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4D Pyritohedral Symmetry

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ABSTRACT: We describe an extension of the pyritohedral symmetry in 3D to 4-dimensional Euclidean space and construct the group elements of the 4D pyritohedral group of order 576 in terms of quaternions. It turns out that it is a maximal subgroup of both the rank-4 Coxeter groups $W(F_4)$ and $W(H_4)$, implying that it is a group relevant to the crystallographic as well as quasicrystallographic structures in 4-dimensions. We derive the vertices of the 24 pseudoicosahedra, 24 tetrahedra and the 96 triangular pyramids forming the facets of the pseudo snub 24-cell. It turns out that the relevant lattice is the root lattice of $W(D_4)$. The vertices of the dual polytope of the pseudo snub 24-cell consists of the union of three sets: 24-cell, another 24-cell and a new pseudo snub 24-cell. We also derive a new representation for the symmetry group of the pseudo snub 24-cell and the corresponding vertices of the polytopes.

Keywords: Pseudoicosahedron; Pyritohedron; Lattice; Coxeter groups and Quaternions.

تماثل متعدد الأوجه ذو التركيب المشابه لمعدن البايريت في الفضاء الاقليدي ذو الأبعاد الأربعة

نظيفة كوجا، أمل القنوبي و محمد كوجا

مستخلص: وصفنا امتدادا لتماثل تركيب متعددات الأوجه من النوع المشابه لبلورة معدن البايرايت (pyritohedral) - و الذي يمكن صياغته في الابعاد الثلاثة - إلى الفضاء الاقليدي ذو الأبعاد الأربعة ثم بنينا عناصر المجموعة المتصلة ببلورة معدن البايرايت رباعية الابعاد باستخدام العدد المركب التخيلي الرباعي (الكواتيرنيونات)، و اتضح أن هذه المجموعة من أكبر المجموعات الفرعية لكل من مجموعات الكوكستر $W(F_4)$ و $W(H_4)$ ذات الرتبة الرابعة مما يعني أنها مجموعة ذات صلة بالتركيبات البلورية فضلا عن اشباه البلورية في الأبعاد الأربعة، كما تم اشتقاق احداثيات رؤوس 24 ذوات العشرون وجها غير المنتظم و 24 رباعي الأسطح و 96 أهراما مثلثية و التي تمثل أسطح لمجسم رباعي الأبعاد يسمى pseudo snub 24-cell ، و اتضح أن الشبيكة الفراغية ذات الصلة هي جذر الشبيكة الفراغية لـ $W(D_4)$ و أن رؤوس المزدوج لـ pseudo snub 24-cell هي اتحاد ثلاث مجموعات: اربعة و عشرون خلية (24cell) و اربعة و عشرون خلية (24-cell) أخرى و pseudo snub 24-cell جديد، كذلك اشتققنا تمثيلا جديدا لمجموعة تماثل pseudo snub 24-cell و رؤوس الأجسام رباعية الأبعاد المقابلة.

كلمات مفتاحية: شبيه ذي العشرين وجها المنتظم، متعدد الأوجه ذو التركيب المشابه لمعدن البايرايت، الشبيكة الفراغية، مجموعات كوكستر، الرباعيات.

1. Introduction

Lattices in higher dimensions described by the affine Coxeter groups, when projected into lower dimensions, may represent the quasicrystal structures [1-5]. It is known that the A_4 lattice projects into the aperiodic lattice with 5-fold symmetry [1]. There is no doubt that the projections of the higher dimensional lattices may have some implications in physics. The exceptional Coxeter-Weyl group $W(F_4)$ describes the symmetry of the unique self-dual polytope, the 24-cell, which is the Voronoi cell (Wigner-Seitz cell) of the F_4 lattice. The noncrystallographic Coxeter group $W(H_4)$ is the symmetry of the famous 600-cell and its dual 120-cell [6-7].

In this work we construct the 4D pyritohedral group from D_4 diagram. In technical terms the group $\left({W(D_4)}/{C_2}\right):S_3$ of order 576 [8] can be expressed in terms of quaternions and we will determine its orbits as the pseudo snub 24-cell and its dual polytope, which are related to the lattice $W(D_4)$. All rank 4 Coxeter-Weyl groups can be represented, in compact forms, by quaternion pairs [9].

This paper is organized as follows. In Section 2 we introduce the 4D - pyritohedral symmetry derived from D_4 diagram. In Section 3 we construct the group $\left(W(D_4)/C_2\right):S_3$ and apply it to a vector to generate the vertices of a polytope which we call "pseudo snub 24-cell". We find the facets of the pseudo snub 24-cell which consist of the pseudoicosahedra, tetrahedra and triangular pyramids. The vertices of the dual polytope of the pseudo snub 24-cell are constructed. Finally, in Section 4 we present a brief discussion on the physical implications of our technique.

2. 4D crystals with the pyritohedral symmetry derived from D_4 diagram

In the paper [10] we discussed the pyritohedral group $[T,\bar{T}]\cup[T,\bar{T}]^*$ of order 24 which is derived from the Coxeter-Dynkin diagram D_3 by the rotation generators and the Dynkin diagram symmetry. The straightforward generalization of this group to 4D is to start with the rotation generators of D_4 and impose the Dynkin diagram symmetry. We will see that the generated group from the diagram D_4 is nothing other than the group $[T,T]\cup[T,T]^*$ of order 576. It represents the symmetry of the snub 24-cell [6], [8]. The snub 24-cell is a convex uniform polytope in four dimensions consisting of 120 regular tetrahedral and 24 icosahedral cells. It has 96 vertices at each of which five tetrahedra and three icosahedra meet.

