Predator-Prey Model with Refuge, Fear and Z-Control

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ABSTRACT: In this paper, we consider a predator-prey model incorporating fear and refuge. Our results show that the predator-free equilibrium is globally asymptotically stable if the ratio between the death rate of predators and the conversion rate of prey into predator is greater than the value of prey in refuge at equilibrium. We also show that the co-existence equilibrium points are locally asymptotically stable if the value of the prey outside refuge is greater than half of the carrying capacity. Numerical simulations show that when the intensity of fear increases, the fraction of the prey inside refuge increases; however, it has no effect on the fraction of the prey outside refuge, in the long run. It is shown that the intensity of fear harms predator population size. Numerical simulations show that the application of Z-control will force the system to reach any desired state within a limited time, whether the desired state is a constant state or a periodic state. Our results show that when the refuge size is taken to be a non-constant function of the prey outside refuge, the systems change their dynamics. Namely, when it is a linear function or an exponential function, the system always reaches the predator-free equilibrium. However, when it is taken as a logistic equation, the system reaches the co-existence equilibrium after long term oscillations.

Keywords: Predator-prey; Refuge; Fear; Z-control and Adaptive control.

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نموذج المفترس والفريسة مع ملجأ وخوف وتحكم Z

إبراهيم مجتبى،كوكب العامري و قمر الجليل أحمد خان

الملخص:في هذه الورقة ، نعتبر نموذج المفترس والفريسة الذي يتضمن الخوف والملجأ. تظهر نتائجنا أن التوازن الخالي من المفترس يكون مستقرًا بشكل مقارب عالميًا إذا كانت النسبة بين معدل وفاة المفترس ومعدل تحويل الفريسة إلى مفترس أكبر من قيمة الفريسة في الملجأ عند التوازن. نظهر أيضًا أن نقطة توازن التعايش مستقرة محليًا بشكل مقارب إذا كانت قيمة الفريسة خارج الملجأ أكبر من نصف القدرة الاستيعابية. تظهر المحاكاة العددية أنه عندما تزداد شدة الخوف، يزداد جزء الفريسة داخل الملجأ ، ولكن ليس لها أي تأثير على جزء الفريسة خارج الملجأ ، على المدى الطويل تبين أن شدة الخوف لها تأثير سلبي على عدد الحيوانات المفترسة. تظهر المحاكاة العددية أن تطبيق التحكم Z سيجبر النظام على الوصول إلى أي حالة مرغوبة خلال فترة زمنية محدودة ، سواء كانت حالة الرغبة حالة ثابتة أو حالة دورية. تظهر نتائجنا أنه عندما يتم اعتبار حجم الملجأ وظيفة غير ثابتة للفريسة خارج الملجأ ، فإن الأنظمة تغير ديناميكياتها. أي عندما تكون دالة خطية أو دالة أسية ، فإن النظام يصل دائمًا إلى التوازن الخالي من الحيوانات المفترسة. ومع ذلك ، عندما يتم أخذها كمعادلة لوجستية ، يصل النظام إلى توازن التعايش بعد التذبذبات طويلة المدى.

الكلمات المفتاحية:المفترس، فريس، لجأ، يخاف،Z- التحكم، التحكم التكيفي.

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PREDATOR-PREY MODEL WITH REFUGE, FEAR AND Z-CONTROL

1. Introduction

Predator-induced stress in prey animals has not been much studied by physiologists or population ecologists. Canon [1] studied the concept of predator induced stress in a prey population. Population ecologists have focused their studies on predator induced stress effects on the birth rate of free-living prey populations and found that it affects the demography process of prey animals seriously. By experiments in laboratories and field studies it has been demonstrated that mere exposure of prey animals to predators affects the birth rate of prey, and this phenomenon of behavioral change of demography is called the "Ecology of Fear", "Degree of Fear" or "Cost of Fear" by ecologists. Only a little work has been done on the subject of stress induced in a prey population due to predators. The concept of stress was limited to humans and it was thought that stress in a prey population is transitory and that it is not lifelong. Ecologists in the 1990's [2-4] verified experimentally that fear of predators is a more powerful cause of demographical change in a prey population than direct killing, shortage of food, or parasitic infection. Fear of predators in prey persists even in the absence of predators and has long-lasting effects on their production of prey species. Sapolsky [5] explained the concept of stress in zebras due to fear of lions. Zanette*et al*. [6] reported, after experimental verification, that the sparrow reduces offspring production by 40% just with intimidation by predators where direct killing is stopped by some means. Zanette*et al*.[6], Eggers *et al*. [7], and Travers *et al*. [8] verified that female sparrows lay fewer eggs and, due to incubation disruption, fail to hatch eggs. Due to fear, they bring less food to their nests and, as a result, a greater proportion of their nestlings starve to death. Creel *et al*. [9], Creeland Christianson [10], and Creel *et al*. [11] reported that in the National Parks, USA, due to intimidation by wolves, elk pregnancy rates decline. Recently, Monclus*et al*. [12] examined an association between predator risk and birth rate of prey.

Field studies have demonstrated that playback of predator sounds can affect the emotions of prey. Remage-Healey *et al*. [13] demonstrated that a playback sound of dolphins affects the emotions of gulf toad fish. Mateo [14] found that playback of the call of predators' alerts squirrels and that they communicate predator risk to each other. The playback sound of predators increases the glucocorticoid level in prey and hence increases the fear or stress in the prey population. Wang*et al*. [15] studied how fear of a predator reduces the reproduction of prey animals and found that it could destabilize the system.

A refuge is an area, such as island, where wild animals obtain protection from predation. In this protected territory the chances of being hunted by predator are reduced and these areas reduce the chances of extinction of prey species due to predatory killing. It is a natural phenomenon of prey species in an ecosystem to seek protection from predation (Cowlishaw [16], Sih [17]). Refuge habitats are of different types, such as burrows, trees, cliff faces, or dense vegetation (Clarke *et al*. [18], Dill and Houtman [19], Berger [20], Cassine[21]). Coral reefs provide refuge for prey fish (Friedlander and Martine [22], Sandin*et al*. [23]).

Various control systems are used to prevent a species from drastic oscillations and avoid extinctions in an ecological system. The literature records various techniques for control in the multi-species Lotka-Volterra System. These controls are called adaptive control, back idea of control, impulsive control and applications of control theory to Lyapunov functions. One can refer to [24,25,26], where Z-control obtains the desired steady state quickly and prevents high amplitude oscillations. The Z-type control method is an error-based dynamic method, and in this method, it is certain that error function converges to zero. The error between desired outputs and actual outputs go to zero exponentially. There are two ways to apply Z-control in a predator-prey system. The first method is called direct control, where both prey and predator are controlled simultaneously to bring the population to a desired level. The second method is called indirect control, where either prey species or predator species is controlled through immigration, emigration or culling. The second species automatically comes to the desired level exponentially.

