS_iC_iC_i$ identifies a plate with edges simply supported along $x = 0, a$, and edges clamped along $y = 0, b$. The subscript i (i=1,…,4) indicates the in-plane constraints according to Table 1. The results for square laminated composite plates are compared with (Moleiro *et al.* 2008). The gtraphite/epoxy material with properties are used. The reference flexural stiffness is $D_o = E_1h^3$ / 12 (1 - $v_{12}v_{21}$).

4.1 Validation and Convergence Studies

Comparison and convergence studies were conducted to evaluate the accuracy of the present formulation. Table 2 shows results for square angle-ply anti-symmetric laminates (θ / - θ) with a $S_1S_1C_1C_1$ boundary conditions and thickness ratio. The results are given in the form of deflection, bending moments at the plate center, and axial forces at one of the corners of the plate. The number of polynomials (M, N) are stepped steadily from 5 to 8 to demonstrate the convergence. In the same table the accuracy and reliability of the present results are demonstrated by comparison with (Sheikh *et al.* 2002) who used a high precision triangular element considering the effect of shear deformation by taking transverse displacement and bending rotations as independent field variables. The comparison shows a very good agreement.

Table 3 shows results of deflections, and moment and transverse force resultants for simply supported cross ply (0/90) square plates with uniform loading. The lamina properties are E_1 / E_2 = 25, G_{12} / E_2 = G_{13} / E_2 = 0.5, G_{23} / E_2 = 0.2, v_{12} = 0.25. The results for different thickness ratios were compared with those published by (Moleiro *et al.* 2008) and a very good agreement was observed.

4.2 Numerical Examples

The developed Ritz formulation is applied in this section to obtain the plate deflections and moment resultants of general laminated plates with several combinations of boundary conditions and stacking sequences. The capability and generality of the present approach is shown through selected examples depicted in Table 4 and Figs. 2-4. Two different combinations of boundary conditions, three thickness ratios, and four lamination schemes have been chosen as examples.

The effect of span to thickness ratio on center deflection for symmetrical and anti-symmetrical cross-ply and angle-ply laminates with $S_1S_1C_2C_2$ boundary conditions is shown in Fig. 5. The non-dimensional deflection decreased as b / h increased, but the decrease was small (beyond b / h = 40). The value of the non-dimensional deflection became, as expected, practically constant for thin plates. Also, the difference between the symmetrical and anti-symmetrical lay-up values was small for cross-ply laminates and much larger for angle-ply laminates.

Finally, antisymmetrical two layered laminates (θ / - θ), with a fiber orientation varying between 0° and 45° were analyzed when simply supported ($S_1S_1S_1S_1$), fully clamped ($C_2C_2C_2C_2$), and when mixed ($S_1S_1C_2C_2$) boundary conditions were used. Variations of non-dimensional deflection are shown in Fig. 6. A change in fiber orientation led to an increase in the non-dimensional deflection in case of fully clamped plates, while for simply supported and mixed boundary conditions, the non-dimensional deflection showed a convex shape.

5. Conclusions

A general algorithm for static analysis of thick, arbitrarily laminated composite plates with multiple combination of boundary conditions is presented in this

Table 2. Convergence and validation results for angle-ply antisymmetric laminates