Snub 24-cell can be constructed from the 24-cell by dividing the edges in the golden ratio and truncating it in a certain way. This truncation transforms the 24 octahedral cells of the 24-cell to the 24 icosahedral cells of the snub 24-cell; the truncated vertices become 24 tetrahedral cells and the gaps in between are filled in by another 96 tetrahedra.

3. Construction of the symmetry group of snub 24-cell $\left(W(D_4)/C_2\right):S_3$

The Coxeter-Dynkin diagram D_4 is shown in Figure 1 with the quaternionic simple roots.

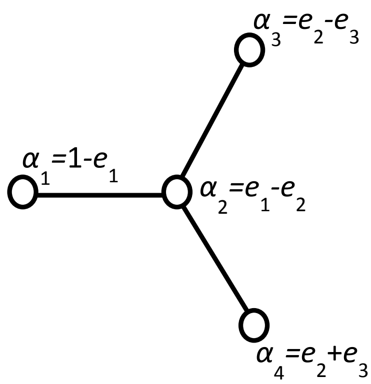


Figure 1. The Coxeter-Dynkin diagram D_4 with the simple roots.

The corresponding weights are determined as

$$\omega_1 = 1,\quad \omega_2 = 1+e_1,\quad \omega_3 = \frac{1}{2}(1+e_1+e_2-e_3),\quad \omega_4 = \frac{1}{2}(1+e_1+e_2+e_3). \tag{1}$$

Note that $\frac{\alpha_i}{\sqrt{2}} \in T'$, $i = 1,2,3,4$; $\frac{\omega_2}{\sqrt{2}} \in T'$; $\omega_j \in T$, j=1,3,4.

The set T is given by the group elements

$$T = \left\{\pm 1, \pm e_1, \pm e_2, \pm e_3, \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3)\right\}, \tag{2}$$

and is called the binary tetrahedral group of order 24 . Another set of 24 quaternions is defined by

$$T' = \begin{Bmatrix} \frac{1}{\sqrt{2}}(\pm 1 \pm e_1), \frac{1}{\sqrt{2}}(\pm 1 \pm e_2), \frac{1}{\sqrt{2}}(\pm 1 \pm e_3), \\ \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2), \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3), \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1) \end{Bmatrix}. \tag{3}$$

In terms of quaternionic simple roots, the group generators of $W(D_4)$ can be written as

$$\begin{aligned} r_1 &= [\frac{1}{\sqrt{2}}(1-e_1),\ -\frac{1}{\sqrt{2}}(1-e_1)]^*, \\ r_2 &= [\frac{1}{\sqrt{2}}(e_1-e_2),\ -\frac{1}{\sqrt{2}}(e_1-e_2)]^*, \\ r_3 &= [\frac{1}{\sqrt{2}}(e_2-e_3),\ -\frac{1}{\sqrt{2}}(e_2-e_3)]^*, \\ r_4 &= [\frac{1}{\sqrt{2}}(e_2+e_3),\ -\frac{1}{\sqrt{2}}(e_2+e_3)]^*. \end{aligned} \tag{4}$$

They generate the Coxeter-Weyl group $W(D_4)$ of order 192 [12]. The subsets of the quaternions $T = V_0 \cup V_+ \cup V_-$ and $T' = V_1 \cup V_2 \cup V_3$ are defined as follows:

$$V_0 = \{\pm 1, \pm e_1, \pm e_2, \pm e_3\}, V_+ = \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \text{, even number of (-) sign,}$$

$$V_- = \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \text{, odd number of (-) sign,}$$

$$\begin{aligned} V_1 &= \{\frac{1}{\sqrt{2}}(\pm 1 \pm e_1), \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3)\}, \\ V_2 &= \{\frac{1}{\sqrt{2}}(\pm 1 \pm e_2), \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1)\}, \\ V_3 &= \{\frac{1}{\sqrt{2}}(\pm 1 \pm e_3), \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2)\}. \end{aligned} \tag{5}$$

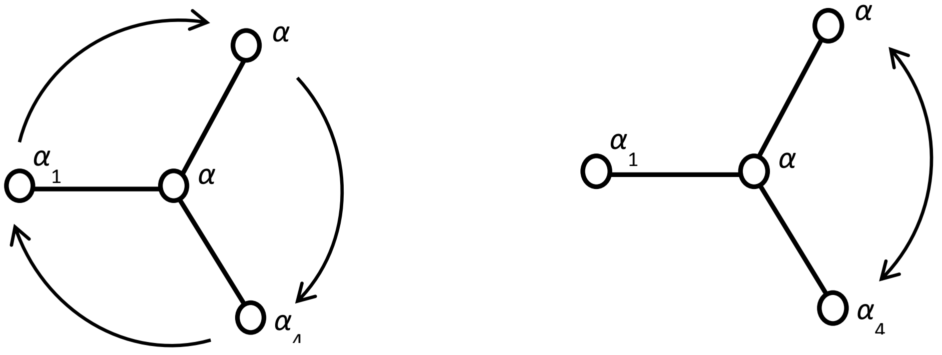
These subsets are useful to denote the Coxeter-Weyl group $W(D_4)$ in a compact form:

$$W(D_4) = \{[V_0, V_0] \cup [V_+, V_-] \cup [V_-, V_+] \cup [V_1, V_1]^* \cup [V_2, V_2]^* \cup [V_3, V_3]^*\}. \tag{6}$$

Note that the subset of the Coxeter-Weyl group

$$W(D_4)/C_2 = \{[V_0, V_0] \cup [V_+, V_-] \cup [V_-, V_+]\} \tag{7}$$

represents the proper subgroup and can be directly generated by the rotation generators r_2r_1, r_2r_3, r_2r_4. Let us impose the Dynkin diagram symmetry which is the permutation group S_3 of the simple roots α_1, α_3 and α_4 as shown in Figure 2.



