Finite Element Convergence Analysis of a Schwarz Alternating Method for Nonlinear Elliptic PDEs

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ABSTRACT: In this paper, we prove uniform convergence of the standard finite element method for a Schwarz alternating procedure for nonlinear elliptic partial differential equations in the context of linear subdomain problems and nonmatching grids. The method stands on the combination of the convergence of linear Schwarz sequences with standard finite element L^{∞} -error estimate for linear problems.

Keywords: Schwarz Method; Finite elements; Convergence.

تحليل تقارب العناصر المحدودة لطريقة شوارز المتناوبة لأجهزة للمعادلات التفاضلية الإخطية الجزئية

مسعود بولبراشن

الملخص: في هذه الورقة ، نثبت التقارب الموحد لطريقة العناصر المحددة القياسية لإجراء تناوب شوارتز للمعادلات التفاضلية الإخطية الجزئية في سياق مشاكل النطاق الفرعي الخطي والشبكات غير المتطابقة. تكمن الطريقة في الجمع بين التقارب بين تسلسل شوارز وتقدير العناصر المحدودة للإرهاب في المشكلات الخطية.

الكلمات المفتاحية: طريقة شوارز، العناصر المحدودة، التقارب.

1. Introduction

The Schwarz alternating method can be used to solve elliptic boundary value problems on domains that consist of two or more overlapping subdomains. The solution is approximated by an infinite sequence of functions that results from solving a sequence of elliptic boundary value problems in each of the subdomains.

There has been extensive analysis of the Schwarz alternating method for nonlinear elliptic boundary value problems [1-4] and the references therein). Also, the effectiveness of Schwarz methods for these problems (especially those in fluid mechanics) has been demonstrated by many authors.

In this paper, we are concerned with the finite element convergence analysis of overlapping Schwarz alternating methods in the context of nonmatching grids for nonlinear PDEs, where the Schwarz sub problems are linear. This study constitutes, to some extent, an improvement of the one achieved in [5], on a Schwarz method with nonlinear sub problems.

For that, we develop an approach which combines the convergence result of Lui [6], with standard finite element error estimate for linear elliptic equations.

For other works on finite element convergence analysis in the maximum norm of overlapping nonmatching Schwarz method, we refer to [7-12].

The rest of the paper is organized as follows. In section 2, we state the continuous alternating Schwarz sub problems and define their respective finite element counterparts in the context of nonmatching overlapping grids. In section 3, we give $L^{\infty}(\Omega)$ - convergence analysis of the method.

2. Preliminaries

We begin by laying down some definitions and classical results related to linear elliptic equations.

2.1 Linear elliptic equations

Let Ω be a bounded polyhedral domain of R^2 or R^3 with sufficiently smooth boundary $\partial\Omega$. We consider the bilinear form

$$a(u,v) = \int_\Omega \nabla u \nabla v dx \tag{1}$$

the linear form

$$(f,v) = \int_\Omega f(x)v(x)dx \tag{2}$$

the right hand side:

$$f \text{ is a regular function,} \tag{3}$$

the space

$$V^{(g)} = \left\{ v \in H^1(\Omega) \text{ such that } v = g \text{ on } \Omega \right\} \tag{4}$$

where g is a regular function defined on $\partial\Omega$.

We consider the linear elliptic equation: Find $\xi \in V^{(g)}$ such that

$$a(\xi,v) + c(\xi,v) = (f,v) \ \forall v \in H^1_0(\Omega) \tag{5}$$

where

$$c \in R, \ c > 0 \text{ such that } c \geq \beta > 0 \tag{6}$$

Let V_h be the space of finite elements consisting of continuous piece-wise linear functions, ϕ_s, $s = 1, 2, ..., m(h)$ be the basis functions of V_h, and $m(h)$ denote the number of vertices of the triangulation in Ω. Let also $\overset{0}{V}_h$ be the subspace of V_h defined by

$$\overset{0}{V}_h = \left\{ v \in V_h \text{ such that } v = 0 \text{ on } \partial\Omega \right\} \tag{7}$$

The discrete counterpart of (.,.) consists of finding $\xi_h \in V_h^g$ } such that

$$a(\xi_h,v) + c(\xi_h,v) = (f,v) \ \forall v \in \overset{0}{V}_h \tag{8}$$

where V_h^g is the space of

$$V_h^{(g)} = \left\{ v \in V_h \text{ such that } v = \pi_h g \text{ on } \partial\Omega \right\} \tag{9}$$

and π_h is the Lagrange interpolation operator on $\partial\Omega$.

Theorem 1. [13] Under suitable regularity of the solution of problem (5), there exists a constant c independent of h such that

$$\left\| \xi - \xi_h \right\|_{L^\infty(\Omega)} \leq Ch^2 \left| \ln h \right|$$

Lemma 1. [5] Let $w \in H^1(\Omega) \cap C(\overline{\Omega})$ satisfy $a(w,\varphi) + c(w,\varphi) \geq 0$ $\forall$ non-negative $\varphi \in H^1_0(\Omega)$, and $w \geq 0$ on $\partial\Omega$. Then $w \geq 0$ on $\overline{\Omega}$.