Sources	$M \times N$	$10^3 wE_2h^3/pa^4$ $x=\frac{a}{2}, y=\frac{b}{2}$	$10^2 M_x h^2/pa^2$ $x=\frac{a}{2}, y=\frac{b}{2}$	$10^2 M_y h^2/pa^2$ $x=\frac{a}{2}, y=\frac{b}{2}$	$10^2 N_x h^2/pa^2$ $x=0, y=0$	$10^2 N_y h^2/pa^2$ $x=0, y=0$
(15º/-15º)						
Present	5x5	7.32100	11.64450	1.35680	36.34265	46.64834
Present	6x6	7.16079	11.65627	1.35659	34.12295	45.44581
Present	7x7	7.15899	11.38646	1.22164	42.46526	45.61578
Present	8x8	7.16724	11.38790	1.21977	43.17482	45.70038
Sheikh *et al.* (2002)		7.16720	11.42900	1.23450	44.10700	45.77800
(30º/-30º)						
Present	5x5	7.86464	7.13211	2.78603	22.00543	31.14831
Present	6x6	7.76537	6.91782	2.66332	22.60426	30.60577
Present	7x7	7.76502	6.91691	2.66246	22.96999	30.68150
Present	8x8	7.77049	6.95257	2.68187	23.04376	30.57181
Sheikh *et al.* (2002)		7.77010	6.95370	2.68280	23.28200	30.66100
(45º/-45º)						
Present	5x5	7.42461	3.90210	3.90210	4.74293	4.74293
Present	6x6	7.33433	3.66572	3.66572	4.49922	4.49922
Present	7x7	7.33426	3.66458	3.66458	4.67624	4.67624
Present	8x8	7.33768	3.68020	3.68020	4.63219	4.63219
Sheikh *et al.* (2002)		7.33830	3.68040	3.68040	4.62080	4.62080

Table 3. Convergence and validation results for $S_2S_2S_2S_2$, (cross-ply laminates)*

Sources	(*)	$M \times N$	$10^2 w^A E_2 h^3/pa^4$	$10 M^A_{yy}/pa^2$	$10 M^B_{xy}/pa^2$	$10\, Q^C_y/pa$
$b/h=10$						
Present		5x5	1.9556	0.6414	-0.1618	3.5091
Present		6x6	1.9470	0.6272	-0.1616	3.4795
Present		7x7	1.9470	0.6272	-0.1611	3.4631
Present		8x8	1.9469	0.6268	-0.1610	3.4657
Moleiro *et al*. (2008)			1.9469	0.6268	-0.1604	3.4703
$b/h=20$						
Present		5x5	1.76747	0.6450	-0.15872	3.5518
Present		6x6	1.75838	0.6298	-0.15814	3.4986
Present		7x7	1.75840	0.6297	-0.15810	3.4786
Present		8x8	1.75825	0.6291	-0.15800	3.4854
Moleiro et al. (2008)			1.7582	0.6291	-0.15760	3.4880
$b/h=100$						
Present		5x5	1.7074	0.6465	-0.1570	4.1771
Present		6x6	1.6982	0.6311	-0.1561	3.8143
Present		7x7	1.6982	0.6310	-0.1561	3.5224
Present		8x8	1.6980	0.6301	-0.1559	3.4948
Moleiro *et al*. (2008)			1.6980	0.6300	-0.1572	3.4926

(*) Results at A, ($x = a/2, y = b/2$); B, ($x = 0, y = 0$); and C, ($x = a/2, y = 0$).

Table 4. Laminated plates with $S_1S_1C_3C_3$, $S_1S_1F_4F_4$, $S_1F_4C_3C_3$ and $C_1F_1S_4S_4$ boundary conditions

