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# The Time Evolution of a Constant Mass of Air Pollutant Emitted by a Point Source

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## التطور الزمني لبؤرة من التلوث أطلقت في الهواء من مصدر ثابت

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**خلاصة :** أطلقت بؤرة تلوث من مصدر ثابت على علو معين في وقت معين ، يقوم البحث بدراسة انتشـار التلـوث باستعمال نموذج رياضي يتكون من معادلة تفاضلية جزئية وشروط محلية ووقتية تحدد حالة الانطلاق. استعملت طرق التحويلات الرياضية التكاملية لإيجاد الحل كما تم تحديد الزمن الكلي لانتشار التلوث الكلي واعتماد ذلك الزمن علـى خواص بؤرة التلوث والاحوال الجوية المصاحبة.

**ABSTRACT:** The transient behaviour of a constant mass (i.e. a blob) of pollutant released from a point source at a height, $h$, above ground level at time $t = 0$ is studied. The time dependent atmospheric diffusion equation in the presence of diffusion in both horizontal and vertical directions is used to model the problem. The model is found to be governed by an initial-boundary-value problem for the concentration of the pollutant. The solution is obtained in closed form using integral transform methods. The solution is illustrated graphically using appropriate numerical integrations. As time passes, the pollutant blob moves with a central point of accumulation of pollutant while the blob increases in volume to spread the pollutant around it. The motion of the accumulation point in space and time is strongly influenced by wind and gravity while the spread of the pollutant is governed by diffusion. The time taken by the blob to diffuse into space is estimated as a function of the parameters governing wind, gravity and diffusion.

KEYWORDS: Initial-boundary value problem, pollutant, diffusion equation, analytic solution, time-dependence, point source.

## 1. Introduction

The study of the transport of small-size particulate matter under dry conditions has been of interest for many years because of its relevance to environmental and industrial applications (Ragland and Dennis, 1975; Vogt, 1977; Beiruti and Alomshy, 1985; El-Baz *et al*., 1990; Keller and Lambrecht, 1995; Yoshikawa *et al*., 1996; Spanomitsios, 2001). It is now known that under idealized conditions of stationary homogeneous turbulence the mean concentration of pollutants emitted from a point source takes the form of a Gaussian distribution. For a comprehensive review which also includes a collection of Gaussian-based formulae the reader is referred to Pasquill and Smith (1984), Chatwin *et al*. (1996), and Seinfeld, 1986). More realistic models have been proposed using the atmospheric diffusion equation. The two main factors affecting the spread of pollutants are the wind speed and the local diffusion coefficients. It is generally accepted that the two factors are not independent. As the wind speed varies, the interaction of the particles is influenced and the turbulence viscosity changes. A number of studies have investigated these two factors and their interrelationship using field data (Vanderhoven, 1976; Semetov *et al*., 1996) and experimental data (Nguyenvanchi and Saab, 1976; Nagai, 2005). However, there is no general agreement, as yet, on the definite form of the dependence of wind and diffusion parameters on position. This is generally due to the fact that the two factors are not independent and as the wind depends on local circumstances as well as on the atmospheric boundary layer conditions the diffusion parameters will accordingly change with the wind conditions. The wind profile has sometimes been taken as logarithmic (Van Ulden and Holtslag, 1985) or as a power law in the vertical distance, Lin and Hildemann (1996). The diffusion coefficients have also been assumed to have a number of different forms. Whereas the wind speed is always assumed to vary with height, the diffusion coefficients are found to depend either on the height for large to moderate wind speeds (in excess of 10 m/s), (Lin and Hildemann, 1997), or depend on the distance along the wind for low wind speeds (about 2m/s), (Sharan *et al*., 1996), or on both, (Lin and Hildemann, 1996). However, in most cases they are assumed to have a power law.

The forms of the wind profile and the diffusion coefficients determine how complicated the study of that model of the atmospheric diffusion equation is. For the logarithmic wind profile the solution of the atmospheric diffusion equation can only be found by numerical integration and the same applies to the general power law profile, although certain special forms of the power law profile can possess analytical solutions (Lin and Hildemann, 1996). Moreover, these solutions have been found only for steady forms of the atmospheric diffusion equation, (Eltayeb and Hassan, 2000; Thompson, 1976; Dumbauld and Bowers, 1976, Brian and Lee, 1998).

Our purpose here is to study the transient evolution of a given mass of pollutant released from a point source with time. This problem has not received much attention in the literature. It requires the solution of the *time dependent* atmospheric diffusion equation as in an initial-boundary- value problem. Such a problem is relevant to industrial accidents when an explosion releases a certain mass of pollutant into the atmosphere, (Vogt, 1977), and it would be interesting to see how it spreads and how long it takes to diffuse away into the surroundings. In order to gain some insight into such a problem, we here study a simple model, as a bench mark for more realistic models, in which the wind speed and diffusion coefficients take simple forms that allow an analytical solution. The advantage of the analytical solution here is that it allows us to isolate the influences of the different factors affecting the solution. Such basic factors are expected to remain qualitatively applicable for more realistic models.

In section 2, we formulate the problem and in section 3 we find the solution. Section 4 is devoted to a discussion of the results and some concluding remarks are made in section 5.