Generator 1: $[p,q]$ Generator 2: $[e_3,-e_3]^*$

Figure 2 **.** The action of permutation group $\boldsymbol{S_3}$.

The permutation group of order 6 can be generated, for example, by two generators $[p,q]$ and $[e_3,-e_3]^*$ where $p=\frac{1}{2}(1+e_1-e_2-e_3)$ and $q=\frac{1}{2}(1-e_1+e_2-e_3)$. They are the elements of $[p,q]\in[V_+,V_+]$ with $[p,q]^3=[1,1]$ and $[e_3,-e_3]^*\in[V_0,V_0]^*$. The group $\left(W(D_4)/C_2\right)$ is invariant under conjugation by the group S_3. We first note that the extension of the group of eq. (7) by the cyclic group of order 3 generated by the generator $[p,q]$ is a group of order 288 which can be denoted by

$$\left(W(D_4)/C_2\right):C_3=[T,T]. \tag{8}$$

The extension of the group by full permutation group S_3 is given as the semi-direct product of two groups as:

$$\left(W(D_4)/C_2\right):S_3=\{[T,T]\cup[T,T]^*\}. \tag{9}$$

As we will see in the next section this is the symmetry group of the snub 24-cell as well as that of any pseudo snub 24-cell.

3.1 Construction of the vertices of the pseudo snub 24-cell

The affine Coxeter group $W_a(D_4)=<r_0,r_1,r_2,r_3,r_4>$ can be generated by five generators by introducing r_0 as shown in Figure 3. r_0 represents the reflection with respect to the hyperplane bisecting the line from the origin to the highest root $\tilde{\alpha}=\alpha_1+2\alpha_2+\alpha_3+\alpha_4=1+e_1=-\alpha_0$.

The action of r_0 on a general vector Λ is given as $r_0\Lambda=\Lambda-\frac{2[(\Lambda,\tilde{\alpha})-1]}{(\tilde{\alpha},\tilde{\alpha})}\tilde{\alpha}$ [13].

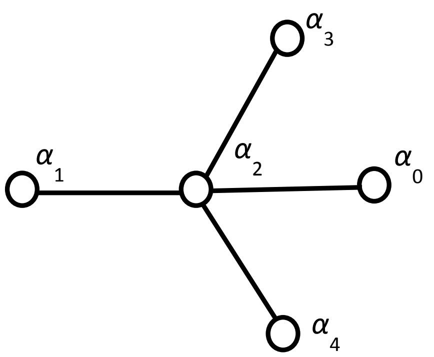


Figure 3. Affine Coxeter-Dynkin diagram $\boldsymbol{W_a(D_4)}$ with simple roots.

Applying the group $W_a\,(D_4)$ on a simple root we can generate the root lattice. We now derive the vertices of pseudo snub 24- cell in terms of the root lattice vectors of D_4. The lattice of D_4 is self-dual so that we can express the dual lattice vectors in terms of the weight vectors [14]. Table 1 shows the lattice vectors in terms of the root & weight vectors and quaternions.

Table 1. The construction of the lattice from $W_a\,(D_4)$.

	$W_a(D_4)$ generates	
Lattice	Root lattice (*Real lattice*)	dual	Weight lattice (*Reciprocal lattice*)
Vectors	$\Lambda = b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 + b_4\alpha_4$ $b_i \in Z$	$\Lambda = a_1\omega_1 + a_2\omega_2 + a_3\omega_3 + a_4\omega_4$ $a_i \in Z$
Vectors in terms of quaternions	$\Lambda = m_1 e_1 + m_2 e_2 + m_3 e_3 + m_4$	$\Lambda = n_1 e_1 + n_2 e_2 + n_3 e_3 + n_4$
Determine the lattice	$\sum_{i=0}^{3} m_i = 2b_4 = \text{even integer}$	dual	$\sum_{i=0}^{3} n_i = \text{even integer}$

We impose the Dynkin-diagram symmetry S_3 on the vector Λ, that is, $S_3\Lambda = \Lambda$ as shown in Figure 2. The group S_3 permutes the weight vectors ω_1, ω_3 *and* ω_4 but leaves ω_2 invariant. Similarly, the generators r_1, r_3 *and* r_4 are permuted and the generator r_2 is left invariant under the group conjugation of S_3.This implies that the vector Λ takes the form

$$\Lambda = a_2\omega_2 + a_1(\omega_1 + \omega_3 + \omega_4),\ a_i \in Z,\ i = 1,2. \tag{10}$$