The proposition below establishes a Lipschitz continuous dependency of the solution with respect to the data.

Notation 1. Let (f,g), $(\tilde{f},\tilde{g})$ be a pair of data, and $\xi = \sigma(f,g)$ and $\tilde{\xi} = \sigma(\tilde{f},\tilde{g})$ be the corresponding solutions to (5).

Proposition 1. Under the conditions of lemma 1, we have:

$$\left\| \xi - \tilde{\xi} \right\|_{L^\infty(\Omega)} \leq \max \left\{ \frac{1}{\beta} \left\| f - \tilde{f} \right\|_{L^\infty(\Omega)}, \left\| g - \tilde{g} \right\|_{L^\infty(\partial\Omega)} \right\} \tag{10}$$

Proof. First, set

$$\theta = \max \left\{ \frac{1}{\beta} \left\| f - \tilde{f} \right\|_{L^\infty(\Omega)}, \left\| g - \tilde{g} \right\|_{L^\infty(\partial\Omega)} \right\}$$

Then

$$\tilde{f} \leq f + \left\| f - \tilde{f} \right\|_{L^{\infty}(\Omega)}$$

$$\leq f + \frac{c}{\beta} \left\| f - \tilde{f} \right\|_{L^{\infty}(\Omega)}$$

$$\leq f + c \max \left\{ \frac{1}{\beta} \left\| f - \tilde{f} \right\|_{L^{\infty}(\Omega)}, \left\| g - \tilde{g} \right\|_{L^{\infty}(\partial\Omega)} \right\}$$

$$\leq f + c\theta$$

So

$$a(\xi,\varphi) + \left(\xi, \varphi \right) \leq a(\xi,\varphi) + c \left(\xi, \varphi \right) + c(\theta, \varphi) \ \forall \varphi \geq 0, \ \varphi \in H_0^1(\Omega)$$

$$\leq a(\xi + \theta, \varphi) + c \left((\xi + \theta), \varphi \right) = \left(f + c\theta, \varphi \right)$$

On the other hand, we have

$$\xi + \theta - \xi \geq 0 \text{ on } \partial\Omega$$

So

$$\begin{cases} a(\xi + \theta - \xi, \varphi) + c(\xi + \theta - \xi, \varphi) \geq 0 \\ \xi + \theta - \xi \geq 0 \text{ on } \partial\Omega \end{cases}$$

Thus, making use of lemma 1, we get

$$\xi + \theta - \xi \geq 0 \ \text{On } \overline{\Omega}$$

Similarly, interchanging the roles of the couples (f, g), $(\tilde{f}, \tilde{g})$, we obtain

$$\xi + \theta - \xi \geq 0 \text{ on } \overline{\Omega}$$

which completes the proof.

Remark 1. Lemma 1 holds true in the discrete case.

Indeed, assume that the discrete maximum principle (d.m.p) holds, i.e. the matrix resulting from the finite element discretization is an M-Matrix. Then we have:

Lemma 2. Let $w \in V_h$ satisfy $a(w, \phi_s) + c(w, \varphi) \geq 0 \ \forall \phi_s = s = 1, 2, ..., m(h)$ and $w \geq 0$ on $\partial\Omega$. Then $w \geq 0$ on $\overline{\Omega}$.

Proof. The proof is a direct consequence of the discrete maximum principle.

Let (f, g) and $(\tilde{f}, \tilde{g})$ be a pair of data and $\zeta_h = \sigma_h(f, g)$ and $\tilde{\zeta}_h = \sigma_h(\tilde{f}, \tilde{g})$ be the corresponding solutions to problem (8).

Proposition 2. Let the *d.m.p* hold. Then, under conditions of lemma 2, we have

$$\left\| \xi_h - \xi_h \right\|_{L^{\infty}(\Omega)} \leq \max \left\{ \frac{1}{\beta} \left\| f - \tilde{f} \right\|_{L^{\infty}(\Omega)}, \left\| g - \tilde{g} \right\|_{L^{\infty}(\partial\Omega)} \right\}$$

Proof. The proof is similar to that of the continuous case. Indeed, as the basis functions $\phi_s > 0$ of the space V_h are positive, it suffices to make use of the discrete maximum principle.

Let (f, g) and $(\tilde{f}, \tilde{g})$ be a pair of data and $\zeta_h = \sigma_h(f, g)$ and $\tilde{\zeta}_h = \sigma_h(\tilde{f}, \tilde{g})$ be the corresponding solutions to problem (8).