		$S_1S_1C_3C_3$				$S_1S_1F_4F_4$			
	b/h	$\frac{w^A D_0}{pa^2b^2}$	$10^2 M_{xx}^A/pa^2$	$10^2 M_{yy}^A/pa^2$	$10^2 M_{xy}^B/pa^2$	$\frac{w^A D_0}{pa^2b^2}$	$10^2 M_{xx}^A/pa^2$	$10^2 M_{yy}^A/pa^2$	$10^2 M_{xy}^B/pa^2$
0/90	10	1.91085	5.16985	3.10277	-0.15931	4.50222	12.44125	0.06638	-0.21960
	20	1.46203	4.70059	3.32673	-0.07967	3.82016	12.42912	0.07668	-0.20928
	100	1.31256	4.49614	3.43055	-0.01331	3.60204	12.42342	0.08115	-0.20198
30/-30									
	10	2.33660	6.73065	2.70848	-0.94762	5.00887	17.53942	5.55785	-7.63619
	20	1.87945	6.51884	2.59216	-0.05395	4.23181	16.90835	5.31326	-6.72788
	100	1.71707	6.37823	2.53948	1.37554	3.92325	16.31041	5.10021	-3.22217
0/45									
	10	2.29114	7.21433	2.09630	-4.67575	3.91459	13.07055	2.20518	3.44731
	20	1.79935	7.26791	2.03974	-3.89374	3.19469	13.04412	2.39501	5.06825
	100	1.62649	7.22313	1.97489	-1.43144	2.96609	13.01622	2.44544	8.02781
0/30									
	10	2.40472	9.45824	1.81181	-3.03480	3.22818	13.21586	1.52171	4.09556
	20	1.86325	9.78735	1.76011	-2.41140	2.46427	13.22517	1.54095	5.58218
	100	1.67733	9.80246	1.72394	-0.57772	2.21681	13.11340	1.52356	8.17810
		$S_1F_4C_3C_3$				$C_1F_1S_4S_4$			
0/90									
	10	2.37292	1.54299	3.92693	-0.41387	3.13604	1.01229	8.65299	-0.14203
	20	1.75334	1.35792	4.02024	-0.20051	2.59387	1.20841	8.43092	-0.06499
	100	1.55141	1.28573	4.06181	-0.02767	2.41851	1.28463	8.32839	-0.00914
30/-30									
	10	-0.02767	4.45259	2.93097	-2.85338	6.30485	2.73118	2.30205	-5.58008
	20	3.91322	4.29930	2.78252	-1.30692	-5.58008	2.90081	2.08945	-2.94748
	100	3.65247	4.19075	2.69676	1.11596	4.63883	3.11313	2.05653	-0.52289
0/45									
	10	4.34409	3.36795	3.50878	-6.12189	6.71910	2.40778	5.42696	-12.42389
	20	3.61144	3.45713	3.57674	-4.79724	5.43032	2.39852	5.01418	-10.41609
	100	3.34215	3.57463	3.58661	-1.99905	4.91878	2.40356	4.82715	-4.00303
0/30									
	10	5.88477	4.53894	3.09778	-8.46556	7.26061	-2.92413	0.17388	-23.32274
	20	5.16109	4.73620	3.05669	-6.20950	5.65240	-3.30340	0.23627	-19.35380
	100	4.88650	4.87695	3.03342	-3.00629	5.05097	-4.15112	-0.04051	-12.55993

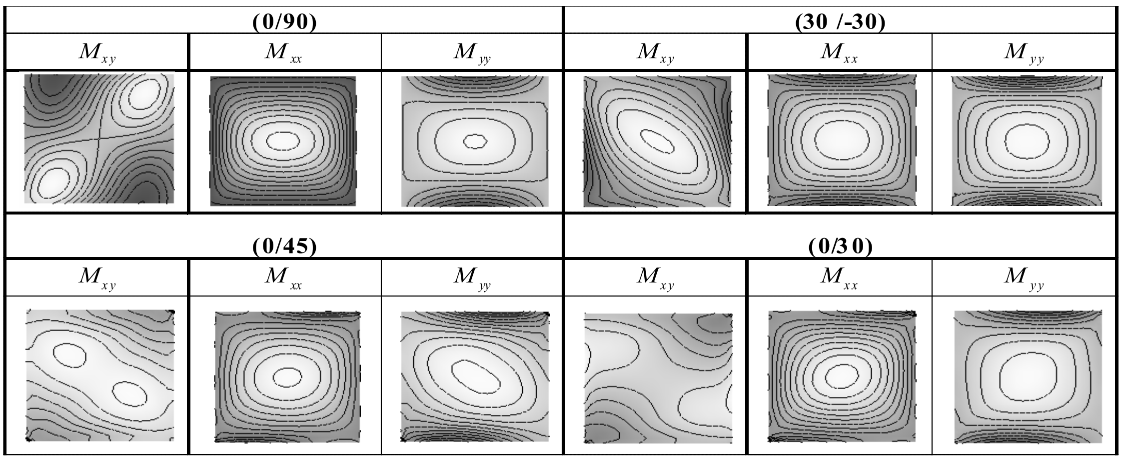

Figure 2. Contour plot of moment resultants for different laminated plates with $S_1S_1C_3C_3$ boundary conditions