Factorizing by a_2 and defining the rational number $x = \frac{a_1}{a_2}$ the vector Λ reads in terms of quaternions

$$\Lambda = a_2[(1+2x) + (1+x)e_1 + xe_2]\ . \tag{11}$$

Note that the sum of the coefficients of the quaternionic units is an even integer as we mentioned earlier. For $x = -1, -\frac{1}{2}$ and 0 the vector Λ belongs to the set of quaternions, $\Lambda \in T'$ which is known to be the 24-cell. For these particular values of x, the group $\left({W(D_4)}/{C_2}\right):S_3 = \{[T,T] \cup [T,T]^*\}$ generates 24 vertices of T'. The 24-cell has 24 vertices and 24 octahedral cells where 6 octahedra meet at one vertex. The polytope 24-cell constitutes the unit cell of the D_4 lattice. In terms of the quaternionic sets when T represents the unit cell of the root lattice, then the set $\frac{T'}{\sqrt{2}}$ represents its Voronoi cell. Applying the group elements represented by eq. (9) on the vector in (11) we obtain 96 vertices of pseudo snub 24-cell as the orbit of the group $\left({W(D_4)}/{C_2}\right):S_3$ as expected for $\frac{576}{6} = 96$. We list 96 vertices of the pseudo snub 24-cell omitting the overall factor a_2 as follows:

$$\begin{aligned} S(x) = \{ & \pm(1+2x) \pm (1+x)e_1 \pm xe_2, \pm x \pm (1+2x)e_1 \pm (1+x)e_2, \pm(1+x) \pm xe_1 \pm (1+2x)e_2, \\ & \pm(1+2x) \pm (1+x)e_2 \pm xe_3, \pm x \pm (1+2x)e_2 \pm (1+x)e_3, \pm(1+x) \pm xe_2 \pm (1+2x)e_3, \\ & \pm(1+2x) \pm xe_1 \pm (1+x)e_3, \pm x \pm (1+x)e_1 \pm (1+2x)e_3, \pm(1+x) \pm (1+2x)e_1 \pm xe_3, \\ & \pm(1+2x)e_1 \pm xe_2 \pm (1+x)e_3, \pm xe_1 \pm (1+x)e_2 \pm (1+2x)e_3, \pm(1+x)e_1 \pm (1+2x)e_2 \pm xe_3 \}. \end{aligned} \tag{12}$$

The mirror image of the pseudo snub 24-cell can be obtained by applying any reflection generator of D_4. For example, applying r_2 on $S(x)$ in (12) interchanges $e_1 \leftrightarrow e_2$ and leaves the other quaternionic units unchanged. Then under the action of the mirror operator r_2 one can obtain the mirror image of the pseudo snub 24-cell in (12). Both the pseudo snub 24-cell and its mirror image lie in the D_4 lattice. It is clear that for $x=-1,-\frac{1}{2}$ and 0 the vertices of (12) reduces to the set T'. Excluding these values of x, the set of vertices represent a pseudo snub 24-cell. Only in the limit $x \to \frac{1\pm\sqrt{5}}{2}$ ($x \to \tau, \sigma$) the vertices given in (13) represent the snub 24-cell [8]. Since in this case x is not a rational number, the vertices do not belong to the lattice D_4.

$$S=\left\{\begin{array}{c}\frac{1}{2}(\pm\tau \pm e_1 \pm \sigma e_3), \frac{1}{2}(\pm\tau \pm e_2 \pm \sigma e_1), \frac{1}{2}(\pm\tau \pm e_3 \pm \sigma e_2), \\ \frac{1}{2}(\pm\sigma \pm e_1 \pm \tau e_3), \frac{1}{2}(\pm\sigma \pm e_2 \pm \tau e_1), \frac{1}{2}(\pm\sigma \pm e_3 \pm \tau e_2), \\ \frac{1}{2}(\pm 1 \pm \tau e_1 \pm \sigma e_2), \frac{1}{2}(\pm 1 \pm \tau e_2 \pm \sigma e_3), \frac{1}{2}(\pm 1 \pm \tau e_3 \pm \sigma e_1), \\ \frac{1}{2}(\pm\sigma e_1 \pm \tau e_2 \pm e_3), \frac{1}{2}(\pm\sigma e_2 \pm \tau e_3 \pm e_1), \frac{1}{2}(\pm\sigma e_3 \pm \tau e_1 \pm e_2).\end{array}\right\}. \tag{13}$$

Table 2 summarizes the action of the group $\left(W(D_4)/C_2\right):S_3$ on Λ given in (11) for certain x values.

Table 2. Action of the group $\left(W(D_4)/C_2\right):S_3$ on Λ.

x	Vertices	Polytope generated	#vertices	#cells	Type of cell
$x=-1,-\frac{1}{2}$ and 0	T' in (3)	24-cell	24	24	24 octahedra
$x=\sigma$	S in (13)	Snub 24-cell	96	144	24 icosahedra, 24 tetrahedra, 96 tetrahedra
$x=\tau$	Mirror image of S	Snub 24-cell	96	144	24 icosahedra, 24 tetrahedra, 96 tetrahedra
Any other x	$S(x)$ in (12)	Pseudo snub 24-cell	96	144	24 pseudoicosahedra, 24 tetrahedra, 96 triangular pyramid

3.2 Determination of facets of pseudo 24-cell

It is known that every vertex of the snub 24-cell is surrounded by three icosahedra and five tetrahedra. We shall prove that the facets of the pseudo snub 24-cell consist of pseudo icosahedra, tetrahedra and triangular pyramids. In the pseudo snub 24-cell three pseudoicosahedra, one tetrahedron and four triangular pyramids meet at the same vertex. Now we discuss the details of this structure. It is evident from the D_4 diagram that each of the following sets of rotation generators (r_1r_2, r_2r_3), (r_3r_2, r_2r_4) and (r_4r_2, r_2r_1) generate a proper subgroup of the tetrahedral group of order 12 as shown in Figure 4.