Our model is similar to that of Wang *et al*. [27], and we examine this model by introducing predator fear. Prey only come out of refuge when they feel there is less predation; otherwise, they go back to the protected area. We use two control measures to bring the model population of prey and predator to a desired level, and thus we can save it from becoming extinct.

2. Model formulation

We studied a model where a prey species lives in two different habitats. One is called the refuge habitat where the prey species is saved from predation. It is assumed that all resources required for the growth of the prey species are available inside the refuge habitat and that their population grows logistically. It is also assumed that the prey species is fully protected from predation inside the refuge habitat. When pressure of predation fear is released, then the prey species moves to a second habitat outside the refuge and in this habitat the prey species can be killed by predators under the law of mass action. As predation fear increases in the prey species due to the presence of predators, the prey species migrates to the refuge habitat. In the absence of a prey species, predators die exponentially, because predators can only consume prey outside the refuge habitat. This predator-prey interaction is modelled by the following diagram and system of differential equations:
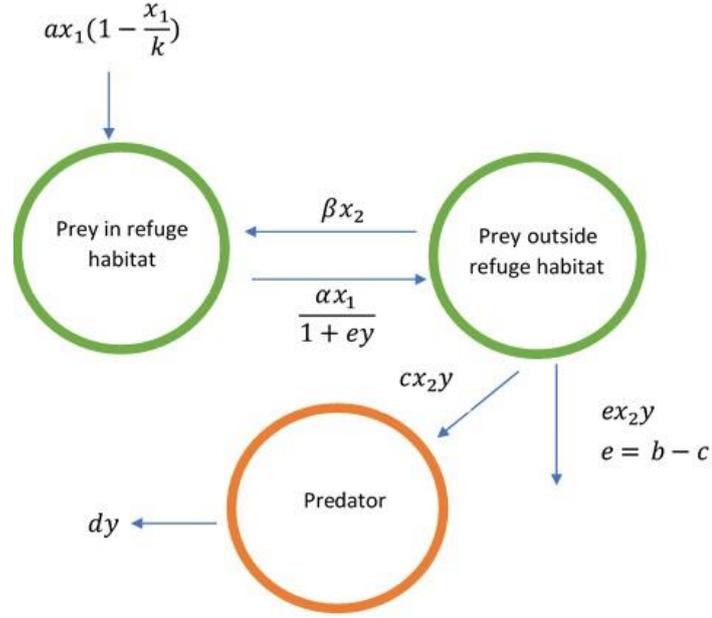
Figure 1. Compartmental representation of the model.

$$\frac{dx_1}{dt} = ax_1\left(1 - \frac{x_1}{k}\right) - \frac{\alpha x_1}{1 + ey} + \beta x_2$$

$$\frac{dx_2}{dt} = \frac{\alpha x_1}{1 + ey} - \beta x_2 - bx_2 y$$

$$\frac{dy}{dt} = cx_2 y - dy \tag{1}$$

All variables and parameters in model (1) are positive and defined below:
x_1 Prey density in the refuge habitat.
x_2 Prey density outside the refuge habitat.
y Abundance of predator species.
d Death rate of the predator.
k Carrying capacity of the prey in the refuge habitat.
b Feeding rate of the predator on the prey outside the refuge habitat.
c Conversion rate of prey to predator.
α Migration rate from the refuge habitat.
β Immigration rate into the refuge habitat.
a Birth rate of prey species inside refuge.
e The fear parameter.

3. Mathematical analysis of the model

3.1 Positivity of solutions

Model (1) describes the dynamics of animal populations and therefore it is very important to prove that all quantities will remain positive for all time. We want to prove that all solutions of the model with positive initial data will remain positive for all time $t>0$. We can easily verify that

$$\frac{dx_1}{dt}\Big|_{x_1=0} = \beta x_2 \geq 0$$

$$\frac{dx_2}{dt}\Big|_{x_2=0} = \frac{\alpha x_1}{1+ey} \geq 0$$

$$\frac{dy}{dt}\Big|_{y=0} = 0 \geq 0 \quad . \tag{2}$$

Hence, all solutions will remain positive for all time.

3.2 Boundedness

Proposition 1 The trajectories of system (1), are bounded.

Proof. Let $w = x_1 + x_2 + y$. Take the time derivative along the solution of model (1)

$$\frac{dw}{dt} = \frac{dx_1}{dt} + \frac{dx_2}{dt} + \frac{dy}{dt} = ax_1\left(1 - \frac{x_1}{k}\right) - bx_2y + cx_2y - dy$$

For any positive constant $q$we have:

$$\frac{dw}{dt} + qw \leq ax_1 - (b-c)x_2y + qx_1 + qx_2 + qy$$

Where $x_1 \leq k$and $x_2 \leq k$. So

$$\frac{dw}{dt} + qw \leq (a + 2q)k - [(b-c)\frac{d}{c} - q]y$$

Because$x_2 > \frac{d}{c}$. Now iff$\frac{bd}{c} > q + d$, then

$$\frac{dw}{dt} + qw \leq (a + 2q)k$$

Let $(a + 2q)k = L$.Therefore, we have

$$w \leq \frac{L}{q} + Ae^{-qt}$$

From which we can deduce that

$$\lim_{t\to\infty} \sup w \leq \frac{L}{q}$$

Independently of the initial conditions. This completes the proof.

Corollary 1: If $\frac{bd}{c} > q + d > 0$, then the region

$$\emptyset = \left\{0 \leq x_1, x_2 \leq k, 0 \leq y, x_1 + x_2 + y \leq \frac{L}{q}\right\}$$

is an invariant region for model 1.

Proof. This is a direct conclusion of Proposition 1

3.3 Equilibrium analysis

Let$\overline{x_1}, \overline{x_2}$and $\bar{y}$ be the equilibrium values of $x_1, x_2$and$y$.We find three biologically meaningful equilibrium points

(i)$\overline{E_0} = (0,0,0)$. The extinction of all populations, this equilibrium always exists.

(ii) $\overline{E_1} = \left(k, \frac{\alpha k}{\beta}, 0\right)$. The prey species survive inside and outside the refuge habitat and the predator goes to extinction.