## 2. Formulation of the problem

Consider a volume $Q$ of pollutant emitted at time $t^* = 0$ from a point at a height $h$ above ground level. We intend to study the distribution, $c^*$, of pollutant concentration as it evolves with time in the presence of a uni-directional horizontal wind blowing with speed $U$. We take a Cartesian system of coordinates

$O(x^*, y^*, z^*)$ such that $Ox^*$ is horizontal and parallel to the direction of the wind, $Oz^*$ is vertically upwards and $Oy^*$ is horizontal and normal to $Ox^*$. Accordingly the source is situated along the $z^*$-axis. The governing equation is the time-dependent diffusion equation

$$\frac{\partial c^*}{\partial t^*} + \mathbf{u}\cdot\nabla c^* = \nabla.\left(\mathbf{D}\cdot\nabla c^*\right) - \mathbf{w}.\nabla c^*, \quad \nabla\cdot\mathbf{u} = 0 \tag{2.1}$$

in which $\mathbf{u}$ is the velocity, $\mathbf{w}\,(=(0,0,W))$ is the settling velocity and $\mathbf{D}$ is the stress tensor. The settling velocity depends on the particle size and can be taken as a measure of the influence of gravity. The second term on the left-hand side of the first of (2.1) represents the advection of the particles by the local fluid motion.

As we mentioned in the introduction above, the wind speed is, in general, a function of height, $z^*$, but in order to make progress with the time dependent initial-boundary-value problem below, we shall assume that the wind speed is uniform so that

$$\mathbf{u} = (U, 0, 0) \tag{2.2}$$

and a stress tensor, $\mathbf{D}$, of the form

$$\mathbf{D} = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \lambda z^* \end{pmatrix}, \tag{2.3}$$

in which $\gamma$, $\beta$ and $\lambda$ are constants. The stress tensor components in the horizontal may not be uniform but we believe that their variations for moderate wind speeds are smaller than those in the vertical direction. These assumptions about the wind speed and diffusion coefficients allow us to find an analytical solution that permits an analysis of the influences of the different components of diffusion as well as that of the wind speed in the process of diffusion and dispersion of the pollutant with time.

We shall now proceed to transform the governing equation into a standard form using a scaling for both independent and dependent variables together with some transformations. Define

$$\begin{aligned} &t = \lambda t^*, \quad x = (\lambda/U)\,x^*, \quad y = y^*, \quad z = z^*, \quad b = \gamma\lambda/U^2 \\ &a = \beta/\lambda, \quad \nu = W/\lambda, \quad c = Uc^*/Q \end{aligned} \tag{2.4}$$

and reduce (2.1) to

$$\frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} = (\nu+1)\frac{\partial c}{\partial z} + z\frac{\partial^2 c}{\partial z^2} + b\frac{\partial^2 c}{\partial x^2} + a\frac{\partial^2 c}{\partial y^2} \tag{2.5}$$

The initial condition of the problem is given by

$$c(x, y, z, 0) = \delta(y)\,\delta(x)\,\delta(h - z) \tag{2.6}$$

Since the pollutant is that released from the source, the solution must decay away from the source so that

$$c(x, y, z, t) \to 0, \quad \text{as} \quad |x|, |y|, z, t \to \infty \tag{2.7}$$

The condition at the top of the roughness layer where $z = z_0$ should realistically be of the form

$$\partial c/\partial z + \sigma c = 0 \quad \text{at} \quad z = z_0 \tag{2.8}$$

where $\sigma$ is a constant. Here the first term represents reflection of the vertical flux of the pollutant. However, this condition makes the solution of the atmospheric diffusion equation very difficult and has not been used in many previous studies. Most studies use either the condition

$$\partial c/\partial z^* = 0 \quad \text{at} \quad z = z_0 \tag{2.9}$$

representing vanishing vertical flux at the top of the roughness layer so that the pollutant is perfectly reflected back into the region above the roughness layer, or the condition

$$c = 0 \quad \text{at} \quad z = z_0 \tag{2.10}$$

representing the vanishing of the concentration at the top of the roughness layer so that the pollutant falling to the top of the roughness layer is completely absorbed.

Here we find that the condition (2.10) permits an exact solution for all values of the thickness of the roughness layer, $z_0$. In the special case in which the thickness of layer is vanishingly small (ie. $z_0 \to 0$) the boundary condition (2.9) will also lead to an exact solution. The general condition (2.8) will not be considered here. We shall then apply the condition

$$c(x, y, z_0, t) = 0 \tag{2.11}$$

We are now required to solve the equation (2.5) subject to the initial condition (2.6) and the two boundary conditions (2.7) and (2.11). We make the transformation

$$u(x, y, \varsigma, t) = \varsigma^{\nu} e^{-x/2b} c\left(x, y, \varsigma, t\right), \qquad \varsigma = 2z^{1/2}, \quad h_0 = 2h^{1/2} \tag{2.12}$$

and reduce (2.5) to

$$\frac{\partial u}{\partial t} = \frac{1}{\varsigma}\frac{\partial}{\partial \varsigma}\left(\varsigma \frac{\partial u}{\partial \varsigma}\right) - \frac{\nu^2}{\varsigma^2}u + b\frac{\partial^2 u}{\partial x^2} - \frac{1}{4b}u + a\frac{\partial^2 c}{\partial y^2} \tag{2.13}$$

The boundary and initial conditions (2.11) can be written in terms of $u$ as

$$\begin{aligned} &\text{(i)} \ \ \mathrm{u}(x, y, \zeta_0, t) = 0 \\ &\text{(ii)} \ \ \mathrm{u}(x, y, \zeta, 0) = e^{-x/2b}\delta\left(y\right)\delta\left(x\right)\zeta^{\nu}\delta\left(h_0 - \zeta\right) \\ &\text{(iii)} \ \ e^{x/2b}\zeta^{-\nu}\mathrm{u}(x, y, \zeta, t) \to 0, \quad \text{as} \ \ |x|, |y|, \zeta, t \to \infty \end{aligned} \tag{2.14}$$

The model then poses the initial-boundary-value problem (2.13) and (2.14).