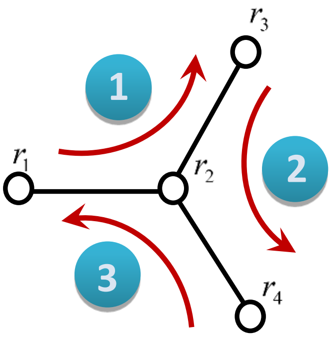


Figure 4. Three proper subgroups of the tetrahedral group from the $\boldsymbol{D_4}$ diagram.

Let us discuss how one of these groups acts on the vertex Λ. Let us take the group $<r_1r_2, r_2r_3> = [T, \bar{\omega}_4\bar{T}\omega_4]$ (case 1 in Figure 4) which implies that the group consists of 12 elements leaving the weight vector $\omega_4 \in T$ invariant. The group generators transform quaternionic units as follows:

$$\begin{aligned} &r_1r_2 : 1 \to e_1 \to e_2 \to 1,\ e_3 \to e_3 \\ &r_2r_3 : 1 \to 1,\ e_1 \to e_2 \to e_3 \to e_1. \end{aligned} \tag{14}$$

The 12 vertices generated from $\Lambda = [(1+2x)+(1+x)e_1 + xe_2]$ (here we dropped the overall scale factor) the group $<r_1r_2, r_2r_3>$ are determined as:

$$\begin{aligned} &(1+2x)+(1+x)e_1 + xe_2,\ x+(1+2x)e_1+(1+x)e_2,\ (1+x)+xe_1+(1+2x)e_2, \\ &(1+2x)+(1+x)e_2 + xe_3,\ x+(1+2x)e_2+(1+x)e_3,\ (1+x)+xe_2+(1+2x)e_3, \\ &(1+2x)+xe_1+(1+x)e_3,\ x+(1+x)e_1+(1+2x)e_3,\ (1+x)+(1+2x)e_1+xe_3, \\ &(1+2x)e_1+xe_2+(1+x)e_3,\ xe_1+(1+x)e_2+(1+2x)e_3,\ (1+x)e_1+(1+2x)e_2+xe_3. \end{aligned} \tag{15}$$

Since the vector ω_4 is left invariant by the generators in (14) the center of the polyhedron of (15) can be taken as ω_4 up to a scale factor. One may check that the set in (15) is also left invariant under the group element $[\omega_4, \omega_4]^*$. Therefore, the set (15) is invariant under the larger group $T_h \approx [T, \bar{\omega}_4\bar{T}\omega_4] \cup [T, \omega_4\bar{T}\omega_4]^*$ of order 24 isomorphic to the pyritohedral group. If we define a new set of unit quaternions $p_0 = \omega_4, p_1 = e_1p_0, p_2 = e_2p_0, p_3 = e_3p_0$ and express the vectors in (15) in terms of the new set of quaternions, then the set of vertices in (15) represent a pseudoicosahedron as shown in Figure 5(a). When the set of generators $<r_3r_2, r_2r_4>$ (case 2 in Figure 4) and $<r_4r_2, r_2r_1>$ (case 3 in Figure 4) respectively are applied to the vector Λ, one generates two more pseudo icosahedra with the centers represented by ω_1 and ω_3 respectively. The groups generating the vertices of the second and the third pseudo icosahedra can be written respectively as $[T, \bar{\omega}_1\bar{T}\omega_1] \cup [T, \omega_1\bar{T}\omega_1]^*$, $\omega_1 \in T$ and $[T, \bar{\omega}_3\bar{T}\omega_3] \cup [T, \omega_3\bar{T}\omega_3]^*$, $\omega_3 \in T$ .These groups are isomorphic to the pyritohedral group $T_h = \{[T, \bar{T}] \cup [T, -\bar{T}]\}$. This is obtained if $\omega_1 = 1$ is substituted above in the basis vectors e_1, e_2 and e_3 in 3D. Note that $[1, -1] \in [T, T]$ is also an element of the group $[T, T]$ which commutes with all elements and sends a quaternion to its negative $q \to -q$. With the generator $[1, -1]$ one obtains a larger group $T_h \times C_2 \approx [T, \pm\bar{t}\bar{T}t] \cup [T, \pm t\bar{T}t]^*$, $t \in T$ of order 48 leaving the vector $\pm t$ invariant so that the group can be embedded in the group $\left(W(D_4)/C_2\right) : S_3$ 12 different ways. The group $[T, T] \cup [T, T]^*$ also leaves the set of quaternions T' invariant. The largest subgroup which leaves $\pm t' \in T'$ invariant can be written as $T_d \times C_2 \approx [T, \pm\bar{t}'\bar{T}t'] \cup [T, \pm t'\bar{T}t']^*$, $t' \in T'$.

Now we continue to discuss the polyhedral facets having Λ as a vertex. Let us consider the following five sets of rotational generators obtained from the generators of the Coxeter-Weyl group $W(D_4)$:

$$(r_1r_3, r_3r_4, r_4r_1),\ (r_2r_1, r_2r_3, r_2r_4),\ (r_1r_2, r_1r_3, r_1r_4),\ (r_3r_1, r_3r_2, r_3r_4),\ (r_4r_1, r_4r_2, r_4r_3). \tag{16}$$

The first two sets of generators are invariant under the conjugation of the permutation group S_3 but the next three sets of generators are permuted among each other. Let us determine the vertices of the polyhedra under the action of five sets.