(iii) $\overline{E_2} = (\overline{x_1}, \overline{x_2}, \bar{y})$. All populations survive. Note that at this equilibrium, and using the third equation of system (1), we get:

$$\overline{x_2} = \frac{d}{c} \tag{3}$$

From the second equation of system (1) at equilibrium we have

$$\overline{x_1} = \frac{\overline{x_2}(1 + e\bar{y})(\beta + b\bar{y})}{\alpha} \ . \tag{4}$$

Using the first equation of system (1) at equilibrium and substituting (4) we get

$$f(\bar{y}) = \frac{a\overline{x_2}}{k\alpha} e^2 b^2 \bar{y}^4 + \frac{2a\overline{x_2}\, eb}{k\alpha}(b + e\beta)\bar{y}^3 + \left[\frac{a\overline{x_2}b^2}{k\alpha} + \frac{a\overline{x_2}e^2\beta^2}{k\alpha} + ae\left(\frac{4\overline{x_2}\beta^2}{k\alpha} - 1\right)\right]\bar{y}^2$$

$$+ \left[ab\left(\frac{2\overline{x_2}\beta}{k\alpha} - 1\right) + ae\beta\left(\frac{2\overline{x_2}\beta}{k\alpha} - 1\right) + 2b\right]\bar{y} + a\beta\left(\frac{\overline{x_2}\beta}{k\alpha} - 1\right) = 0 \tag{5}$$

If $\frac{1}{2} < \frac{\overline{x_2}\beta}{k\alpha} < 1$, then there will be only one positive root of $\bar{y}$ of (5), because $f(0) = a\beta\left(\frac{\overline{x_2}\beta}{k\alpha} - 1\right) < 0$, and $\lim_{\bar{y}\to\infty} f(\bar{y}) = \infty$, and then by the continuity of $f(\bar{y})$ and zero-point theorem, $f(\bar{y}) = 0$ has one positive solution, so there will be a unique positive coexistence equilibrium.

3.4 Stability analysis

In this subsection, we examine the stability of the system about the equilibrium points found in the previous subsection.

(a) Stability analysis of the equilibrium (i): Consider a small perturbation about the equilibrium $x_1 = \overline{x_1} + u$, $x_2 = \overline{x_2} + v$ and $y = \bar{y} + w$. Substituting these into the system (1), and neglecting products of small quantities, we obtain the stability matrix:

$$\begin{pmatrix} a - \alpha & \beta & 0 \\ \alpha & -\beta & 0 \\ 0 & 0 & -d \end{pmatrix}$$

The corresponding characteristic equation is:

$$-(\lambda + d)(\lambda^2 + \lambda(2 + \beta - a) - a\beta) = 0 \tag{6}$$

One of the values of λ is positive, so (0,0,0) is unstable and hence the both populations will never be extinct.

(b) Stability analysis of the equilibrium (ii):

Theorem 1. *The predator free equilibrium* $\overline{E_1} = (k, \frac{\alpha k}{\beta}, 0)$ *of system (1) is locally asymptotically stable if* $\frac{c\alpha k}{\beta} <$ *d, and unstable if* $\frac{c\alpha k}{\beta} > d$ *. In fact, we can prove that* $\overline{E_1}$ *is globally asymptotically stable if* $\frac{c\alpha k}{\beta} < d$.

Proof. Using the above mentioned, we obtain the stability matrix

$$\begin{pmatrix} -a - \alpha & \beta & \alpha bk \\ \alpha & -\beta & \dfrac{-\alpha bk}{\beta} - \alpha bk \\ 0 & 0 & \dfrac{c\alpha k}{\beta} - d \end{pmatrix}$$

and the corresponding characteristic equation is

$$\left(\frac{c\alpha k}{\beta} - d - \lambda\right) [\lambda^2 + (a + \alpha + \beta)\lambda + a\beta] \tag{7}$$

All roots will be negative if $\frac{c\alpha k}{\beta} < d$, and hence this equilibrium will be stable.

If $g(t)$ is a continuous and bounded function, then we define:

$$g^\infty \triangleq \lim_{t\to\infty} Sup\ g(t), \qquad g_\infty \triangleq \lim_{t\to\infty} Inf\ g(t)$$

For a system (1) with initial conditions $x_1 = x_1(t), x_2 = x_2(t)$ and $= y(t)$, we have

$$0 \le x_{1\infty} \le x_1^\infty \le \infty, \quad 0 \le x_{2\infty} \le x_2^\infty \le \infty, \quad 0 \le y_\infty \le y^\infty \le \infty \ .$$

Using fluctuation lemma [28], we can say that there is a sequence $\{t_n\}$, and when $t_n \to \infty$ we have $x_1(t_n) \to x_1^\infty$ and $\frac{dx_1(t_n)}{t_n} \to 0$ as $n \to \infty$. On substituting t_n into the third equation of system (1),
We have

$$\frac{dy(t_n)}{t_n} = (cx_2(t_n) - d)y(t_n)$$

Taking the limit on both sides

$$\lim_{n\to\infty} \frac{dy(t_n)}{t_n} = (c \lim_{n\to\infty} x_2(t_n) - d) \lim_{n\to\infty} y(t_n)$$

Which gives

$$0 = (cx_2^\infty - d)y^\infty$$

Therefore $y^\infty = 0$ or $x_2^\infty = \frac{d}{c}$.

Adding the first and second equations of system (1), we obtain

$$\frac{dx_1(t_n)}{t_n} + \frac{dx_2(t_n)}{t_n} \le ax_1(t_n)\left(1 - \frac{x_1(t_n)}{k}\right)$$

Taking the limit on both sides

$$\lim_{n\to\infty} \frac{dx_1(t_n)}{t_n} + \frac{dx_2(t_n)}{t_n} \le \lim_{n\to\infty} ax_1(t_n)\left(1 - \frac{x_1(t_n)}{k}\right)$$

$$0 \leq ax_1^\infty \left(1 - \frac{x_1^\infty}{k}\right)$$

Therefore either $x_1^\infty = 0$ or $0 < x_1^\infty \leq k$. According to the limit Theorem [29], we get $\lim_{t \to \infty} x_1(t) = k$. The second equation of system (1) yields

$$\frac{dx_2(t_n)}{t_n} \leq ax_1(t_n) - \beta x_2(t_n)$$

Taking the limit on both sides

$$\lim_{n \to \infty} \frac{dx_2(t_n)}{t_n} \leq \alpha \lim_{n \to \infty} x_1(t_n) - \beta \lim_{n \to \infty} x_2(t_n)$$

Which gives

$$x_2^\infty \leq \frac{\alpha k}{\beta}$$

Therefore, and using the limit Theorem we have

$$\lim_{t \to \infty} x_2 = \frac{\alpha k}{\beta}$$

Again, using the second equation of system (1), we have

$$\frac{dx_2(t_n)}{t_n} \leq ax_1(t_n) - \beta x_2(t_n)$$

Similarly, $0 < ax_1^\infty - \beta x_2^\infty$; i.e. $ax_1^\infty > \beta x_2^\infty$. If $x_1^\infty = 0$, then $x_2^\infty < 0$, which is not possible. Therefore, we take, $x_1^\infty = k$. If we consider, $x_2^\infty = \frac{d}{c}$, then

$$\frac{dx_2(t_n)}{t_n} \leq ax_1(t_n) - \beta x_2(t_n)$$

Taking the limit on both sides

$$0 \leq \alpha k - k\frac{d}{c}$$

Hence

$$d \leq \frac{c\alpha k}{\beta}$$

This inequality is not true also because for stability of $\overline{E_1}$, we need $d > \frac{c\alpha k}{\beta}$. Therefore, the predator free equilibrium is globally asymptomatically stable if $k > \frac{\beta d}{c\alpha}$; i.e. maximum prey population inside the refuge is greater than $\frac{\beta d}{c\alpha}$.