## 3. The solution

We adopt Weber's transform, in $\varsigma$, defined by

$$\overline{u}\left(x, y, p, t\right) = \int_{\varsigma_0}^{\infty} u\left(x, y, \varsigma, t\right) H_{\nu}\left(p\varsigma\right)\varsigma d\varsigma \tag{3.1}$$

together with its inverse

$$u\left((x, y, \varsigma, t\right) = \int_0^{\infty} \frac{\overline{u}(x, y, p, t)\, H_{\nu}(p\varsigma)}{J_{\nu}^2(p\varsigma_o) + Y_{\nu}^2(p\varsigma_o)}\, pdp \tag{3.2}$$

in which

$$H_{\nu}(p\varsigma) = J_{\nu}(p\varsigma)Y_{\nu}(p\varsigma_o) - J_{\nu}(p\varsigma_o)Y_{\nu}(p\varsigma) \tag{3.3}$$

Here $J_{\nu}(x)$ and $Y_{\nu}(x)$ are Bessel functions of the First and Second Kinds with order $\nu$ and argument $x$. We note here that

$$\begin{aligned} &H_{\nu}\left(p\varsigma_0\right) = 0, \qquad \frac{dH_{\nu}\left(p\varsigma_0\right)}{d\varsigma} = -\frac{2}{\pi\varsigma_0}, \qquad \varsigma_0 = 2z_0^{1/2}, \\ &\frac{d^2}{d\varsigma^2}H_{\nu}\left(p\varsigma\right) + \frac{1}{\varsigma}\frac{d}{d\varsigma}H_{\nu}\left(p\varsigma\right) + \left(p^2 - \frac{\nu^2}{\varsigma^2}\right)H_{\nu}\left(p\varsigma\right) = 0 \end{aligned} \tag{3.4}$$

The application of the Weber transform (3.1) to equation (2.13) and the utilization of conditions (2.14), iii and (3.4) yields the equation

$$\frac{\partial \overline{u}}{\partial t} = b\frac{\partial^2 \overline{u}}{\partial x^2} + a\frac{\partial^2 \overline{u}}{\partial y^2} - \left( p^2 + \frac{1}{4b} \right)\overline{u} \tag{3.5}$$

We next use the double Fourier transform in $x$ and $y$ as defined by

$$\tilde{\tilde{\overline{u}}}\left(\omega, s, p, t\right) = \frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{u}\left(x, y, p, t\right) e^{-i(\omega x + sy)} dxdy \tag{3.6}$$

and its inverse defined by

$$\overline{u}\left(x, y, p, t\right) = \frac{2}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\tilde{\overline{u}}}\left(\omega, s, p, t\right) e^{i(\omega x + sy)} d\omega ds \tag{3.7}$$

The application of the double Fourier transform to (3.5) and the use of (2.14),(iii) leads to

$$\frac{\partial \tilde{\tilde{\overline{u}}}}{\partial t} = -\left( p^2 + \frac{1}{4b} + b\omega^2 + as^2 \right)\tilde{\tilde{\overline{u}}} \tag{3.8}$$

The solution of the first order equation (3.8) in $t$ is

$$\tilde{\tilde{\overline{u}}}\left(\omega, s, p, t\right) = \tilde{\tilde{\overline{u}}}\left(\omega, s, p, 0\right)\exp\left\{ -\left( p^2 + \frac{1}{4b} + b\omega^2 + as^2 \right)t \right\} \tag{3.9}$$

The application of the two transforms to (2.14),(ii) yields

$$\tilde{\tilde{\overline{u}}}\left(\omega, s, p, 0\right) = \left(h_0\right)^{\nu+1} H_\nu\left(ph_0\right) \tag{3.10}$$

Equations (3.9) and (3.10) then give

$$\tilde{\tilde{\overline{u}}}\left((\omega, s, p, t\right) = \left(h_0\right)^{\nu+1} H_\nu\left(ph_0\right)\exp\left\{ -\left( p^2 + \frac{1}{4b} + b\omega^2 + as^2 \right)t \right\} \tag{3.11}$$

The solution now proceeds by inverting the two transforms (see Duffy, 1994]). It is expedient to invert the Fourier transforms first. This is carried out in a straightforward manner using standard tables:

$$\overline{u}\left(x, y, p, t\right) = \left(h_0\right)^{\nu+1} H_\nu\left(ph_0\right).\frac{1}{\sqrt{2bt}}.\exp\left\{ -\left( p^2 + \frac{1}{4b} \right)t - \frac{x^2}{4bt} - \frac{y^2}{4at} \right\} \tag{3.12}$$