1. The set of vertices

$$\begin{aligned}
\Lambda &= (1+2x)+(1+x)e_1+xe_2,\\
r_1r_3\Lambda &= (1+x)+(1+2x)e_1+xe_3,\\
r_3r_4\Lambda &= (1+2x)+(1+x)e_1-xe_2,\\
r_4r_1\Lambda &= (1+x)+(1+2x)e_1-xe_3,
\end{aligned} \tag{17}$$

represents a tetrahedron of edge length $2x$ as shown in Figure 5(b). Its center can be represented by $p(1)=\omega_2=1+e_1$ up to some scale factor. Since the Dynkin diagram symmetry S_3 also leaves $\omega_2=1+e_1$ invariant, the group $C_2\times C_2$ generated by the generators (r_1r_3, r_3r_4, r_4r_1) can be extended by the S_3 symmetry to a group $[T,\bar{\omega}_2\bar{T}\omega_2]\cup[T,\omega_2\bar{T}\omega_2]^*$ of order 24 isomorphic to the tetrahedral group [10]. Since $S_3\Lambda=\Lambda$, the group $T_d\approx[T,\bar{\omega}_2\bar{T}\omega_2]\cup[T,\omega_2\bar{T}\omega_2]^*, \omega_2\in T'$ leaves the vertices of the tetrahedron in (17) invariant (Figure 5b). Note that this not a pyritohedral symmetry. This tells us that the number of tetrahedra generated by the conjugate tetrahedral groups is also 24. Extension of the group by the generator $[1,-1]$ leads to the group $[T,\pm\bar{\omega}_2\bar{T}\omega_2]\cup[T,\pm\omega_2\bar{T}\omega_2]^*, \omega_2\in T'$ that is isomorphic to the octahedral group [10].

2. The set of vertices

$$\begin{aligned}
\Lambda &= (1+2x)+(1+x)e_1+xe_2,\\
r_2r_1\Lambda &= (1+x)+xe_1+(1+2x)e_2,\\
r_2r_3\Lambda &= (1+2x)+(1+x)e_2+xe_3,\\
r_2r_4\Lambda &= (1+2x)+(1+x)e_2-xe_3.
\end{aligned} \tag{18}$$

determines a triangular pyramid with a base of equilateral triangle with sides $2x$ and the other edges of length $\sqrt{2(x^2+x+1)}$. The triangular pyramid is depicted in Figure 5 (c). The hyperplane determined by these four vertices in (18) is orthogonal to the vector $p(2)=(2+x)+e_1+(1+x)e_2$.

The group generated by these generators extended by the group S_3 is the full group of symmetry $\left(W(D_4)/C_2\right):S_3$.

3. The set of vertices

$$\begin{aligned}
\Lambda &= (1+2x)+(1+x)e_1+xe_2,\\
r_1r_2\Lambda &= x+(1+2x)e_1+(1+x)e_2,\\
r_1r_3\Lambda &= (1+x)+(1+2x)e_1+xe_3,\\
r_1r_4\Lambda &= (1+x)+(1+2x)e_1-xe_3.
\end{aligned} \tag{19}$$

defines another triangular pyramid with the same edge lengths as above. The vector orthogonal to the hyperplane determined by the vertices in (19) is $p(3)=(1+x)+(2+x)e_1+e_2$.

4. The set of vertices

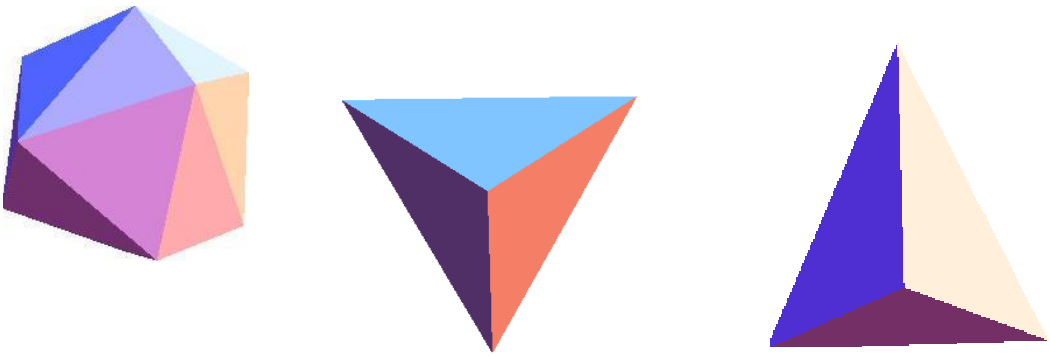
$$\begin{aligned}
\Lambda &= (1+2x)+(1+x)e_1+xe_2,\\
r_3r_1\Lambda &= (1+x)+(1+2x)e_1+xe_3,\\
r_3r_2\Lambda &= (1+2x)+xe_1+(1+x)e_3,\\
r_3r_4\Lambda &= (1+2x)+(1+x)e_1-xe_2.
\end{aligned} \tag{20}$$

also determines a triangular pyramid as above. The vector which is orthogonal to the hyperplane determined by the vectors of (20) can be computed as $p(4)=(2+x)+(1+x)e_1+e_3$.

5. The set of vertices

$$\begin{aligned}
\Lambda &= (1+2x)+(1+x)\,e_1+xe_2\,,\\
r_4r_1\Lambda &= (1+x)+(1+2x)\,e_1-xe_3\,,\\
r_4r_2\Lambda &= (1+2x)+xe_1-(1+x)\,e_3\,,\\
r_4r_3\Lambda &= (1+2x)+(1+x)\,e_1-x\,e_2\,.
\end{aligned} \tag{21}$$

defines another triangular pyramid with the same edge lengths as above. The vector orthogonal to the hyperplane of the vectors in (21) is $p(5)=(2+x)+(1+x)\,e_1-e_3$.