(a) Stability analysis of the equilibrium(iii):
Theorem 2. *The equilibrium point $\overline{E_2}$ is locally asymptotically stable if $\overline{x_1} > \frac{k}{2}$.*
Proof. The stability matrix of the system (1) around the equilibrium point $\overline{E_2}$ is

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ 0 & c\bar{y} & 0 \end{pmatrix}$$

The corresponding characteristic equation is

$$\lambda^3 + (-p_1 - q_2)\lambda^2 + (p_1 q_2 - p_2 q_1 - c\bar{y}q_3)\lambda + (c\bar{y}q_3 p_1 - c\bar{y}p_3 q_1) = 0 \tag{8}$$

Where

$$p_1 = -\frac{\alpha \overline{x_1}}{k} - \frac{\beta \overline{x_2}}{\overline{x_1}} < 0$$
$$p_2 = \beta > 0$$

$$p_3 = \frac{\alpha e \overline{x_1}}{(1 + e\bar{y})^2} > 0$$

$$q_1 = \frac{\alpha}{1 + e\bar{y}} > 0 \tag{9}$$

$$q_2 = -\beta - b\bar{y} < 0$$

$$q_3 = -\frac{\alpha e \overline{x_1}}{(1 + e\bar{y})^2} - b\overline{x_2} < 0$$

The equation (8) can be written as

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{10}$$

Where $a_1 = (-p_1 - q_2)$, $a_2 = (p_1 q_2 - p_2 q_1 - c\bar{y}q_3)$ and $a_3 = c\bar{y}(q_3 p_1 - p_3 q_1)$. The Routh-Hurwitz criteria for the third order system is given by: $a_1 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$.

Here $a_1 = (-p_1 - q_2) > 0$

$$a_3 = c\bar{y}(q_3 p_1 - p_3 q_1)$$

$$= \left(-a + \frac{2a\overline{x_1}}{k}\right)\left(\frac{\alpha e \overline{x_1}}{(1 + e\bar{y})^2} + b\overline{x_2}\right) + \frac{b\alpha \overline{x_2}}{1 + e\bar{y}} > 0$$

if $\overline{x_1} > \frac{k}{2}$.

Now to show that $a_1 a_2 > a_3$; i.e.

$$-(p_1 + q_2)(p_1 q_2 - p_2 q_1 - c\bar{y}q_3) > c\bar{y}(q_3 p_1 - p_3 q_1)$$

Which, after simplification, gives

$$(p_1 q_2 - p_2 q_1)(-p_1 - q_2) > c\bar{y}(q_2 q_3 + p_3 q_1) > 0$$

Now

$$p_1 q_2 - p_2 q_1 = \left(\frac{\alpha \overline{x_1}}{k} + \frac{\beta \overline{x_2}}{\overline{x_1}}\right)(\beta + b\bar{y}) - \frac{\beta \alpha}{1 + e\bar{y}}$$

$$= \frac{\alpha \overline{x_1}}{k}(\beta + b\bar{y})$$

So, clearly,$p_1 q_2 - p_2 q_1 > 0$. Note that $-p_1 - q_2 > 0$; therefore, $a_1 a_2 > a_3$. Hence the co-existing equilibrium$\overline{E_2} = (\overline{x_1}, \overline{x_2}, \overline{y})$ will be asymptotically stable if $\overline{x_1} > \frac{k}{2}$. Hence the co-existing equilibrium will be stable if therefuge prey population at equilibrium is higher than the carrying capacity of the prey in the refuge habitat.

4. Z-control

To achieve predator population and prey population inside and outside the refuge to a desire level, direct Z-control strategy is used. The direct Z-control are functions that are incorporated in each equation of the system (1). This system is then described as follows

$$\frac{dx_1}{dt} = ax_1\left(1 - \frac{x_1}{k}\right) - \frac{\alpha x_1}{1 + ey} + \beta x_2 - u_1(t)x_1$$

$$\frac{dx_2}{dt} = \frac{\alpha x_1}{1 + ey} - \beta x_2 - bx_2 y - u_2(t)x_2 \qquad (11)$$

$$\frac{dy}{dt} = cx_2 y - dy - u_3(t)y$$

Then we define the error functions as $x_1 - x_{1d} = e_1 = u_1$, $x_2 - x_{2d} = e_2 = u_2$, $y - y_d = e_3 = u_3$, where x_{1d}, x_{2d} and y_d are desiredstates of prey inside the refuge, prey outside the refuge and the predator population respectively. These functions decay exponentially with time, i.e. e_1, e_2 and e_3 tends to zero. For achieving our purpose we adopt $\dot{e}_1 =$

$-\lambda_1 e_1, \dot{e}_2 = -\lambda_2 e_2$and $\dot{e}_3 = -\lambda_3 e_3$, with$\lambda_1, \lambda_2, \lambda_3 > 0$.Now we have

$$\dot{x}_1 - \dot{x}_{1d} = -\lambda_1(x_1 - x_{1d})$$

$$\dot{x}_2 - \dot{x}_{2d} = -\lambda_2(x_2 - x_{2d}) \qquad (12)$$

$$\dot{y} - \dot{y}_d = -\lambda_3(y - y_d)$$

Finally, with (11) and (12) we get the following three control functions

$$u_1 = \frac{1}{x_1}\left[ax_1\left(1 - \frac{x_1}{k}\right) - \frac{\alpha x_1}{1 + ey} + \beta x_2 - \dot{x}_{1d} + \lambda_1(x_1 - x_{1d})\right]$$

$$u_2 = \frac{1}{x_2}\left[\frac{\alpha x_1}{1 + ey} - \beta x_2 - bx_2 y - \dot{x}_{2d} + \lambda_2(x_2 - x_{2d})\right] \qquad (13)$$

$$u_2 = \frac{1}{y}[cx_2 y - dy - \dot{y}_d + \lambda_3(y - y_d)]$$

5. Adaptive non-linear control

We start by first non-dimensionalizing the system (1) by using the following transformations:$\frac{x_1}{k} = X_1, \frac{x_2}{k} = X_2, ey = Y, \frac{\alpha}{a} = \alpha_1, \frac{\beta}{a} = \beta_1, \frac{b}{a} = b_1, at = \tau, \frac{ck}{ae} = c_1, \frac{d}{a} = d_1$.Now the system (1) takes the non-dimensional form

$$\frac{dX_1}{d\tau} = X_1(1 - X_1) - \frac{\alpha_1 X_1}{1 + Y} + \beta_1 X_2$$

$$\frac{dX_2}{d\tau} = \frac{\alpha_1 X_1}{1 + Y} - \beta_1 X_2 - b_1 X_2 Y \qquad (14)$$

$$\frac{dY}{d\tau} = c_1 X_2 Y - d_1 Y$$

We use non-linear feedback control for the system (14). This system can be represented as

$$\frac{dX_1}{d\tau} = X_1(1 - X_1) - \frac{\alpha_1 X_1}{1 + Y} + \beta_1 X_2 + u_1$$

