

Realization in Generalized State Space form for 2-D Polynomial System Matrices

*M.S. Boudellioua and **B. Chentouf

*Department of Mathematics and Statistics, College of Science, Sultan Qaboos University, P.O. Box 36, Al-Khod 123, Sultanate of Oman, Email: *boudell@squ.edu.om, **chentouf@squ.edu.om.*

(zero coprime system equivalence)

ABSTRACT: In this paper, a direct realization procedure is presented that brings a general 2-D polynomial system matrix to generalized state space (GSS) form, such that all the relevant properties including the zero structure of the system matrix are retained. It is shown that the transformation linking the original 2-D polynomial system matrix with its associated GSS form is zero coprime system equivalence. The exact nature of the resulting system matrix in GSS form and the transformation involved are established.

KEYWORDS: 2-D Systems, system matrix, generalized state space form, zero coprime system equivalence, invariant polynomials, invariant zeros, grobner bases.

1. Introduction

State space models play an important role in the theory of 1-D finite-dimensional linear systems. In recent years attempts have been made toward extending the state space representation to more general systems, e.g. time-delay systems or systems described by partial differential equations. Another extension from 1-D to 2-D is the discrete linear state space model which has a number of variants as given by Givone and Roesser (1972), Attasi (1973) or Fornasini and Marchesini (1976). One of the limitations of these models is that they can only be used to describe 2-D proper transfer functions. In other words, they are suitable only for the representation of northeast quarter plane 2-D systems. Several authors have suggested a generalized state space description for 2-D systems. Zak

(1984) suggested a generalized model based on Roesser's model while Kaczorek (1988) proposed a model based on that of Fornasini-Marchesini. The natural description of a system is not necessarily in a state space form, and it is often desirable to reduce such a description into a simpler but equivalent form. The reduction of an arbitrary 2-D polynomial system matrix to 2-D generalized state space (GSS) form was first studied by Pugh *et al.* (1998). Their algorithm involves the application of a two stage reduction procedure which includes the removal of factors from certain matrices to ensure that the transformations linking the original system matrix with the final GSS form are polynomial. The method does not give a priori the form of neither the resulting 2-D GSS system matrix nor the transformation linking it to the original 2-D polynomial system matrix. In the present work, we present a direct and simple procedure for the realization of a 2-D polynomial system matrix by an equivalent 2-D polynomial system matrix in GSS form. The exact nature of the resulting GSS system matrix and the transformation linking it with the original system matrix will be given. The transformation linking the original system matrix to its corresponding GSS form is shown to be zero coprime system equivalence. This type of equivalence has been studied by Levy (1981), Johnson (1993) and Pugh *et al.* (1996) and has been shown by Pugh *et al.* (1998) to provide the connection between all least order polynomial realizations of a given 2-D transfer function matrix.

2. 2-D System Matrices and System Equivalence

Consider the 2-D system matrix in the general form:

$$P(s, z) = \begin{bmatrix} T(s, z) & U(s, z) \\ -V(s, z) & W(s, z) \end{bmatrix} \quad (2.1)$$

where $T(s, z)$, $U(s, z)$, $V(s, z)$ and $W(s, z)$ are respectively $r \times r$, $r \times n$, $m \times r$ and $m \times n$ polynomial matrices with $T(s, z)$ invertible, in which case the system matrix in (2.1) is said to be regular. The transfer function matrix of the system matrix in (2.1) is given by

$$G(s, z) = V(s, z)T^{-1}(s, z)U(s, z) + W(s, z) \quad (2.2)$$

A special case of (2.1) is obtained from the system described by the following 2-D generalized state space discrete equations (Kaczorek 1988),

$$\begin{aligned} Ex(i+1, j+1) &= A_1x(i+1, j) + A_2x(i, j+1) \\ &+ A_0x(i, j) + B_1u(i+1, j) \\ &+ B_2u(i, j+1) + B_0u(i, j), \end{aligned} \quad (2.3a)$$

$$y(i, j) = Cx(i, j) + Du(i, j) \quad (2.3b)$$

where $x(i, j)$ is the state vector, $u(i, j)$ is the input vector, $y(i, j)$ is the output vector, $E, A_0, A_1, A_2, B_0, B_1, B_2, C$ and D are constant real matrices of appropriate dimensions and E may be singular. Then, taking the 2-D z -transform of (2.3a) and (2.3b) and assuming zero boundary conditions yields

$$\left[\begin{array}{c|c} szE - sA_1 - zA_2 - A_0 & sB_1 + zB_2 + B_0 \\ \hline -C & D \end{array} \right] \begin{bmatrix} \bar{x}(s, z) \\ -\bar{u}(s, z) \end{bmatrix} = \begin{bmatrix} 0 \\ -\bar{y}(s, z) \end{bmatrix} \quad (2.4)$$