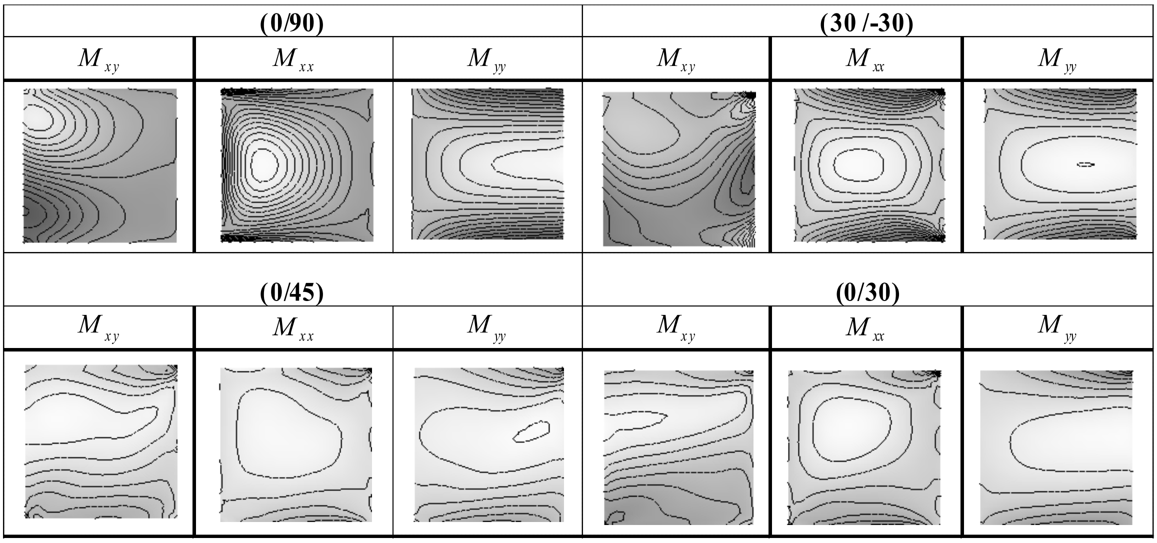

Figure 3. Contour plot of moment resultants for different laminated plates with $S_1F_4C_3C_3$ boundary conditions

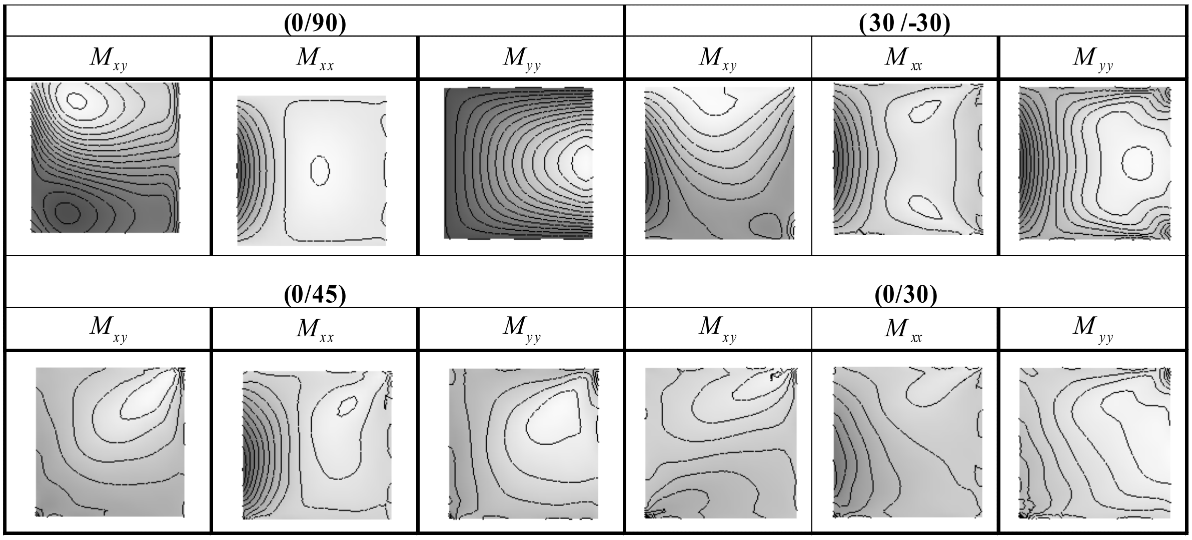

Figure 4. Contour plot of moment resultants for different laminated plates with $C_1F_1S_4S_4$ boundary conditions