(a) (b) (c)

Figure 5. Facets of pseudo snub 24-cell: (a) Pseudoicosahedron for $\boldsymbol{x}=\boldsymbol{\tau}$, (b) Tetrahedron with edges for $x=1$, (c) Triangular pyramid.

Since the group is the full symmetry group, all vectors $p(i), i=2,3,4,5$ lie in the same orbit of the group $\left({W(D_4)}/{C_2}\right):S_3$.

Below we list these five vectors up to a some scale factors representing the centers of the above polyhedra expressed in terms of the weight vectors $\omega_i, i=1,2,3,4$:

$$\begin{aligned}
p(1) &= \omega_2\,,\\
p(2) &= (1+x)(\omega_1+\omega_3+\omega_4)-x\omega_2\,,\\
p(3) &= -\omega_1+\omega_3+\omega_4+(1+x)\omega_2\,,\\
p(4) &= \omega_1-\omega_3+\omega_4+(1+x)\omega_2\\
p(5) &= \omega_1+\omega_3-\omega_4+(1+x)\omega_2\,.
\end{aligned} \tag{22}$$

Now their symmetries are more transparent under the permutation group: S_3 permutes ω_1, ω_3 and ω_4 but leaves $\omega_2$invariant; $p(1)$ and $p(2)$ are invariant under the group S_3 but the others are permuted to each other. Pseudo snub 24-cell consists of $N_0=96$ vertices, $N_3=24+24+96=144$ cells consisting of pseudo icosahedra, tetrahedra and triangular pyramids respectively. It has $N_1=432$ edges and $N_2=480$ faces. These numbers satisfy the Euler characteristic formula $N_0-N_1+N_2-N_3=0$. Table 3 summarizes the facets of pseudo snub 24-cell.

Table 3. Facets of pseudo snub 24-cell.

Group generators	Group which generates the vertices	Vertices	Center of polyhedron	Polyhedron (facet) generated
$< r_1r_2, r_2r_3 >$	$[T,\bar{\omega}_4\bar{T}\omega_4]\cup[T,\omega_4\bar{T}\omega_4]^*$	in (15)	ω_4	pseudoicosahedron
$< r_3r_2, r_2r_4 >$	$[T,\bar{\omega}_1\bar{T}\omega_1]\cup[T,\omega_1\bar{T}\omega_1]^*$		ω_1	pseudoicosahedron
$< r_4r_2, r_2r_1 >$	$[T,\bar{\omega}_3\bar{T}\omega_3]\cup[T,\omega_3\bar{T}\omega_3]^*$		ω_3	pseudoicosahedron
(r_1r_3, r_3r_4, r_4r_1)	$[T,\pm\bar{\omega}_2\bar{T}\omega_2]\cup[T,\pm\omega_2\bar{T}\omega_2]^*$	in (17)	$P(1)=\omega_2$	tetrahedron

(r_2r_1, r_2r_3, r_2r_4)	$\left(W(D_4)/C_2\right): S_3$	in (18)	$P(2) = (1+x)(\omega_1 + \omega_3 + \omega_4) - x\omega_2$	triangular pyramid
(r_1r_2, r_1r_3, r_1r_4)		in (19)	$P(3) = -\omega_1 + \omega_3 + \omega_4 + (1+x)\omega_2$	
(r_3r_1, r_3r_2, r_3r_4),		in (20)	$P(4) = \omega_1 - \omega_3 + \omega_4 + (1+x)\omega_2$	
(r_4r_1, r_4r_2, r_4r_3)		in (21)	$P(5) = \omega_1 + \omega_3 - \omega_4 + (1+x)\omega_2$	

Table 3. Contd.

3.3 Construction of the vertices of the dual polytope of the pseudo snub-24 cell

To construct the dual of the pseudo snub 24-cell we need to determine the centers (orthogonal vectors to the hyperplane) of the pseudoicosahedra, the tetrahedron and the four pyramids up to some scale factors. The centers of the first three pseudo icosahedra can be taken as the weight vectors ω_1, ω_3 *and* ω_4 as shown in Table 3. The other vectors in (22) can be taken as the center (orthogonal vector to the hyperplane) of the tetrahedron and the centers of the four pyramids up to some scale vectors. Let us denote by $c(i),\ i = 1,2,\dots,5$, the centers of the respective tetrahedron and the pyramids and define

$$c(1) = \lambda p(1),\ \ c(i) = \eta p(i),\ \ i = 2,\ 3,\ 4,\ 5 \tag{23}$$

So we have eight vertices including ω_1, ω_3 *and* ω_4. To determine the actual centers of these polyhedra the hyperplane defined by the eight vectors must be orthogonal to the vector Λ. This will determine the scale factors and the centers of the cells which can be written as:

$$\begin{aligned} &\omega_1 = 1,\ \omega_3 = \frac{1}{2}(1+e_1+e_2-e_3),\ \omega_4 = \frac{1}{2}(1+e_1+e_2+e_3), \\ &c(1) = \frac{1+2x}{2+3x}(1+e_1), \\ &c(i) = \frac{1+2x}{3+7x+3x^2}\, p(i),\ i = 2,3,4,5. \end{aligned} \tag{24}$$

These eight vectors now determine the vertices of one facet of the dual polytope of the pseudo snub 24-cell. The center of this facet is the vector Λ. This is a convex solid with 8 vertices 15 edges and 9 faces possessing S_3 symmetry.