The inversion of the Weber transform gives

$$u\left(x, y, \varsigma, t\right) = \frac{\left(h_0\right)^{\nu+1}}{\sqrt{2bt}}\exp\left\{ -\left( \frac{t}{4b} + \frac{x^2}{4bt} + \frac{y^2}{4at} \right) \right\}\int_0^\infty F\left(\varsigma, t; \varsigma_0, h_0, \nu\right) dp \tag{3.13}$$

in which

$$F\left(\varsigma, t; \varsigma_0, h_0, \nu\right) = \frac{H_\nu(ph_0)H_\nu\left(p\varsigma\right)}{J_\nu^2\left(p\varsigma_0\right) + Y_\nu^2\left(p\varsigma_0\right)} pe^{-tp^2} \tag{3.14}$$

The concentration $c(x, y, \varsigma, t)$ can now be written in closed form as

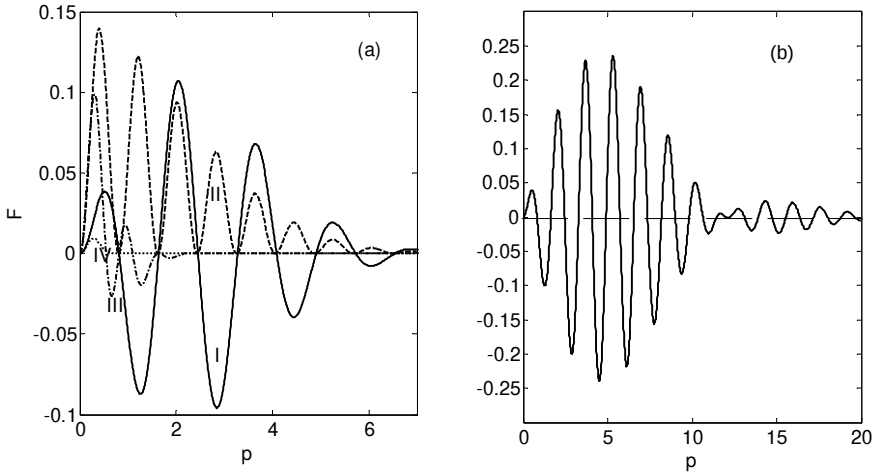


Figure 1. A sample of representative profiles of the integrand $F\left(\varsigma,t;\varsigma_0,h_0,\nu\right)$ as a function of the integration variable $p$ when $h=5.0,\nu=0.5$ and $z_0=0.1,b=0.0$. In (a), the four curves are such that I (continuous) , II ( discontinuous), III (discontinuous-dotted) and IV ( dotted) refer to the pair $(z-z_0,t)$ taking the values (0.1,0.1), (5.0,0.1), (0.1,10.0) and (10.0,1.0), respectively, while (b) represents $z-z_0=0.1$ and $t=0.01$.

$$c\left(x,y,\varsigma,t\right)=\frac{\left(h_0\right)^{\nu+1}}{\varsigma^{\nu}\sqrt{2bt}}\exp\left\{-\frac{\left(x-t\right)^2}{4bt}-\frac{y^2}{4at}\right\}\int_0^{\infty}F(\varsigma,t;\varsigma_0,h_0,\nu)dp \tag{3.15}$$

for all $|x|,|y|,z(=\varsigma^2/4),t>0.$ The dependence on the longitudinal distance $x$ appears in the exponential factor outside the integral and is coupled only with the time and the diffusion coefficient in the same direction so that it is separated from the height and the diffusion coefficient in the vertical direction. A similar behaviour applies to the dependence on the lateral distance $y$. The dependence on the vertical distance $z(=\varsigma^2/4)$ is more complicated; it is coupled with the parameters representing the height, $h_0$ , the settling velocity, $\nu$, and the roughness height, $z_0$ and depends on the integral

$$I=\int_0^{\infty}F\left(\varsigma,t;\varsigma_0,h_0,\nu\right)dp \tag{3.16}$$

For non-zero values of the roughness height, as represented by $\varsigma_0$, the integral (3.16) must be evaluated numerically. As the interval of integration is semi-infinite, the question of convergence of the integral must be addressed. Since the integrand is an analytic function we have chosen to examine the nature of the function first before adopting a complicated automatic routine for its evaluation. An examination of the integrand $F\left(\varsigma,t;\varsigma_0,h_0,\nu\right)$ shows that it is oscillatory with the period as well as decay rate depending crucially on the time $t$ (see Figure 1). It is bounded, possesses no singularities and decays to zero for all values of the relevant

parameters. Except for small times (i.e. $t < 0.1$), the function decays to a negligible value for $p \lesssim 10.0$. This makes it possible to limit the interval of integration to (0.0,10.0) except for small values of $t$. The integral is then evaluated using A NAG subroutine which uses an adaptive integrator based on a method due to Piessens *et al.*(1983). For very small values of $t$ convergence is not good, as the integrand oscillates very rapidly and sometimes erratically until $p$ increases to large values before it decays to zero. This is not a serious matter as we know that for very small times, the blob has not changed its shape much and is approximately in its original shape.

The numerical integrations have shown excellent agreement with the analytical result (3.18) obtained for the case $\varsigma_0 \to 0$ below.

It is always useful to identify any special cases where fully analytical solutions can be obtained in order to compare the numerical results with those particular cases in order to build confidence in the numerical results or to identify solutions which exhibit particular properties that are easily interpreted. In the present situation, we find two such cases.