The polynomial matrix over $\mathbb{R}[s, z]$,

REALIZATION IN GENERALIZED STATE SPACE FORM

$$P(s, z) = \left[\begin{array}{c|c} \frac{szE - sA_1 - zA_2 - A_0}{-C} & \frac{sB_1 + zB_2 + B_0}{D} \\ \hline & \end{array} \right] \quad (2.5)$$

in (2.4), is a system matrix in GSS form and the corresponding transfer function is given by

$$G(s, z) = C(szE - sA_1 - zA_2 - A_0)^{-1}(sB_1 + zB_2 + B_0) + D \quad (2.6)$$

Definition 2.1 Two polynomial matrices $P_1(s, z)$ and $S_1(s, z)$ of appropriate dimensions, are said to be zero left coprime if the matrix

$$[P_1(s, z) \quad S_1(s, z)] \quad (2.7)$$

has full rank for all $(s, z) \in \mathbb{C}^2$.

Similarly, $P_2(s, z)$ and $S_2(s, z)$, of appropriate dimensions, are said to be zero right coprime if the matrix

$$\begin{bmatrix} P_2^T(s, z) & S_2^T(s, z) \end{bmatrix}^T \quad (2.8)$$

has full rank for all $(s, z) \in \mathbb{C}^2$.

Following the results of Youla and Gnani (1979), it follows that the polynomial matrices $P_1(s, z)$ and $S_1(s, z)$ are *zero left coprime* if and only if there exist *zero right coprime* polynomial matrices $X(s, z)$ and $Y(s, z)$ of appropriate dimensions satisfying the Bezout's relation

$$P_1(s, z)X(s, z) + S_1(s, z)Y(s, z) = I \quad (2.9)$$

One immediate result given by Sontag (1980) is that a necessary and sufficient condition for the matrices $P_1(s, z)$ and $S_1(s, z)$ to be *zero left coprime* is that the matrix in (2.7) is unimodular equivalent to the matrix $[I \ 0]$. Similar results can be stated for *zero right coprime* matrices.

Definition 2.2 Given a $p \times q$ polynomial matrix $P(s, z)$, the i th order invariant polynomial $\phi_i(s, z)$ of $P(s, z)$ is defined by :

$$\phi_i(s, z) = \begin{cases} d_i(s, z) & \text{if } 1 \leq i \leq t \\ d_{i-1}(s, z) & \text{if } 1 \leq i \leq \min(p, q) \\ 0 & \end{cases} \quad (2.10)$$

where t is the normal rank of $P(s, z)$, $d_0(s, z) = 1$, $d_1(s, z) = 1$ and $d_i(s, z)$ is the greatest common divisor of all the i th order minors of the given matrix $P(s, z)$.

As in 1-D case, the zero structure of 2-D systems is a crucial indicator of the system behavior. Zerz (1996) has shown that the controllability and observability of a system is connected to the zero structure of the associated polynomial matrix. However, unlike the 1-D case, the zero structure of a multivariate polynomial matrix is not completely captured by the invariant polynomials. Therefore the following concept of invariant zeros as given by Pugh and El-Nabrawy (2003) is introduced.

Definition 2.3 Given a $p \times q$ polynomial matrix $P(s, z)$, the i th order invariant zeros of $P(s, z)$ are the elements of the variety $V_{\mathbb{R}}(I_i^{[P]})$ defined by the ideal $I_i^{[P]}$ generated by the i th order minors of

$P(s,z)$.