One can generate the vertices of the dual polytope by applying the group $\left(W(D_4)/C_2\right): S_3$ on the vertices (24) representing one of the facets of the dual polytope. One can display them as the union of three sets

$$T \cup \left(\frac{1+2x}{2+3x}\right)\sqrt{2}T' \cup \left(\frac{1+2x}{3+7x+3x^2}\right)R(x) \tag{25}$$

where T and T' are given in (2-3) and $R(x)$ can be written as follows

$$\begin{aligned} &R(x) = \\ &\{\pm(2+x)\pm e_1 \pm (1+x)e_2, \pm(1+x)\pm(2+x)e_1 \pm e_2, \pm 1 \pm (1+x)e_1 \pm (2+x)e_2, \\ &\pm(2+x)\pm e_2 \pm (1+x)e_3, \pm(1+x)\pm(2+x)e_2 \pm e_3, \pm 1 \pm (1+x)e_2 \pm (2+x)e_3, \\ &\pm(2+x)\pm e_3 \pm (1+x)e_1, \pm(1+x)\pm(2+x)e_3 \pm e_1, \pm 1 \pm (1+x)e_3 \pm (2+x)e_1, \\ &\pm(2+x)e_1 \pm e_2 \pm (1+x)e_3, \pm e_1 \pm (1+x)e_2 \pm (2+x)e_3, \pm(1+x)e_1 \pm (2+x)e_2 \pm e_3\}. \end{aligned} \tag{26}$$

As we substitute $x=-\frac{1}{2}$ in (25) what we obtain is set T, as expected because it is the dual of the set T'. If $x=-\frac{2}{3}$, the dual polytope does not exist. The roots of the quadratic equation $3+7x+3x^2=0$ are irrational numbers which are already excluded since the vertices of the pseudo snub 24-cell and its dual must remain in the D_4 lattice for x to be rational number. One wonders whether the vertices in (26) represent any familiar polytope. If we replace x by $x=-\frac{1}{y+1}$, the set $R(x\rightarrow-\frac{1}{y+1})$, in (26) takes exactly the same form of $S(y)$ in (13) apart from a scale factor. This implies that $R(x)$ represents another pseudo snub 24-cell. This proves that every dual of a pseudo snub 24-cell includes another pseudo snub 24-cell in addition to the two sets of 24-cells T and T'. The pseudo snub 24-cell $S(x)$ turns out to be snub 24-cell whose cells are regular icosahedra and tetrahedra when $x=\tau$ or $x=\sigma$. However, $R(\tau)$ and $R(\sigma)$ represent two pseudo snub 24-cells, the mirror images of each other. In brief, when $S(x)$ represents a snub 24-cell in its dual, there exists an orbit with 96 vertices representing another pseudo snub 24-cell, albeit the coefficients of the unit quaternions are irrational numbers [8]. Table 4 summarizes the way to obtain the dual of the snub 24-cell in terms of its vertices.

Table 4. Dual polytope of the pseudo snub 24-cell.

One facet (polyhedron) of pseudo snub 24-cell	vertices of one facet of the dual of the pseudo snub 24-cell = Centers of polyhedron	Apply $\left(W(D_4)/C_2\right):S_3$ on the vertices	The dual of the pseudo snub 24-cell
Pseudoicosahedron (1)	$\boldsymbol{\omega_4}$	Elements belong to T	24-cells (*24 vertices*)
Pseudoicosahedron (2)	$\boldsymbol{\omega_1}$		
Pseudoicosahedron (3)	$\boldsymbol{\omega_3}$		
Tetrahedron	$c(1)=\eta\, p(1)$	Elements belong to $\left(\frac{\mathbf{1+2x}}{\mathbf{2+3x}}\right)\sqrt{\mathbf{2}}\boldsymbol{T'}$	24-cells (*24 vertices*)
Triangular pyramid (1)	$c(2)=\eta\, p(2)$	Elements belong to $\eta\, R(x)$	Pseudo snub 24-cell (*96 vertices*)
Triangular pyramid (2)	$c(3)=\eta\, p(3)$		
Triangular pyramid (3)	$c(4)=\eta\, p(4)$		
Triangular pyramid (4)	$c(5)=\eta\, p(5)$		

4. Conclusion

We have studied the extension of the pyritohedral group $\left(W(D_3)/C_2\right):C_2'$ in 3D to the group $\left(W(D_4)/C_2\right):S_3$ acting in 4D Euclidean space. We have constructed the 4D polytope with 96 vertices (pseudo snub 24-cell) with the facets as 24 pseudo icosahedron, 24 tetrahedron and 96 triangular pyramids. We also derived its dual polytope with 144 vertices forming three orbits under the group. The explicit construction of the group in terms of quaternions has been worked out. The relevance of the group and the polytopes to the root lattice of the affine Coxeter group $W_a(D_4)$ has been pointed out. We have pointed out that the dual of the pseudo snub 24-cell is the union of the sets $T\ and\ T'$ representing two 24-cells dual to each other and the set $S(x)$ with 96 vertices. For rational values of x the vertices belong to the lattice D_4. The snub 24-cell is no more in the lattice D_4 but belongs to the quasi-lattice of the Coxeter group $W(H_4)$ which can be obtained by projection of the lattice E_8 into 4D.

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