(i) $\varsigma_0 = 0$ (no roughness height)

Here the integral (3.16) can be evaluated analytically by using the asymptotic forms of Bessel functions in the integrand (3.14) to get

$$I = \frac{1}{2t}\exp\left\{-\frac{1}{4t}\left(\varsigma^2 + h_0^2\right)\right\} I_\nu(\varsigma h_0 / 2t) \tag{3.17}$$

so that the concentration is given by

$$c(x,\varsigma,t) = \frac{(h_0)^{\nu+1}}{\varsigma^\nu \sqrt{2bt}}\frac{1}{2t}\exp\left\{-\frac{1}{4t}\left[\varsigma^2 + h_0^2 + (x-t)^2/b + y^2/a\right]\right\} I(\varsigma h_0 / 2t) \tag{3.18}$$

in which $I_\nu(x)$ is the modified Bessel function with argument $x$ and order $\nu$. The expression (3.17) illustrates the complicated nature of the integral (3.16) as a function of $\varsigma$ and $t$ and shows the complex nature of the interaction between vertical diffusion and gravity. The asymptotic behaviour of the modified Bessel function $I_\nu(\alpha)$ shows that the solution decays to zero as $t \to \infty$. If we take the limit $t \to 0$, we recover the initial condition (2.11),(ii).

This solution was used to compare the results for small $\varsigma_0$ obtained using the numerical integration of the integral (3.16) to find excellent agreement. It can also be used to test the boundary condition (2.11), (i) demanding the vanishing of the concentration at the top of the roughness layer with the other relevant one of vanishing flux of concentration there. Using the transformation (2.12), we have

$$\frac{\partial c}{\partial z} = 2\varsigma^{-\nu-1}e^{x/2b}\left[\zeta\frac{\partial u}{\partial \zeta} - \nu u\right], \quad \text{at } \zeta = \zeta_0 \tag{3.19}$$

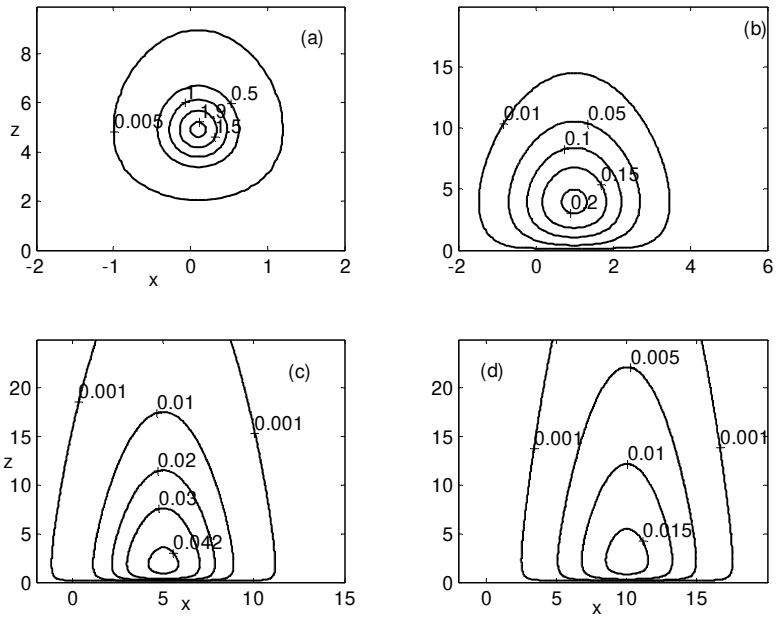


Figure 2. The contours of pollutant concentration when $h = 5.0, z_0 = 0.1, b = 0.5$ and $\nu = 0.5$ at different times: (a) $t = 0.1$, (b) $t = 1.0$, (c) $t = 5.0$, (d) $t = 10.0$. Note the spread of pollutant upwind even for moderate values of $t$.

When $\zeta_0 \to 0$, the condition $\partial c / \partial z = 0$ at $\zeta_0 = 0$ gives the condition

$$u(x, y, 0, t) = 0 \tag{3.20}$$

which is identical with (2.14),(i) when $\zeta_0 = 0$.

(ii) $b = 0$ ( no longitudinal diffusion)

As $b \to 0$, the presence of the factor $(x-t)^2$ in the exponential has the consequence that the solution decays to zero unless $|x-t|$ is very small. We find that

$$c(x, \varsigma, t) = \frac{(h_0)^{\nu+1}}{\varsigma^{\nu}} \delta(x-t) \exp\left\{-\frac{y^2}{4at}\right\} \int_0^{\infty} \frac{H_{\nu}(ph_0) H_{\nu}(p\varsigma)}{J_{\nu}^2(p\varsigma_0) + Y_{\nu}^2(p\varsigma_0)} e^{-tp^2} p dp \tag{3.21}$$

It is noteworthy here that the delta function present initially remains in the solution at subsequent times and maintains the local nature of the distribution. This solution can be obtained by applying the Weber and Laplace transforms directly to (2.13) and (2.14) with $b = 0$.

The solution (3.21) represents a mass of pollutant traveling as a front with uniform (dimensional) speed $U$ and distributed vertically and laterally. The lack of spread of pollutant along the wind direction is due to the absence of diffusion in that direction.