$$\frac{dX_2}{d\tau} = \frac{\alpha_1 X_1}{1+Y} - \beta_1 X_2 - b_1 X_2 Y + u_2 \qquad (15)$$

$$\frac{dY}{d\tau} = c_1 X_2 Y - d_1 Y + u_3$$

Where u_1, u_2 and u_3 are adaptive nonlinear feedback control functions which will be the functions of x_1, x_2 and y. If these feedback functions stabilize the system, then in an infinitely long time state the variables converge to zero. Let $e_{\alpha_1} = \alpha_1 - \hat{\alpha}_1, e_{\beta_1} = \beta_1 - \hat{\beta}_1, e_{b_1} = b_1 - \hat{b}_1, e_{c_1} = c_1 - \hat{c}_1, e_{d_1} = d_1 - \hat{d}_1$ be unknown estimators, which give $\dot{e}_{\alpha_1} = -\dot{\hat{\alpha}}_1, \dot{e}_{\beta_1} = -\dot{\hat{\beta}}_1, \dot{e}_{b_1} = -\dot{\hat{b}}_1, \dot{e}_{c_1} = -\dot{\hat{c}}_1, \dot{e}_{d_1} = \widehat{-\dot{d}}_1$. To prove the global stability, we choose Lyapunov function as

$$V(X_1, X_2, Y) = \frac{1}{2}X_1^2 + \frac{1}{2}X_2^2 + \frac{1}{2}Y^2 + \frac{1}{2}e_{\alpha_1}^2 + \frac{1}{2}e_{\beta_1}^2 + \frac{1}{2}e_{b_1}^2 + \frac{1}{2}e_{c_1}^2 + \frac{1}{2}e_{d_1}^2$$

Which has the derivative

$$\dot{V} = X_1\left[X_1 - X_1^2 - \frac{\alpha X_1}{1 + Y} + \beta X_2 + u_1\right] + X_2\left[\frac{\alpha X_1}{1 + Y} - \beta X_2 - b_1 X_2 Y + u_2\right] +$$

$$Y[c_1 X_2 Y - d_1 Y + u_3] + e_{\alpha_1}\dot{e}_{\alpha 1} + e_{\beta_1}\dot{e}_{\beta_1} + e_{b_1}\dot{e}_{b_1} + e_{c_1}\dot{e}_{c_1} + e_{d_1}\dot{e}_{d_1}$$

Choosing adaptive non-linear controls

$$u_1 = -2X_1 + X_1^2 + \frac{\hat{\alpha}_1 X_1}{1 + Y} - \hat{\beta}_1 X_2$$

$$u_2 = -\frac{\hat{\alpha}_1 X_1}{1+Y} + \hat{\beta}_1 X_2 + \hat{b}_1 X_2 Y - X_2 \qquad (16)$$

$$u_3 = -\hat{c}_1 X_2 Y + \hat{d}_1 Y - Y$$

Using (16) in $\dot{V}$, we have

$$\dot{V} = (-X_1^2 - X_2^2 - Y^2) + e_{\alpha_1}\frac{X_1 X_2}{1 + Y} - e_{\beta_1}X_2^2 - e_{b_1}X_2 Y + e_{c_1}X_2 Y - e_{d_1}Y^2 - e_{\alpha_1}\dot{\hat{\alpha}}_1$$

$$-e_{\beta_1}\dot{\hat{\beta}}_1 - e_{b_1}\dot{\hat{b}}_1 - e_{c_1}\dot{\hat{c}}_1 - e_{d_1}\dot{\hat{d}}_1 \qquad (17)$$

$$= (-X_1^2 - X_2^2 - Y^2) + e_{\alpha_1}\left(-\frac{X_1^2}{1 + Y} + \frac{X_1 X_2}{1 + Y} - \dot{\hat{\alpha}}_1\right) + e_{\beta_1}\left(X_1 X_2 - X_2^2 - \dot{\hat{\beta}}_1\right)$$

$$+e_{b_1}\left(-X_2^2 Y - \dot{\hat{b}}_1\right) + e_{c_1}\left(X_2 Y^2 - \dot{\hat{c}}_1\right) + e_{d_1}\left(-Y^2 - \dot{\hat{d}}_1\right)$$

Considering parameter estimators

$$\dot{\hat{\alpha}}_1 = -\frac{X_1^2}{1 + Y} + \frac{X_1 X_2}{1 + Y} + e_{\alpha_1}$$

$$\dot{\hat{\beta}}_1 = X_1 X_2 - X_2^2 + e_{\beta_1} \qquad (18)$$

$$\dot{\hat{b}}_1 = -X_2^2 Y + e_{b_1}$$

$$\dot{\hat{c}}_1 = X_2 Y^2 + e_{c_1}$$

$$\dot{\hat{d}}_1 = -Y^2 + e_{d_1}$$

Using dynamics of unknown estimators (18) in (17) we will find

$$\dot{V} = (-X_1^2 - X_2^2 - Y^2) - e_{\alpha_1}^2 - e_{\beta_1}^2 - e_{b_1}^2 - e_{c_1}^2 - e_{d_1}^2$$

Clearly the system will be globally stable, because $\dot{V} < 0$.

6. Numerical simulation

In this section, we performed several numerical simulations for the system (1) to confirm our theoretical results and to acquire more knowledge about its dynamics and general behavior. The parameter values used are listed in the following table, some of them might be changed in order to study their effect:

Table 1. Parameter values used for simulations.

Parameter	value
a	0.07
α	0.035
β	0.0119
k	0.8
b	0.0112
c	0.04
d	0.07
e	50

6.1 Effect of adaptive control

To study the effect of adaptive control on the system, we look at the stable co-existence equilibrium point. (Figure 2) shows that without the adaptive control the system took a long time to converge to this stable equilibrium point. However, with the use of adaptive control it is clear that the time to reach the equilibrium point is very short, and the system almost instantly started to reach this stable equilibrium point, as seen from (Figure 3).
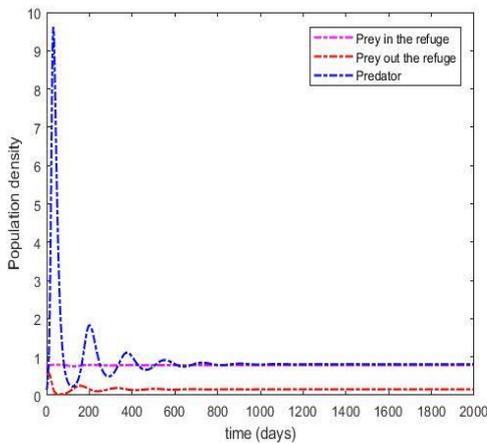
Figure 2. The convergence of the co-existence equilibrium point. The parameters are $a = 0.1, \alpha = 0.035, \beta = 0.00119, k = 0.8, b = 0.0112, c = 0.45, d = 0.07$ and $e = 20$.
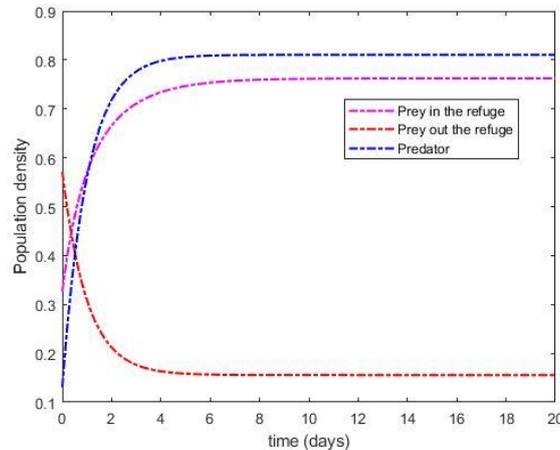
Figure 3. The effect of adaptive control on the convergence of the co-existence equilibrium point. The parameters are $a = 0.1, \alpha = 0.035, \beta = 0.00119, k = 0.8, b = 0.0112, c = 0.45, d = 0.07$ and $e = 20$.