3. Schwarz Alternating Methods for Nonlinear PDEs

3.1 The nonlinear PDE

Consider the nonlinear PDE: Find $u \in C^2(\Omega)$ such that

$$\begin{cases} -\Delta u + cu = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{11}$$

or in its weak form: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) + c(u,v) = (f(u),v) \ \forall v \in H_0^1(\Omega) \tag{12}$$

where

$$f(.) \text{ is a non-decreasing nonlinearity}$$

We assume that $f(.)$ is Lipschitz continuous, that is

$$|f(x) - f(y)| \le k |x - y| \ \forall x, y \in \mathbb{R}$$

such that

$$k < \beta$$

where β is defined in (6).

Theorem 2. [14] Under the above assumptions, Problem (11) has a unique solution.

3.2 The Linear Schwarz Sub problems

We decompose Ω into two overlapping smooth subdomains Ω_1 and Ω_2 such that:

$$\Omega = \Omega_1 \cup \Omega_2 \tag{13}$$

We denote by $\partial \Omega_i$ the boundary of Ω_i, $\gamma_i = \partial \Omega_i \cap \Omega_j$, and $\Gamma_i = \partial \Omega \cap \Omega_i$. We assume that the intersection of $\overline{\gamma}_i$ and $\overline{\gamma}_j$; $i \neq j$ is empty. Let u_2^0 be an initial guess. We define the alternating Schwarz sequences (u_1^{n+1}) on Ω_1 such that $u_1^{n+1} \in C^2(\Omega_1)$ solves

$$\begin{cases} -\Delta u_1^{n+1} + c u_1^{n+1} = f(u_1^n) & \text{in } \Omega_1 \\ u_1^{n+1} = 0 & \text{on } \Gamma_1 \\ u_1^{n+1} = u_2^{n+1} & \text{on } \gamma_2 \end{cases} \tag{14}$$

and the sequence $\left(u_2^{n+1}\right)$ such that $u_2^{n+1} \in C^2(\Omega_2)$ solves

$$\begin{cases} -\Delta u_2^{n+1} + c u_2^{n+1} = f(u_2^n) & \text{in } \Omega_2 \\ u_2^{n+1} = 0 & \text{on } \Gamma_2 \\ u_2^{n+1} = u_1^{n+1} & \text{on } \gamma_2 \end{cases} \tag{15}$$

Note that Schwarz subdomain problems (14) and (15) are linear.

Theorem 3. [6] The sequences (14) and (15) converge uniformly in $C^2(\Omega_1)$ and $C^2(\Omega_2)$, respectively, to $u_i = u / \Omega_i$, $i = 1, 2$, where u is the solution of (11).

3.3 The variational Linear Schwarz Sub problems

The corresponding variational problems read as follows: $u_1^{n+1} \in V_1$ solves

$$\begin{cases} b_1\left(u_1^{n+1},v\right) = \left(f\left(u_1^n\right),v\right) \ \forall v \in \overset{0}{V}_1 \\ u_1^{n+1} / \gamma_1 = u_2^n / \gamma_1 \end{cases} \tag{16}$$

and $u_2^{n+1} \in V_2$ solves

$$\begin{cases} b_2\left(u_2^{n+1},v\right)=\left(f\left(u_2^{n}\right),v\right) \ \forall v \in \overset{0}{V}_2 \\ u_2^{n+1} / \gamma_2 = u_2^{n} / \gamma_2 \end{cases} \tag{17}$$

where

$$V_i = H^1\left(\Omega_i\right); \ \overset{0}{V}_i = H_0^1\left(\Omega_i\right) \tag{18}$$

$$b_i\left(u,v\right)= \int\limits_{\Omega_i} \left(\nabla u \square \nabla v + cuv\right)dx \tag{19}$$

$$\left(u,v\right)_i = \int\limits_{\Omega_i} u(x)v(x)dx \tag{20}$$

3.4 The Discretization

For $i = 1,2$, let be τ^{h_i} be a standard regular and quasi-uniform finite element triangulation in Ω_i; h_i, being the mesh size. The two meshes being mutually independent on $\Omega_1 \cap \Omega_2$, a triangle belonging to one triangulation does not necessarily belong to the other.

Let us define the discrete analog of spaces V_i and $\overset{0}{V}_i$, respectively, that is

$$V_{h_i} = \left\{v \in C(\Omega_i) \text{such that } v /_K \in P_1 \ \forall K \in \tau^{h_i}\right\}$$

$$\overset{0}{V}_{h_i} = \left\{v \in V_{h_i} \text{ such that } v = 0 \text{ on } \partial\Omega_i\right\}$$

and let π_{h_i} denote the Lagrange interpolation operator on γ_i

The discrete Maximum principle (see [15,16]).
We assume that the respective matrices resulting from the discretization of problems (16) and (17) are M-matrices.