## 4. Discussion

The solution (3.15) gives the evolution of the mass of pollutant released from $(0,0,h)$ at $t=0$ for all subsequent times. The variation of the solution in the vertical direction is more complicated than those in the horizontal because of the interaction with gravity. Furthermore, the horizontal variations in the wind direction are different from those in the lateral direction despite that the diffusion coefficients in both directions are uniform because of the interaction with the wind speed. This illustrates the crucial role played by diffusion and its interaction with the wind. The lack of wind speed in the presence of a uniform diffusion component in the lateral direction results in the concentration decaying exponentially on either side of the central plane $y=0$. If we express the concentration (3.15) in the original quantities $x^*, y^*$ and time $t^*$

$$c^*\left(x^*, y^*, \varsigma^*, t^*\right) = \frac{\left(h_0\right)^{\nu+1} Q}{\lambda \varsigma^{*\nu}\sqrt{2\gamma t^*}} \exp\left\{-\frac{\left(x^* - Ut^*\right)^2}{4\gamma t^*} - \frac{y^{*2}}{\beta t^*}\right\} \tag{3.22}$$

$$\int_0^\infty F(\varsigma^*, \lambda t^*; \varsigma *_0, h_0, \nu)dp$$

We see that the pollutant diffuses a distance $O\left(\beta t^*\right)^{1/2}$ on either side of the plane $y=0$. In the direction of the wind, the pollutant travels in a front of thickness $O\left(\gamma t^*\right)^{1/2}$ traveling with the speed of the wind $U$. In all cases, the volume of pollutant, in the central plane $y=0$, moves in the direction of the wind diffusing both vertically and horizontally about a central point, at which the concentration is the highest. We shall refer to this as the point of accumulation. As time increases from 0, the mass of pollutant begins to diffuse in all directions. The dependence of the concentration on the parameters $h, z_0, \nu$ is illustrated in Figures 2-5 and Table 1.

The evolution of the mass of pollutant with time is illustrated in Figure 2 for a given set of parameters. We note that the pollutant diffuses downwind as well as upwind and the location of the source is subject to pollution, albeit in decreasing concentrations, for quite some time after the release of the mass. In Figure 3, the concentration is illustrated for different sets of the parameters at a given time. We note that the position of the height of the source influences the position of the accumulation point and the effect of increasing the roughness height, $z_0$, is to raise the level of the accumulation point.

The point of accumulation is always situated at the centre of the pollutant mass and at a height $z_m$. It moves horizontally with the flow speed, $U$, at all times so that its horizontal position $x_m = t$. Its downward motion is mainly due to the influence of gravity, as represented by $\nu$ (see also Table 1).

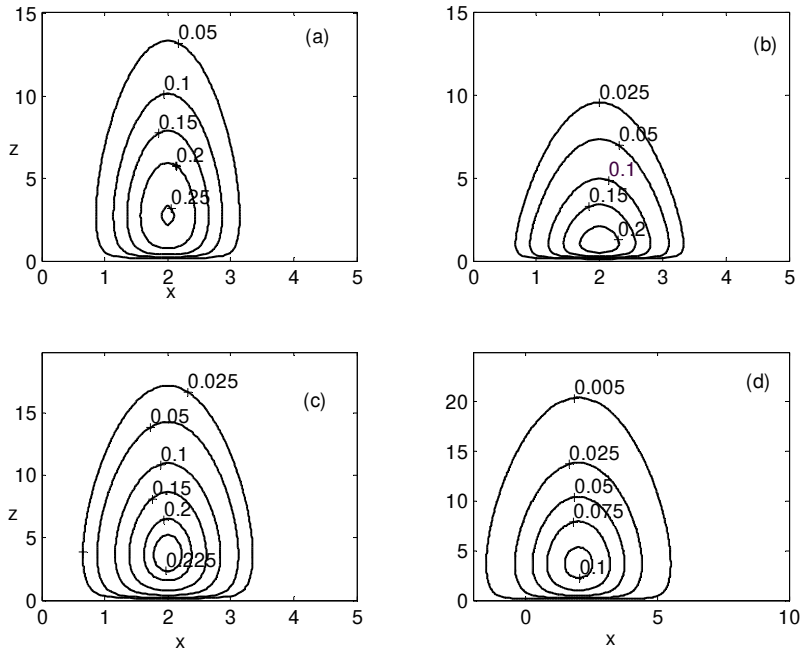


Figure 3. The contours of the concentration of pollutant at a fixed time, $t = 2.0,$ for different sets of values of the pollutant parameters: (a) $h = 5.0,\ z_0 = 0.1,\ \nu = 0.5,\ b = 0.1;$ (b) $h = 2.0,\ z_0 = 0.1,\ \nu = 0.5,\ b = 0.1;$ (c) $h = 5.0, z_0 = 0.1,\ \nu = 0.1, b = 0.1;$ (d) $h = 5.0,\ z_0 = 0.1,\ \nu = 0.1,\ b = 0.5.$ Note the marked horizontal spread of the pollutant volume as $b$ is increased from 0.1 in (c) to 0.5 in (d).