To study the effect of the intensity of fear, we simulate our model with parameter values taken from Table1,

and the intensity of fear taken between 0.1 to 100. It is clear that when the intensity of fear increases, the fraction of the prey in the refuge increases. However, it has no impact on the fraction of prey outside the refuge, as shown from Figures 4-5. On the other hand, when the intensity of fear increases, the fraction of predators decreases as shown in (Figure6), which dictates that fear is not in the favor of the predator.
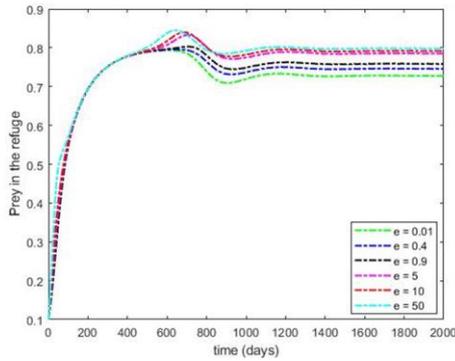
Figure 4. The effect of the fear intensity on the prey in the refuge. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04$ and $d = 0.07$.
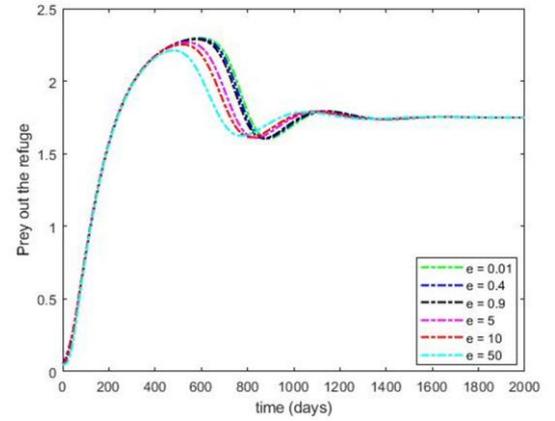
Figure 5. The effect of fear intensity on the prey out the refuge. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04$ and $d = 0.07$.
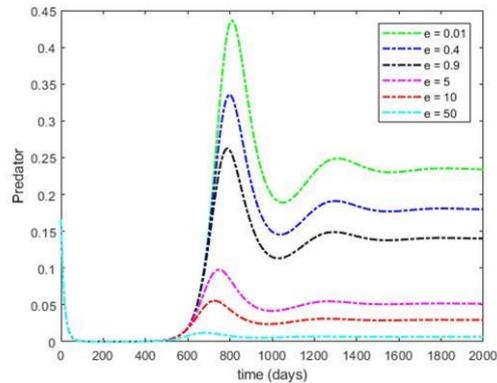
Figure 6. The effect of fear intensity on the predator. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04$ and $d = 0.07$.

**6.3 Application of Z-type control with constant desired state**

Figures 7-9 show that with the help of z-type control all three populations, the prey in the refuge, the prey out of the refuge and the predators reaches the desired states as indicated. Clearly the time needed to reach the desired states is very short, and this is due to the power of z-type control, which takes the output to the desired state rapidly. Figures 10-12 show the control profile of all three populations, where (Figure13) shows the error profiles of all three control profiles.
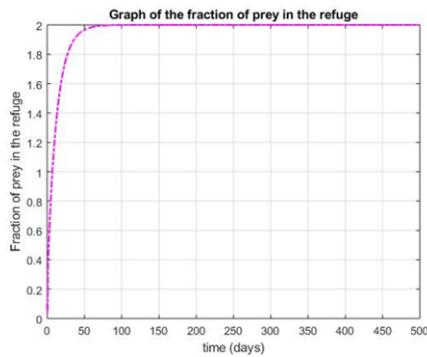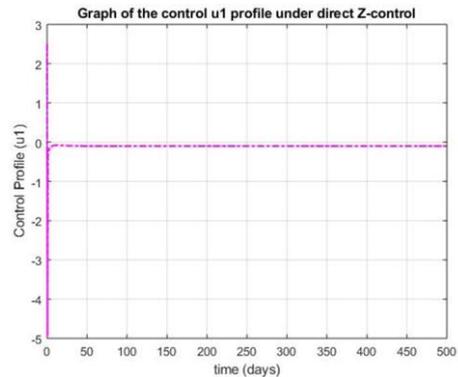
Figure 7. The convergence of the prey in the refuge to its constant desired states $x_{1d} = 1.5$. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07$ and $e = 50$.
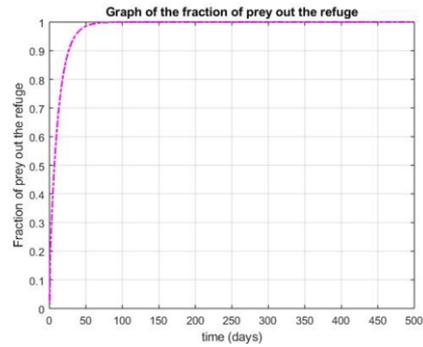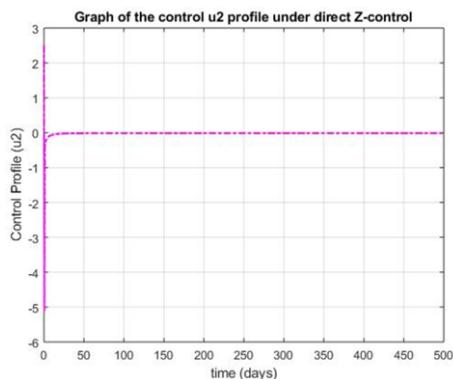