3.5 The Finite Element Linear Schwarz Sub problems

Let $u_{h_2}^0$ be the discrete analog of u_2^0 that is, $u_{h_2}^0 = r_{h_2}\left(u_2^0\right)$ where r_{h_2} denotes the finite element interpolation operator in Ω. We define the sequence $\left(u_{h_1}^{n+1}\right)$ such that $u_{h_1}^{n+1} \in V_{h_1}$ solves

$$\begin{cases} b_1\left(u_{h_1}^{n+1},v\right)=\left(f\left(u_{h_1}^{n}\right),v\right) \ \forall v \in \overset{0}{V}_{h_1} \\ u_{h_1}^{n+1} / \gamma_1 = \pi_{h_1}\left(u_{h_2}^{n}\right) \end{cases} \tag{21}$$

and the sequence $(u_{h_2}^{n+1})$ such that $u_{h_2}^{n+1} \in V_{h_2}$ solves

$$\begin{cases} b_2\left(u_{h_2}^{n+1},v\right)=\left(f\left(u_{h_2}^{n}\right),v\right) \ \forall v \in \overset{0}{V}_{h_2} \\ u_{h_2}^{n+1} / \gamma_2 = \pi_{h_2}\left(u_{h_1}^{n+1} / \gamma_2\right) \end{cases} \tag{22}$$

4. L^∞ - Convergence Analysis

This section is devoted to the proof of the main result of the present paper. To that end, we begin by introducing two discrete auxiliary Schwarz sequences and prove a fundamental lemma.

4.1 Auxiliary Discrete Schwarz Sub problems

We construct a sequence $\left(\omega_{h_1}^{n+1}\right)$ such that $\omega_{h_1}^{n+1} \in V_{h_1}$ solves

$$\begin{cases} b_1\left(\omega_{h_1}^{n+1},v\right) = \left(f\left(u_1^n\right),v\right) & \forall v \in \overset{0}{V}_{h_1} \\ u_{h_1}^{n+1}/\gamma_1 = \pi_{h_1}\left(u_2^n/\gamma_1\right) \end{cases}$$

(23)

and the sequence $\left(\omega_{h_2}^{n+1}\right)$ such that $\omega_{h_2}^{n+1} \in V_{h_2}$ solves

$$\begin{cases} b_2\left(\omega_{h_2}^{n+1},v\right) = \left(f\left(u_2^n\right),v\right) & \forall v \in \overset{0}{V}_{h_2} \\ u_{h_2}^{n+1}/\gamma_2 = \pi_{h_2}\left(u_1^{n+1}/\gamma_2\right) \end{cases}$$

(24)

Then, it is clear that $\omega_{h_1}^{n+1}$ and $\omega_{h_2}^{n+1}$ are the finite element approximation of u_1^{n+1} and u_2^{n+1} defined in (16) and (17), respectively.

Notation 2. From now on, we shall adopt the following notations:

$$|.|_1 = L^\infty\left(\gamma_1\right), \quad |.|_2 = L^\infty\left(\gamma_2\right)$$

(25)

$$\|.\| = L^\infty\left(\Omega\right), \quad \|.\|_1 = L^\infty\left(\Omega_1\right), \quad \|.\|_2 = L^\infty\left(\Omega_2\right)$$

(26)

$$\pi_{h_1} = \pi_{h_2} = \pi_h$$

(27)

4.2 The Main Result

The following lemma will play a key role in proving the main result of this paper.

Lemma 3.

$$\left\|u_1^n - u_{1h}^n\right\|_1 \le \sum_{i=0}^{n}\left\|u_1^i - \omega_{1h}^i\right\|_1 + \sum_{i=0}^{n-1}\left\|u_2^i - \omega_{2h}^i\right\|_2$$

$$\left\|u_2^n - u_{2h}^n\right\|_2 \le \sum_{i=0}^{n}\left\|u_1^i - \omega_{1h}^i\right\|_1 + \sum_{i=0}^{n}\left\|u_2^i - \omega_{2h}^i\right\|_2$$

Proof. The proof will be carried out by induction. For n=1, we have in Ω_1 .

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \le \left\|u_1^1 - \omega_{1h}^1\right\|_1 + \left\|\omega_{1h}^1 - u_{1h}^1\right\|_1$$

and, making use of Proposition 2, we obtain

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \le \left\|u_1^1 - \omega_{1h}^1\right\|_1 + \max\left\{\frac{1}{\beta}\left\|f\left(u_1^0\right) - f\left(u_{1h}^0\right)\right\|_1, \left|\pi_h u_2^0 - \pi_h u_{2h}^0\right|_1\right\}$$

$$\le \left\|u_1^1 - \omega_{1h}^1\right\|_1 + \max\left\{\frac{k}{\beta}\left\|u_1^0 - u_{1h}^0\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\right\}$$

We then have to distinguish between two cases

$$1: \max\left\{\frac{k}{\beta}\left\|u_1^0 - u_{1h}^0\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\right\} = \frac{k}{\beta}\left\|u_1^0 - u_{1h}^0\right\|_1$$

and

$$2 : \max\left\{\frac{k}{\beta}\left\|u_1^0 - u_{1h}^0\right\|_1, \left\|u_2^0 - u_{2h}^0\right\|_2\right\} = \left\|u_2^0 - u_{2h}^0\right\|_2$$