However, the speed with which it moves vertically is also affected by vertical diffusion and by the height of the initial point at $h$, as illustrated in Figure 4. At the early stages of the evolution, the pollutant is still confined to a small volume and the influence of gravity is dominant. Its rate of descent is relatively higher than at later times. When the pollutant has diffused and occupied a larger volume, its vertical speed downwards slows down and eventually decreases to zero and for large times it starts to move upwards. As the volume of pollution diffuses out in all directions, the value of the concentration at the accumulation point naturally decreases. The influence of gravity is now reduced and the accumulation point tends to rise with time. The height $z_m$ of the accumulation point is also dependent on $z_0$ as the accumulation point maintains a certain distance above the top of the roughness layer. As time increases the pollutant spreads over a larger volume which is more extended in the vertical than in the horizontal direction due to the presence of the roughness layer and consequently the accumulation point moves up slowly with time (see Figure 4).

In the absence of longitudinal diffusion (i.e. when $b = 0.0$) the volume of pollutant moves horizontally in the form of a vertical front with speed $U$ and diffuses vertically due to the influence of the vertical diffusion.

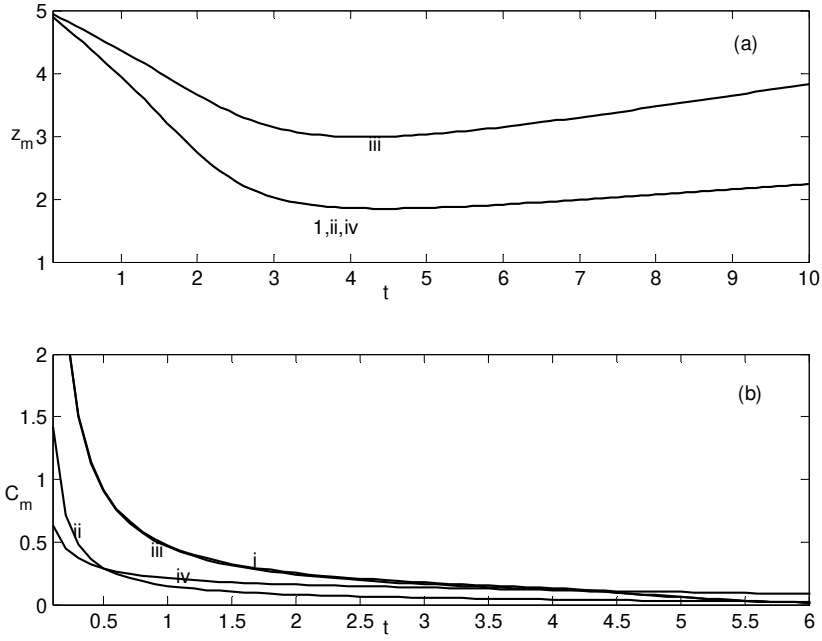


Figure 4. The variation of the properties of the point of accumulation (i.e. the centre of the pollutant mass) with time for different values of the parameters for (a) the height $z_m$ and (b) the value of the concentration $C_m$. The horizontal position of the accumulation point, $x_m = t$ for all times. The curves correspond to $h = 5.0$ and are labeled such that $i$ refers to $b = 0.1,\ z_0 = 0.1,\ \nu = 0.5$, $ii$ for $b = 0.5,\ z_0 = 0.1,\ \nu = 0.5$; $iii$ for $b = 0.5,\ z_0 = 0.5, \nu = 0.5$; and $iv$ for $b = 0.5, z_0 = 0.1, \nu = 0.1$. Note the influence of the roughness height $z_0$ in raising the level of the accumulation point.

In Figure 5, the distribution of the pollutant in the moving front is illustrated in the $(t, z)$ plane. We see that the vertical front also possesses an accumulation point whose height decreases with time.

The expression (3.15) indicates that the concentration $c\left(x, y, z, t\right)$ decays exponentially with time. However, at the accumulation point $x = t$ and $y = 0$, the exponential factor outside the integral vanishes and the nature of decay becomes dependent on the integral (3.16). The rate of decay at the accumulation point is important because it can provide an estimate of the time the pollutant takes to diffuse out into space. In Table 1 we give the time, $t_0$, it takes for the concentration at the accumulation point to reduce to $1.00\text{X}10^{-4}$ (as compared with 1.00 at $t = 0$) as a function of the parameters $h, \nu, z_0, b$. This means that at time $t_0$, the scaled concentration $c\left(x, y, z, t\right)$ is less than that value at every point in space.