Figure 8. The convergence of the prey out the refuge to its constant desired states $x_{2d} = 1$. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07$ and $e = 50$.
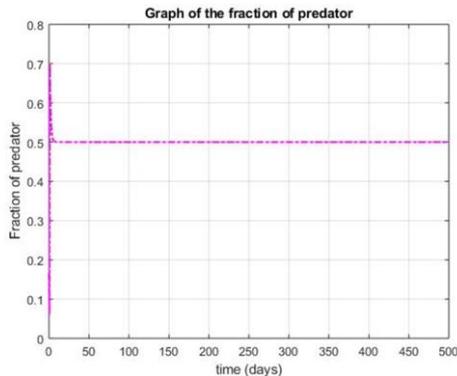
Figure 9. The convergence of the predator to its constant desired states $y_d = 0.5$. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07$ and $e = 50$.
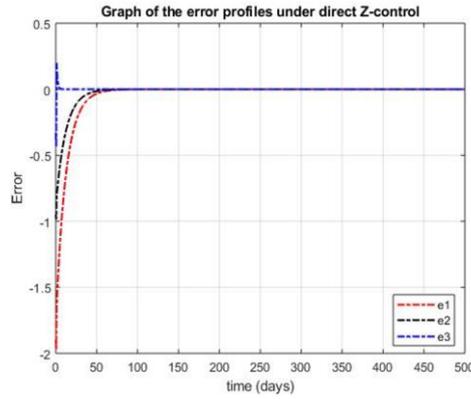

Figure 10. The control profile of the prey in the refuge. The parameters are $a = 0.07, \alpha\ 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07$ and $e = 50$.

Figure 11. The control profile of the prey out the refuge. The parameters are $a = 0.07, \alpha\ 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07$ and $e = 50$.
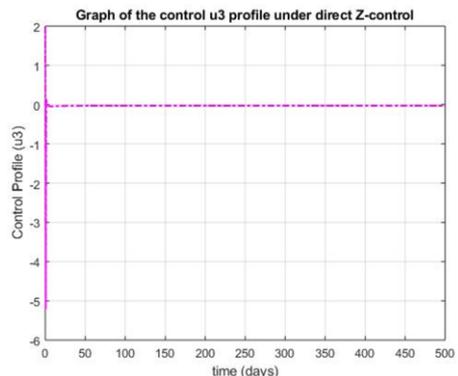
Figure 12. The control profile of the predator. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07$ and $e = 5$.

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Figure 13. The error profile of all three controls. The parameters are $a = 0.07, \alpha = 0.035,$ $\beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07$ and $e = 50$.

6.4 Application of Z-type control with periodic desired state

Figures 14-16 show that Z-type control could be used to achieve a periodic desired state. From a biological point of view, it is very important to be able to reach a stable limit cycle instead of a constant equilibrium point, as periodic solutions (i.e. limit cycles) are of great interest for ecosystems and more generally for conservation biology. For the purpose of simulation, we consider the desired state to be of the form
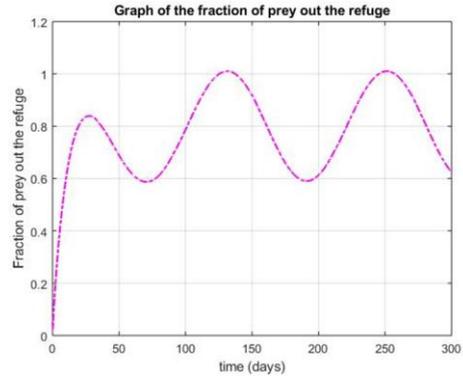
$$x_{1d} = r_1 + \omega_1 \cos \frac{\pi t}{100}$$

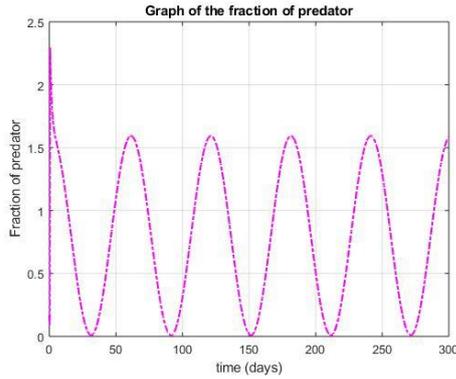$$x_{2d} = r_2 + \omega_2 \cos \frac{\pi t}{200}$$

$$y_d = r_3 + \omega_3 \cos \frac{\pi t}{100}$$
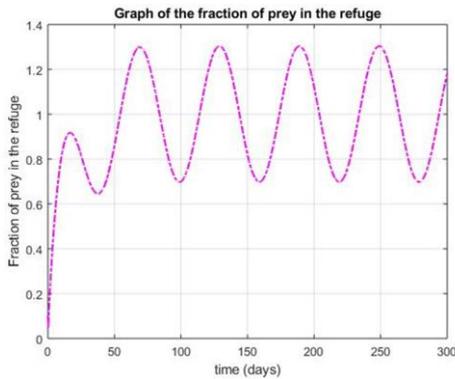
Figure 14. The convergence of the prey in the refuge to its periodic desired states. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07, e = 50, r_1 = 1, \ r_2 = 0.8, \ r_3 = 0.8 , \omega_1 = 0.5, \omega_2 = 0.25, \omega_3 = 0.8$.
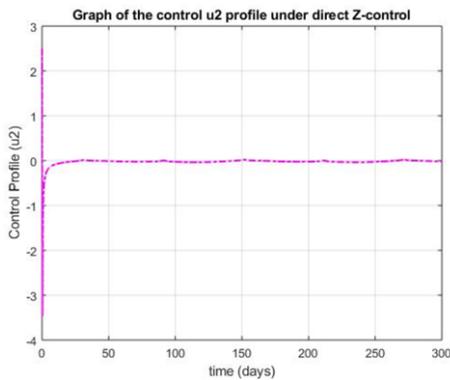
Figure 15. The convergence of the prey out the refuge to its periodic desired states. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07, e = 50, r_1 = 1, \ r_2 = 0.8, \ r_3 = 0.8 , \omega_1 = 0.5, \omega_2 = 0.25, \omega_3 = 0.8$.
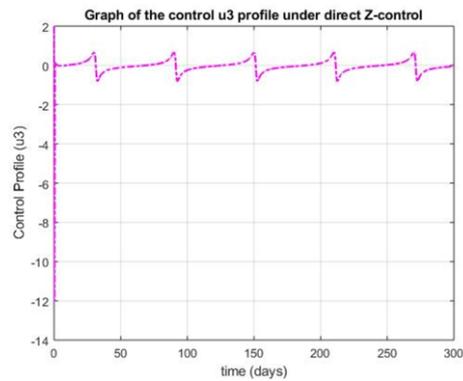
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Figure 16. The convergence of the predator to its periodic desired states. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07, e = 50, r_1 = 1, r_2 = 0.8, r_3 = 0.8, \omega_1 = 0.5, \omega_2 = 0.25, \omega_3 = 0.8.$
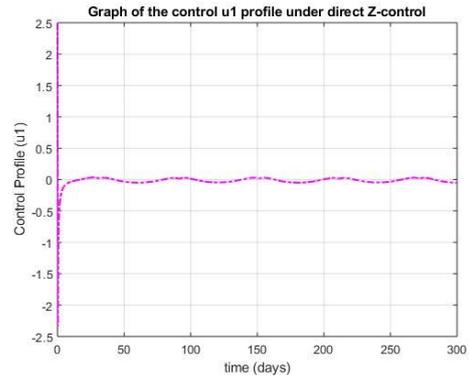
Figure 17. The control profile of the prey in the refuge. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07, e = 50, r_1 = 1, r_2 = 0.8, r_3 = 0.8, \omega_1 = 0.5, \omega_2 = 0.25, \omega_3 = 0.8.$