Case 1. implies that

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \left\|u_1^1 - \omega_{1h}^1\right\|_1 + \frac{k}{\beta}\left\|u_1^0 - u_{1h}^0\right\|_1$$

and hence

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \left\|u_1^1 - \omega_{1h}^1\right\|_1 + \left\|u_1^0 - u_{1h}^0\right\|_1$$

Case 2. implies that

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \left\|u_1^1 - \omega_{1h}^1\right\|_1 + \left\|u_2^0 - u_{2h}^0\right\|_2$$

and, in both cases, we have

$$\left\|u_1^1 - u_{1h}^1\right\|_1 \leq \left\|u_1^1 - \omega_{1h}^1\right\|_1 + \left\|u_1^0 - u_{1h}^0\right\|_1 + \left\|u_2^0 - u_{2h}^0\right\|_2 \tag{28}$$

Similarly, we have in Ω_2

$$\left\|u_2^1 - u_{2h}^1\right\|_2 \leq \left\|u_2^1 - \omega_{2h}^1\right\|_2 + \left\|\omega_{2h}^1 - u_{2h}^1\right\|_2$$

$$\leq \left\|u_2^1 - \omega_{2h}^1\right\|_2 + \max\left\{\frac{k}{\beta}\left\|u_2^0 - u_{2h}^0\right\|_2, \left|\pi_h u_1^1 - \pi_h u_{1h}^1\right|_2\right\}$$

$$\leq \left\|u_2^1 - \omega_{2h}^1\right\|_2 + \max\left\{\frac{k}{\beta}\left\|u_2^0 - u_{2h}^0\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\right\}$$

Here also we need to consider the following two cases:

$$1 : \quad \max\left\{\frac{k}{\beta}\left\|u_2^0 - u_{2h}^0\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\right\} = \frac{k}{\beta}\left\|u_2^0 - u_{2h}^0\right\|_2$$

$$2 : \quad \max\left\{\frac{k}{\beta}\left\|u_2^0 - u_{2h}^0\right\|_2, \left\|u_1^1 - u_{1h}^1\right\|_1\right\} = \left\|u_1^1 - u_{1h}^1\right\|_1$$

Case 3. implies that

$$\left\|u_2^1 - u_{2h}^1\right\|_2 \leq \left\|u_2^1 - \omega_{2h}^1\right\|_2 + \frac{k}{\beta}\left\|u_2^0 - u_{2h}^0\right\|_2$$

so

$$\left\|u_2^1 - u_{2h}^1\right\|_2 \leq \left\|u_2^1 - \omega_{2h}^1\right\|_2 + \left\|u_2^0 - u_{2h}^0\right\|_2$$

Case 4. implies that

$$\left\|u_2^1 - u_{2h}^1\right\|_2 \leq \left\|u_2^1 - \omega_{2h}^1\right\|_2 + \left\|u_1^1 - u_{1h}^1\right\|_1$$

$$\leq \left\|u_2^1 - \omega_{2h}^1\right\|_2 + \left\|u_1^1 - \omega_{1h}^1\right\|_1 + \left\|u_1^0 - u_{1h}^0\right\|_1 + \left\|u_2^0 - u_{2h}^0\right\|_2$$

Thus, in both cases, we have

$$\left\|u_2^1 - u_{2h}^1\right\|_2 \leq \left\|u_2^1 - \omega_{2h}^1\right\|_2 + \left\|u_1^1 - \omega_{1h}^1\right\|_1 + \left\|u_1^0 - u_{1h}^0\right\|_1 + \left\|u_2^0 - u_{2h}^0\right\|_2 \qquad (29)$$

For n = 2

$$\left\|u_1^2 - u_{1h}^2\right\|_1 \leq \left\|u_1^2 - \omega_{1h}^2\right\|_1 + \left\|\omega_{1h}^2 - u_{1h}^2\right\|_1$$

$$\leq \left\|u_1^2 - \omega_{1h}^2\right\|_1 + \max\left\{\frac{1}{\beta}\left\|f(u_1^1) - f(u_{1h}^1)\right\|_1 , \left|\pi_h u_2^1 - \pi_h u_{2h}^1\right|_1\right\}$$

$$\leq \left\|u_1^2 - \omega_{1h}^2\right\|_1 + \max\left\{\frac{k}{\beta}\left\|u_1^1 - u_{1h}^1\right\|_1 , \left\|u_2^1 - u_{2h}^1\right\|_2\right\}$$

Case 1.