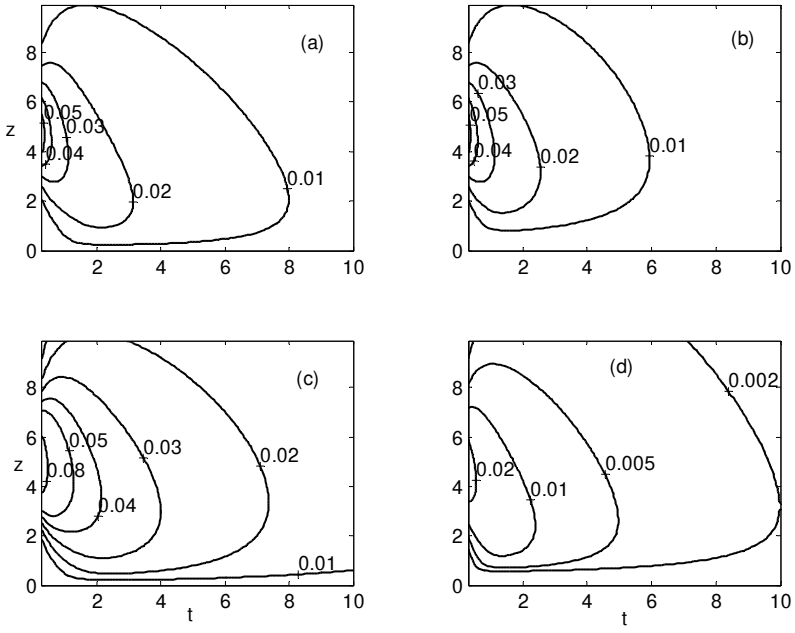


Figure 5. The contours of the concentration in the $t-z$ plane for the case of no horizontal diffusion ($b=0$) when the mass is released from a height $h=5.0$. The subfigures $(a)-(d)$ correspond, respectively, to $z_0=0.1, \nu=0.5$; $z_0=0.5, \nu=0.5$; $z_0=0.1, \nu=0.1$; $z_0=0.5, \nu=0.1$, respectively. The concentration profile for any $x=x_0$, say, is obtained by drawing a vertical line at $t=x_0$ and the intersection points of the line with the contours give the profile of the concentration with the vertical at the advancing front $x=t$ at that instant. Note that the point of accumulation moves down with a rate depending on the settling velocity in the same fashion as in the case $b\neq 0$.

The table shows that $t_0$ is affected by variations in any one of the parameters $h,\nu,z_0,b$. We note that the height of the accumulation point increases with time for moderate and large times for all the parameters $h,\nu,z_0,b$.

## 5. Concluding remarks

A prescribed mass of pollutant particles is released from a point at time $t=0$ and its evolution has been investigated for all times. The gas 'blob' is subject to diffusion in both vertical and horizontal directions and the influence of gravity is also included. The uniform coefficients of diffusion in the wind and cross-wind directions are $\gamma$ and $\beta$, respectively. The solution for the concentration as a function of both position and time has been obtained in closed form. The solution describes the transient movement of the mass of pollutant with time. The solution reduces to simpler forms in special cases where the solution can be obtained independently. The model used is a simple one with uniform wind speed and diffusion that is linear in height vertically and uniform horizontally. The solution obtained exhibits certain characteristics which we believe may apply qualitatively to more realistic models.

Table 1. The time taken for the maximum scaled concentration ($c^*U/Q$) to reduce to $1.00\text{X}10^{-4}$ for different values of the parameters. Note that the position of the maximum concentration occurs on the line $y = 0$, $x = t$.

| h | $\nu$ | $z_0$ | b | $z_m$ | $t_0$ |
|---|---|---|---|---|---|
| 5.0 | 0.1 | 0.1 | 0.2 | 48.75 | 372.00 |
| | 0.1 | 0.1 | 0.5 | 38.92 | 281.00 |
| | 0.1 | 0.5 | 0.2 | 57.04 | 293.00 |
| | 0.1 | 0.5 | 0.5 | 46.31 | 222.00 |
| | 0.5 | 0.1 | 0.2 | 12.02 | 215.00 |
| | 0.5 | 0.1 | 0.5 | 10.41 | 170.00 |
| | 0.5 | 0.5 | 0.2 | 20.29 | 182.00 |
| | 0.5 | 0.5 | 0.5 | 17.62 | 143.00 |
| 2.0 | 0.1 | 0.1 | 0.2 | 33.26 | 232.00 |
| | 0.1 | 0.1 | 0.5 | 26.75 | 176.00 |
| | 0.1 | 0.5 | 0.2 | 35.96 | 159.00 |
| | 0.1 | 0.5 | 0.5 | 29.26 | 120.00 |
| | 0.5 | 0.1 | 0.2 | 8.69 | 128.00 |
| | 0.5 | 0.1 | 0.5 | 7.52 | 101.00 |
| | 0.5 | 0.5 | 0.2 | 13.92 | 96.00 |
| | 0.5 | 0.5 | 0.5 | 12.54 | 75.54 |

The solution shows that:

(1) The diffusion coefficient, $\gamma$, in the direction of the wind has a strong influence on the spread of the pollution along the wind direction. If $\gamma = 0$, the mass of pollutant moves as a vertical front of negligible $x$-thickness with the speed of the wind and the pollutant can only diffuse vertically and in the cross-wind direction. For non-zero $\gamma$, the front has an $x$-thickness of $O\left(\gamma t^*\right)^{1/2}$ about the location $x^* = Ut^*$.

(2) The extent of spread of the pollutant in the cross-wind direction extends a distance $O\left(\beta t^*\right)^{1/2}$ about the plane $y = 0$.

(3) The concentration of pollutant is always maximum at the centre (where $y = 0$, $x = t$) but the actual value of this maximum decreases as time increases. Also the centre moves horizontally with the flow but vertically it descends, for small times, in a manner determined by both gravity and diffusion and rises for moderate to large times.

(4) The time the pollutant takes to diffuse completely into the atmosphere is reduced, to varying degrees, by an increase in any of the parameters $\gamma, z_0, b, h$.

The solution of the atmospheric diffusion equation obtained here examined the evolution of a given mass of pollutant released from a fixed location with time. It would be interesting to compare the results with field and/or experimental data but the available data seem to be averaged over time and no sets of data which measure the concentration of such a model at certain points at different times are available to us. Perhaps this study would motivate some field and /or experimental studies on such transient behaviour, as this may be important to some environment and safety applications.

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