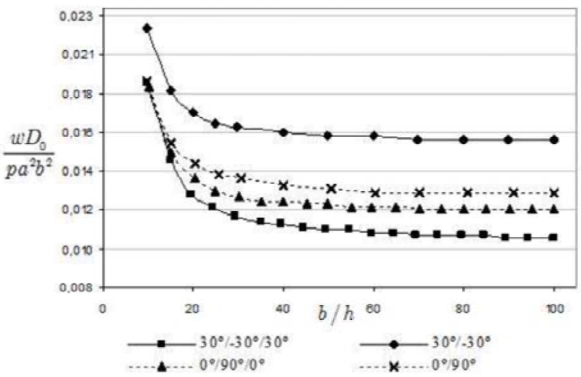

Figure 5. Variation of non-dimensional static deflection at the plate centre with b / h ratio

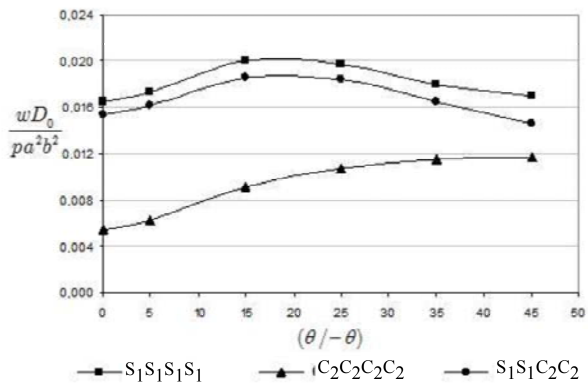

Figure 6. Variation of non-dimensional deflection with fiber orientation angle ($b / h = 20$)

work. The algorithm is based on the Ritz variational method and the first order shear deformation theory. The five components of the displacement field were approximated by sets of characteristic orthogonal polynomials generated using the Gram-Schmidt procedure. The algorithm developed was very general and allowed the analysis of moderately thick plates with different materials, laminate lay-ups, span-to-thickness ratios, and boundary conditions. The computational implementation is also very simple, and the formulation is such that we could appreciate how the various geometrical and mechanical parameters involved influenced the static response.

Consequently, the developed formulation constitutes an efficient tool in design and optimization problems. Comparisons of the present results with a variety of other published results were made with very good agreement. Some typical results were given for cross-ply, angle-ply, and arbitrarily laminated plates with various thickness ratios ranging from moderately thick to thin, and for different boundary conditions.

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Appenxid A

Elements in Eq. (9)

$$
\begin{aligned}
K_{ijkh}^{uu} &= A_{11}\iint\limits_R p_i'^{(u)} p_k'^{(u)} q_j^{(u)} q_h^{(u)}\, dxdy + A_{16}\iint\limits_R \left(p_i'^{(u)} p_k^{(u)} q_j^{(u)} q_h'^{(u)} + p_i^{(u)} p_k'^{(u)} q_j'^{(u)} q_h^{(u)}\right) dxdy \\
&\quad + A_{66}\iint\limits_R p_i^{(u)} p_k^{(u)} q_j'^{(u)} q_h'^{(u)}\, dxdy
\end{aligned}
$$

$$
\begin{aligned}
K_{ijkh}^{uv} &= A_{12}\iint\limits_R p_i'^{(u)} p_k^{(v)} q_j^{(u)} q_h'^{(v)} dxdy + A_{16}\iint\limits_R p_i'^{(v)} p_k'^{(u)} q_j^{(v)} q_h^{(u)} dxdy \\
&\quad + A_{26}\iint\limits_R p_i^{(v)} p_k^{(u)} q_j'^{(v)} q_h'^{(u)} dxdy + A_{66}\iint\limits_R p_i'^{(v)} p_k^{(u)} q_j^{(v)} p_h'^{(u)} dxdy
\end{aligned}
$$