$$\max\left\{\frac{1}{\beta}\left\|u_1^1 - u_{1h}^1\right\|_1 , \left\|u_2^1 - u_{2h}^1\right\|_2\right\} = \frac{k}{\beta}\left\|u_1^1 - u_{1h}^1\right\|_1$$

$$\left\|u_1^2 - u_{1h}^2\right\|_1 \leq \left\|u_1^2 - \omega_{1h}^2\right\|_1 + \frac{k}{\beta}\left\|u_1^1 - u_{1h}^1\right\|_1$$

$$\leq \left\|u_1^2 - \omega_{1h}^2\right\|_1 + \left\|u_1^1 - u_{1h}^1\right\|_1$$

$$\leq \left\|u_1^2 - \omega_{1h}^2\right\|_1 + \left\|u_1^1 - \omega_{1h}^1\right\|_1 + \left\|u_1^0 - u_{1h}^0\right\|_1 + \left\|u_2^0 - u_{2h}^0\right\|_2$$

Case 2.

$$\max\left\{\frac{1}{\beta}\left\|u_1^1 - u_{1h}^1\right\|_1 , \left\|u_2^1 - u_{2h}^1\right\|_2\right\} = \left\|u_2^1 - u_{2h}^1\right\|_2$$

$$\left\|u_1^2 - u_{1h}^2\right\|_1 \leq \left\|u_1^2 - \omega_{1h}^2\right\|_1 + \left\|u_2^1 - \omega_{2h}^1\right\|_2 + \left\|u_1^1 - \omega_{1h}^1\right\|_1 + \left\|u_1^0 - u_{1h}^0\right\|_1 + \left\|u_2^0 - u_{2h}^0\right\|_2$$

So in both cases

$$\left\|u_1^2 - u_{1h}^2\right\|_1 \leq \left\|u_1^2 - \omega_{1h}^2\right\|_1 + \left\|u_2^1 - \omega_{2h}^1\right\|_2 + \left\|u_1^1 - \omega_{1h}^1\right\|_1 + \left\|u_1^0 - u_{1h}^0\right\|_1 + \left\|u_2^0 - u_{2h}^0\right\|_2 \qquad (30)$$

or

$$\left\| u_1^2 - u_{1h}^2 \right\|_1 \leq \sum_{i=0}^{2} \left\| u_1^i - \omega_{1h}^i \right\|_1 + \sum_{i=0}^{1} \left\| u_2^i - \omega_{2h}^i \right\|_2 \tag{31}$$

On the other hand

$$\left\| u_2^2 - u_{2h}^2 \right\|_2 \leq \left\| u_2^2 - \omega_{2h}^2 \right\|_2 + \left\| \omega_{2h}^2 - u_{2h}^2 \right\|_2$$

$$\leq \left\| u_2^2 - \omega_{2h}^2 \right\|_2 + \max\left\{ \frac{k}{\beta} \left\| u_2^1 - u_{2h}^1 \right\|_2 , \left\| u_1^2 - u_{1h}^2 \right\|_2 \right\}$$

Case 1.

$$\left\| u_2^2 - u_{2h}^2 \right\|_2 \leq \left\| u_2^2 - \omega_{2h}^2 \right\|_2 + \frac{k}{\beta} \left\| u_2^1 - u_{2h}^1 \right\|_2$$

$$\leq \left\| u_2^2 - \omega_{2h}^2 \right\|_2 + \left\| u_2^1 - u_{2h}^1 \right\|_2$$

$$\leq \left\| u_2^2 - \omega_{2h}^2 \right\|_2 + \left\| u_2^1 - \omega_{2h}^1 \right\|_2 + \left\| u_1^1 - \omega_{1h}^1 \right\|_1 + \left\| u_1^0 - u_{1h}^0 \right\|_1 + \left\| u_2^0 - u_{2h}^0 \right\|_2$$

Case 2.

$$\left\| u_2^2 - u_{2h}^2 \right\|_2 \leq \left\| u_2^2 - \omega_{2h}^2 \right\|_2 + \left\| u_1^2 - u_{1h}^2 \right\|_1$$

$$\leq \left\| u_2^2 - \omega_{2h}^2 \right\|_2 + \left\| u_1^2 - \omega_{1h}^2 \right\|_1 + \left\| u_2^1 - \omega_{2h}^1 \right\|_2 + \left\| u_1^1 - \omega_{1h}^1 \right\|_1 + \left\| u_1^0 - u_{1h}^0 \right\|_1 + \left\| u_2^0 - u_{2h}^0 \right\|_2$$

So in both cases

$$\left\| u_2^2 - u_{2h}^2 \right\|_2 \leq \left\| u_2^2 - \omega_{2h}^2 \right\|_2 + \left\| u_1^2 - \omega_{1h}^2 \right\|_1 + \left\| u_2^1 - \omega_{2h}^1 \right\|_2 + \left\| u_1^1 - \omega_{1h}^1 \right\|_1 + \left\| u_1^0 - u_{1h}^0 \right\|_1 + \left\| u_2^0 - u_{2h}^0 \right\|_2$$

or

$$\left\| u_2^2 - u_{2h}^2 \right\|_2 \leq \sum_{i=0}^{2} \left\| u_1^i - \omega_{1h}^i \right\|_1 + \sum_{i=0}^{2} \left\| u_2^i - \omega_{2h}^i \right\|_2 \tag{32}$$