Figure 18. The control profile of the prey out the refuge. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07, e = 50, r_1 = 1, r_2 = 0.8, r_3 = 0.8, \omega_1 = 0.5, \omega_2 = 0.25, \omega_3 = 0.8.$
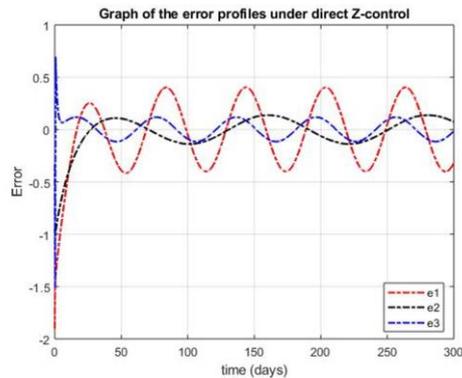

Figure 19. The control profile of the predator. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07, e = 50, r_1 = 1, r_2 = 0.8, r_3 = 0.8, \omega_1 = 0.5, \omega_2 = 0.25, \omega_3 = 0.8.$


Figure 20. The error profile of all three controls. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04, d = 0.07, e = 5, r_1 = 1, r_2 = 0.8, r_3 = 0.8, \omega_1 = 0.5, \omega_2 = 0.25, \omega_3 = 0.8.$

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6.5 Effect of different forms of β

It is more realistic than not to assume the refuge size (i.e. β) is not constant. Figures 21-23 show the fractions of prey in the refuge, the prey outside the refuge and the predator, where we took different forms of β. It is clear that when β is taken as a linear function (i.e. $\beta(t) = (a + \beta_0)x_2$) or as an exponential function of the prey outside the refuge (i.e. $\beta(t) = ax_2 \exp(\beta_0)$), then both the prey outside the refuge and in the refuge reach their stable equilibrium and the predator goes to extinction. However, when β is taken as a logistic function of the prey outside the refuge (i.e. $\beta(t) = a\beta_0 x_2 \left(1 - \frac{x_2}{k}\right)$), then all three populations co-exist together after initial small oscillations. Note that β_0 is a positive constant and represents the base-line value for β.
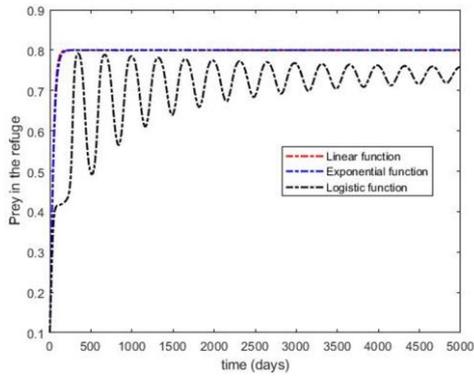
Figure 21. Effect of different forms of β on the prey in the refuge. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04$, $d = 0.07$ and $e = 20$.
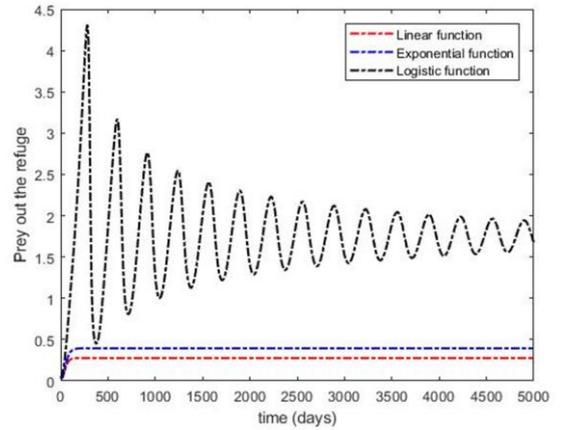
Figure 22. Effect of different forms of β on the prey out the refuge. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04$, $d = 0.07$ and $e = 20$.
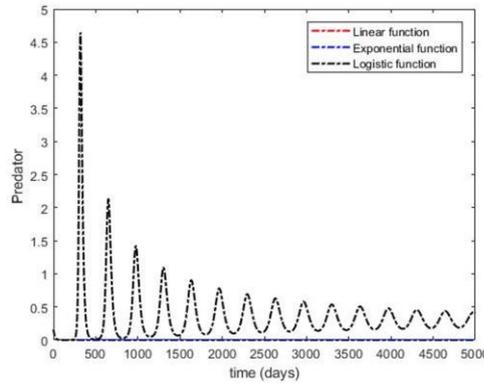
Figure 23. Effect of different forms of β on the predator population. The parameters are $a = 0.07, \alpha = 0.035, \beta = 0.0119, k = 0.8, b = 0.0112, c = 0.04$, $d = 0.07$ and $e = 20$.

7. Conclusion

All animals are threatened by predators and face the risk of predation. Prey populations change their behavior due to their fear of predators. In this paper we have studied the dynamics of predator-prey interaction where prey reside in two habitats, namely refuge and out of refuge. In refuge, the prey is safe from predatory killing and has sufficient resources to survive, and the population in the refuge grows logistically. Out of refuge, predators interact with the prey and may kill them. Prey live under the fear of predation, but when predator fear is diluted, prey come out of their refuge. On increasing predation fear, prey take shelter in the refuge.

We obtain three biologically feasible equilibria and discuss their stability. The equilibrium free from prey and predator population will always exist and it is unstable i.e. prey, and predator populations will never be extinct. The equilibrium having zero predator population and non-zero prey population will always exist, and it will be globally stable if the maximum prey population inside the refuge is greater

than$\frac{\beta d}{c\alpha}$. Otherwise it will be unstable. The co-existing equilibrium will be asymptotically stable if $\overline{x_1}$ is bigger than half of the carrying capacity, and otherwise will be unstable.

To bring the population to the desired level and to protect it from extinction, we use z-control, where the population reaches the desired level in a short time. We performed a simulation where the desired state was limit cycles instead of an equilibrium point. Numerically it is shown that as the intensity of fear increases, the population of the prey in refuge increases, while the population of the predator decreases i.e. fear is not in the favor of predator populations. To make our study more ecologically realistic, we took different forms of refuge size (β) i.e. linear, exponential, and logistic instead of constant. We observed that when refuge size β is linear or exponential, the prey out of the refuge and in the refuge tend to attend their stable equilibrium and predators go to extinction. If we consider refuge size as a logistic function of the prey out of refuge, then after a little oscillation all three populations co-exist. The adaptive control inputs for asymptotic stability are obtained as non-linear feedback. We examined the stability of the system with and without control and noted that the system with control approaches stability faster than the system without control.

Conflict of interest

The authors declare no conflict of interest.

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