$$
K_{ijkh}^{uw} = 0,\; K_{ijkh}^{vw} = 0
$$

$$
\begin{aligned}
K_{ijkh}^{u\phi_x} &= B_{11}\iint\limits_R p_i'^{(u)} \left(p_k'^{(\phi_x)}/a\right) q_j^{(u)} q_h^{(w)} dxdy + B_{16}\iint\limits_R p_i'^{(u)} \left(p_k^{(\phi_x)}/a\right) q_j^{(u)} q_h'^{(w)} dxdy \\
&\quad + B_{16}\iint\limits_R p_i^{(u)} \left(p_k'^{(\phi_x)}/a\right) q_j'^{(u)} q_h^{(w)} dxdy + B_{66}\iint\limits_R p_i^{(u)} \left(p_k^{(\phi_x)}/a\right) q_j'^{(u)} p_h'^{(w)} dxdy
\end{aligned}
$$

$$
\begin{aligned}
K_{ijkh}^{u\phi_y} &= B_{12}\iint\limits_R p_i'^{(u)} p_k^{(w)} q_j^{(u)} \left(q_h'^{(\phi_y)}/b\right) dxdy + B_{16}\iint\limits_R p_i'^{(u)} p_k'^{(w)} q_j^{(u)} \left(q_h^{(\phi_y)}/b\right) dxdy \\
&\quad + B_{26}\iint\limits_R p_i^{(u)} p_k^{(w)} q_j'^{(u)} \left(q_h'^{(\phi_y)}/b\right) dxdy + B_{66}\iint\limits_R p_i^{(u)} p_k'^{(w)} q_j'^{(u)} \left(q_h^{(\phi_y)}/b\right) dxdy
\end{aligned}
$$

$$
\begin{aligned}
K_{ijkh}^{vv} &= A_{22}\iint\limits_R p_i^{(v)} p_k^{(v)} q_j'^{(v)} q_h'^{(v)}\, dxdy + A_{26}\iint\limits_R \left(p_i^{(v)} p_k'^{(v)} q_j'^{(v)} q_h^{(v)} + p_i'^{(v)} p_k^{(v)} q_j^{(v)} q_h'^{(v)}\right) dxdy \\
&\quad + A_{66}\iint\limits_R p_i'^{(v)} p_k'^{(v)} q_j^{(v)} q_h^{(v)}\, dxdy
\end{aligned}
$$

$$
\begin{aligned}
K_{ijkh}^{v\phi_x} &= B_{12}\iint\limits_R p_i^{(v)} p_k'^{(\phi_x)} q_j'^{(v)} \left(q_h^{(w)}/a\right) dxdy + B_{16}\iint\limits_R p_i'^{(v)} p_k'^{(\phi_x)} q_j^{(v)} \left(q_h^{(w)}/a\right) dxdy \\
&\quad + B_{26}\iint\limits_R p_i^{(v)} p_k^{(\phi_x)} q_j'^{(v)} \left(q_h'^{(w)}/a\right) dxdy + B_{66}\iint\limits_R p_i'^{(v)} p_k^{(\phi_x)} q_j^{(v)} \left(q_h'^{(w)}/a\right) dxdy
\end{aligned}
$$

$$
\begin{aligned}
K_{ijkh}^{v\phi_y} &= B_{22}\iint\limits_R p_i^{(v)} p_k^{(w)} q_j'^{(v)} \left(q_h'^{(\phi_y)}/b\right) dxdy + B_{26}\iint\limits_R p_i^{(v)} p_k'^{(w)} q_j'^{(v)} \left(q_h^{(\phi_y)}/b\right) dxdy \\
&\quad + B_{26}\iint\limits_R p_i'^{(v)} p_k^{(w)} q_j^{(v)} \left(q_h'^{(\phi_y)}/b\right) dxdy + B_{66}\iint\limits_R p_i'^{(v)} p_k'^{(w)} q_j^{(v)} \left(q_h^{(\phi_y)}/b\right) dxdy
\end{aligned}
$$