Now assume that

$$\left\| u_1^n - u_{1h}^n \right\|_2 \leq \sum_{i=0}^{n} \left\| u_1^i - \omega_{1h}^i \right\|_1 + \sum_{i=0}^{n-1} \left\| u_2^i - \omega_{2h}^i \right\|_2$$

and

$$\left\| u_2^n - u_{2h}^n \right\|_2 \leq \sum_{i=0}^{n} \left\| u_1^i - \omega_{1h}^i \right\|_1 + \sum_{i=0}^{n} \left\| u_2^i - \omega_{2h}^i \right\|_2$$

and let us prove that

$$\left\| u_1^{n+1} - u_{1h}^{n+1} \right\|_1 \leq \sum_{i=0}^{n+1} \left\| u_1^i - \omega_{1h}^i \right\|_1 + \sum_{i=0}^{n} \left\| u_2^i - \omega_{2h}^i \right\|_2$$

and

$$\left\| u_2^{n+1} - u_{2h}^{n+1} \right\|_2 \leq \sum_{i=0}^{n+1} \left\| u_1^i - \omega_{1h}^i \right\|_1 + \sum_{i=0}^{n+1} \left\| u_2^i - \omega_{2h}^i \right\|_2$$

Indeed, we have

$$\left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \le \left\|u_1^{n+1} - \omega_{1h}^{n+1}\right\|_1 + \left\|\omega_{1h}^{n+1} - u_{1h}^{n+1}\right\|_1$$

$$\le \left\|u_1^{n+1} - \omega_{1h}^{n+1}\right\|_1 + \max\left\{\frac{k}{\beta}\left\|u_1^n - u_{1h}^n\right\|_1, \left|\pi_h u_2^n - \pi_h u_{2h}^n\right|_1\right\}$$

$$\le \left\|u_1^{n+1} - \omega_{1h}^{n+1}\right\|_1 + \max\left\{\frac{k}{\beta}\left\|u_1^n - u_{1h}^n\right\|_1, \left\|u_2^n - u_{2h}^n\right\|_2\right\}$$

and, as above, we need to distinguish between two cases:

$$1:\ \max\left\{\frac{k}{\beta}\left\|u_1^n - u_{1h}^n\right\|_1, \left\|u_2^n - u_{2h}^n\right\|_2\right\} = \frac{k}{\beta}\left\|u_1^n - u_{1h}^n\right\|_1$$

and

$$2:\ \max\left\{\frac{k}{\beta}\left\|u_1^n - u_{1h}^n\right\|_1, \left\|u_2^n - u_{2h}^n\right\|_2\right\} = \left\|u_2^n - u_{2h}^n\right\|_2$$

Case 1. implies that

$$\left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \le \left\|u_1^{n+1} - \omega_{1h}^{n+1}\right\|_1 + \frac{k}{\beta}\left\|u_1^n - u_{1h}^n\right\|_1$$

$$\le \left\|u_1^{n+1} - \omega_{1h}^{n+1}\right\|_1 + \left\|u_1^n - u_{1h}^n\right\|_1$$

$$\le \left\|u_1^{n+1} - \omega_{1h}^{n+1}\right\|_1 + \sum_{i=0}^{n}\left\|u_1^i - \omega_{1h}^i\right\|_1 + \sum_{i=0}^{n-1}\left\|u_2^i - \omega_{2h}^i\right\|_2$$

while **Case 2**. implies that

$$\left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \le \left\|u_1^{n+1} - \omega_{1h}^{n+1}\right\|_1 + \left\|u_2^n - u_{2h}^n\right\|_2$$

$$\le \left\|u_1^{n+1} - \omega_{1h}^{n+1}\right\|_1 + \sum_{i=0}^{n}\left\|u_1^i - \omega_{1h}^i\right\|_1 + \sum_{i=0}^{n}\left\|u_2^i - \omega_{2h}^i\right\|_2$$

So in both cases

$$\left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \le \left\|u_1^{n+1} - \omega_{1h}^{n+1}\right\|_1 + \sum_{i=0}^{n}\left\|u_1^i - \omega_{1h}^i\right\|_1 + \sum_{i=0}^{n}\left\|u_2^i - \omega_{2h}^i\right\|_2$$

$$\le \sum_{i=0}^{n+1}\left\|u_1^i - \omega_{1h}^i\right\|_1 + \sum_{i=0}^{n}\left\|u_2^i - \omega_{2h}^i\right\|_2$$

Likewise

$$\left\|u_2^{n+1} - u_{2h}^{n+1}\right\|_2 \le \left\|u_2^{n+1} - \omega_{2h}^{n+1}\right\|_2 + \left\|u_2^{n+1} - u_{2h}^{n+1}\right\|_2$$

$$\le \left\|u_2^{n+1} - \omega_{2h}^{n+1}\right\|_2 + \max\left\{\frac{k}{\beta}\left\|u_2^n - u_{2h}^n\right\|_2, \left|\pi_h u_1^1 - \pi_h u_{2h}^1\right|_2\right\}$$

$$\le \left\|u_2^{n+1} - \omega_{2h}^{n+1}\right\|_2 + \max\left\{\frac{k}{\beta}\left\|u_2^n - u_{2h}^n\right\|_2, \left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1\right\}$$

Here also we need to discuss two cases:

$$\text{Case 1: } \max\left\{\frac{k}{\beta}\left\|u_2^n - u_{2h}^n\right\|_2, \left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1\right\} = \frac{k}{\beta}\left\|u_2^n - u_{2h}^n\right\|_2$$

implies

$$\left\|u_2^{n+1} - u_{2h}^{n+1}\right\|_2 \leq \left\|u_2^{n+1} - \omega_{2h}^{n+1}\right\|_2 + \left\|u_2^n - u_{2h}^n\right\|_2$$

$$\leq \left\|u_2^{n+1} - \omega_{2h}^{n+1}\right\|_2 + \sum_{i=0}^{n}\left\|u_1^i - \omega_{1h}^i\right\|_1 + \sum_{i=0}^{n}\left\|u_2^i - \omega_{2h}^i\right\|_2$$

$$\text{Case 2: } \max\left\{\frac{k}{\beta}\left\|u_2^n - u_{2h}^n\right\|_2, \left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1\right\} = \left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1$$

implies

$$\left\|u_2^{n+1} - u_{2h}^{n+1}\right\|_2 \leq \left\|u_2^{n+1} - \omega_{2h}^{n+1}\right\|_2 + \left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1$$

$$\leq \left\|u_2^{n+1} - \omega_{2h}^{n+1}\right\|_2 + \left\|u_1^{n+1} - \omega_{1h}^{n+1}\right\|_1 + \sum_{i=0}^{n}\left\|u_1^i - \omega_{1h}^i\right\|_1 + \sum_{i=0}^{n}\left\|u_2^i - \omega_{2h}^i\right\|_2$$

Then, in both cases, we have

$$\left\|u_2^{n+1} - u_{2h}^{n+1}\right\|_2 \leq \left\|u_2^{n+1} - \omega_{2h}^{n+1}\right\|_2 + \left\|u_1^{n+1} - \omega_{1h}^{n+1}\right\|_1 + \sum_{i=0}^{n}\left\|u_1^i - \omega_{1h}^i\right\|_1 + \sum_{i=0}^{n}\left\|u_2^i - \omega_{2h}^i\right\|_2$$

$$\leq \sum_{i=0}^{n+1}\left\|u_1^i - \omega_{1h}^i\right\|_1 + \sum_{i=0}^{n+1}\left\|u_2^i - \omega_{2h}^i\right\|_2$$

which completes the proof.

Theorem 4. There exists $h_n > 0$ with $\lim\limits_{n\to\infty} h_n = 0$, such that

$$\lim_{n\to\infty}\left\|u_1^i - u_{1h}^i\right\|_{L^\infty(\Omega_i)} = 0, \; i = 1, 2$$

Proof. Let us give the proof for i=1, the case i=2 being similar.
Indeed, as

$$\left\|u_1^n - u_{1h}^n\right\|_1 \leq \sum_{i=0}^{n}\left\|u_1^i - \omega_{1h}^i\right\|_1 + \sum_{i=0}^{n-1}\left\|u_2^i - \omega_{2h}^i\right\|_2$$

and

$$\left\|u_1^n - \omega_{1h}^n\right\|_1 \leq Ch^2\left|\ln h\right|$$

$$\left\|u_2^n - \omega_{2h}^n\right\|_2 \leq Ch^2\left|\ln h\right|$$

Then,

$$\left\|u_1^n - u_{1h}^n\right\|_1 \leq (2n+1)Ch^2\left|\ln h\right| \tag{33}$$

Also

$$\left\| u_1^n - u_{1h}^n \right\|_1 \le \left\| u_{1h}^n - u_1^n \right\|_1 + \left\| u_1^n - u_1 \right\|_1$$

Let $\varepsilon > 0$. Theorem 3 implies that there exists $n_0 \in N$ such that, $\forall \, n \ge n_0$

$$\left\| u_1^n - u_1 \right\|_1 \le \frac{\varepsilon}{2}$$

Taking account of (33), the Theorem follows by choosing $h_n > 0$ such that

$$h_n^2 \left| \ln h_n \right| \le \frac{\varepsilon}{(2n+1)C} , \quad \forall n \ge n_0$$

Conclusion

We have proved convergence of the standard finite element approximation for alternating Schwarz procedure in the context of nonmatching grids. Other type of discretizations may also be considered like mixing finite elements and finite differences. Also, the knowledge of a rate of convergence of the Schwarz procedure will enable derivation of error estimate, in each subdomain, between the discrete Schwarz sequence and the exact solution of the nonlinear PDE.

Conflict of interest

The author declares no conflict of interest.

Acknowlegement

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