$$
\begin{aligned}
K_{ijkh}^{ww} &= A_{44}\iint\limits_R p_i^{(w)} p_k^{(w)} q_j'^{(w)} q_h'^{(w)} dxdy + A_{55}\iint\limits_R p_i'^{(w)} p_k'^{(w)} q_j^{(w)} q_h^{(w)} dxdy \\
&\quad + A_{45}\iint\limits_R \left(p_i'^{(w)} p_k^{(w)} q_j^{(w)} q_h'^{(w)} + p_i^{(w)} p_k'^{(w)} q_j'^{(w)} q_h^{(w)}\right) dxdy
\end{aligned}
$$

$$K_{ijkh}^{w\phi_x} = K\left(A_{45}\iint_R p_i^{(w)}(p_k^{(\phi_x)}/a)q_j'^{(w)}q_h^{(w)}dxdy + A_{55}\iint_R p_i'^{(w)}(p_k^{(\phi_x)}/a)q_j^{(w)}q_h^{(w)}dxdy\right)$$

$$K_{ijkh}^{w\phi_y} = K\left(A_{44}\iint_R p_i^{(w)}p_k^{(w)}q_j'^{(w)}(q_h^{(\phi_y)}/b)dxdy + A_{45}\iint_R p_i'^{(w)}p_k^{(w)}q_j^{(w)}(q_h^{(\phi_y)}/b)dxdy\right)$$

$$\begin{aligned}K_{ijkh}^{\phi_x\phi_x} &= KA_{55}\iint_R (p_i^{(\phi_x)}/a)(p_k^{(\phi_x)}/a)q_j^{(w)}q_h^{(w)}dxdy + D_{11}\iint_R (p_i'^{(\phi_x)}/a)(p_k'^{(\phi_x)}/a)q_j^{(w)}q_h^{(w)}dxdy\\ &+D_{16}\iint_R ((p_i'^{(\phi_x)}/a)(p_k^{(\phi_x)}/a)q_j^{(w)}q_h'^{(w)} + (p_i^{(\phi_x)}/a)(p_k'^{(\phi_x)}/a)q_j'^{(w)}q_h^{(w)})dxdy\\ &+D_{66}\iint_R (p_i^{(\phi_x)}/a)(p_k^{(\phi_x)}/a)q_j'^{(w)}q_h^{(w)}dxdy\end{aligned}$$

$$\begin{aligned}K_{ijkh}^{\phi_x\phi_y} &= KA_{45}\iint_R (p_i^{(\phi_x)}/a)p_k^{(w)}q_j^{(w)}(q_h^{(\phi_y)}/b)dxdy + D_{12}\iint_R (p_i'^{(\phi_x)}/a)p_k^{(w)}q_j^{(w)}(q_h'^{(\phi_y)}/b)dxdy\\ &+D_{16}\iint_R (p_i'^{(\phi_x)}/a)p_k'^{(w)}q_j^{(w)}(q_h^{(\phi_y)}/b)dxdy + D_{26}\iint_R (p_i^{(\phi_x)}/a)p_k^{(w)}q_j'^{(w)}(q_h'^{(\phi_y)}/b)dxdy\\ &+D_{66}\iint_R (p_i^{(\phi_x)}/a)p_k'^{(w)}q_j'^{(w)}(q_h^{(\phi_y)}/b)dxdy\end{aligned}$$

$$\begin{aligned}K_{ijkh}^{\phi_y\phi_y} &= KA_{44}\iint_R p_i^{(w)}p_k^{(w)}(q_j^{(\phi_y)}/b)(q_h^{(\phi_y)}/b)dxdy + D_{22}\iint_R p_i^{(w)}p_k^{(w)}(q_j'^{(\phi_y)}/b)(q_h'^{(\phi_y)}/b)dxdy\\ &+D_{26}\iint_R (p_i^{(w)}p_k'^{(w)}(q_j'^{(\phi_y)}/b)(q_h^{(\phi_y)}/b) + p_i'^{(w)}p_k^{(w)}(q_j^{(\phi_y)}/b)(q_h'^{(\phi_y)}/b))dxdy\\ &+D_{66}\iint_R p_i'^{(w)}p_k'^{(w)}(q_j^{(\phi_y)}/b)(q_h^{(\phi_y)}/b)dxdy\end{aligned}$$