Definition 2.4 Let $\mathbb{P}(m,n)$ denote the class of $(r+m) \times (r+n)$ polynomial matrices where m, n are fixed positive integers and $r > \min(m, n)$. Two polynomial system matrices $P_1(s,z)$ and $P_2(s,z)$ are said to be zero coprime equivalent if there exist polynomial matrices $S_1(s,z), S_2(s,z)$ of appropriate dimensions such that

$$S_2(s,z)P_1(s,z) = P_2(s,z)S_1(s,z) \quad (2.11)$$

where $P_1(s,z), S_1(s,z)$ are zero left coprime and $P_2(s,z), S_2(s,z)$ are zero right coprime. Pugh *et al.* (1996) and Pugh and El-Nabrawy (2003) have shown that zero coprime equivalence exhibits fundamental algebraic properties amongst its invariants:

Lemma 2.1 (Pugh *et al.* 1996) Suppose that two polynomial matrices $P(s,z)$ and $Q(s,z) \in \mathbb{P}(m, n)$, are related by zero coprime equivalence and let $\phi_1^{[P]}, \phi_2^{[P]}, \dots, \phi_h^{[P]}$, where $h = \min(r^{[P]} + m, r^{[P]} + n)$, denote the invariant polynomials of $P(s,z)$ and $\phi_1^{[Q]}, \phi_2^{[Q]}, \dots, \phi_k^{[Q]}$, where $k = \min(r^{[Q]} + m, r^{[Q]} + n)$, denote the invariant polynomials of $Q(s,z)$, then

$$\phi_{h-i}^{[P]} = c_i \phi_{k-i}^{[Q]} \quad \text{for } i = 0, 1, \dots, \max(k-1, h-1) \quad (2.12)$$

where $\phi_j^{[P]} = 1, \phi_j^{[Q]} = 1$ for any $j < 1, c_i \in R \setminus \{0\}$

Lemma 2.2 (Pugh and El-Nabrawy 2003) Suppose that two polynomial matrices $P(s,z)$ and $Q(s,z) \in \mathbb{P}(m,n)$, are related by zero coprime equivalence and let $I_j^{[P]}$ for $j = 1, \dots, h = \min(r^{[P]} + m, r^{[P]} + n)$ denote the ideal generated by the $j \times j$ minors of $P(s,z)$ and $I_i^{[Q]}$, for $i = 1, \dots, k = \min(r^{[Q]} + m, r^{[Q]} + n)$ denote the ideal generated by the $i \times i$ minors of $Q(s,z)$. Then

$$I_{h-i}^{[P]} = I_{k-i}^{[Q]}, i = 0, \dots, \bar{h} \quad (2.13)$$

where $\bar{h} = \min(h-1, k-1)$ and for any $i > h, I_{h-i}^{[P]} = \langle 1 \rangle$ in case $i < h$ or $i < k$.

A basic transformation proposed for the study of 2-D systems is *zero coprime system equivalence* given by Levy (1981) and Johnson (1993). This transformation, based on *zero coprime equivalence* is characterized by the following definition.

Definition 2.5 Two polynomial system matrices $P_1(s,z)$ and $P_2(s,z) \in \mathbb{P}(m,n)$, are said to be zero coprime system equivalent if they are related by the following

$$\underbrace{\begin{bmatrix} M(s,z) & 0 \\ X(s,z) & I_m \end{bmatrix}}_{S_1(s,z)} \underbrace{\begin{bmatrix} T_1(s,z) & U_1(s,z) \\ -V_1(s,z) & W_1(s,z) \end{bmatrix}}_{P_2(s,z)}$$

$$= \underbrace{\begin{bmatrix} T_2(s, z) & U_2(s, z) \\ -V_2(s, z) & W_2(s, z) \end{bmatrix}}_{P_1(s, z)} \underbrace{\begin{bmatrix} N(s, z) & Y(s, z) \\ 0 & I_n \end{bmatrix}}_{S_2(s, z)} \quad (2.14)$$

where $P_1(s, z)$, $S_1(s, z)$ are zero left coprime and $P_2(s, z)$, $S_2(s, z)$ are zero right coprime and $M(s, z)$, $N(s, z)$, $X(s, z)$ and $Y(s, z)$ are polynomial matrices of appropriate dimensions.

The transformation of zero coprime system equivalence is an extension of Fuhrmann's strict system equivalence from the 1-D to the 2-D setting and has been shown by Levy (1981), Johnson (1993) and Pugh *et al.* (1996, 1998) to preserve important properties of the system matrix $P(s, z)$ and plays a key role in certain aspects of 2-D systems theory.

Lemma 2.3 (Johnson 1993) the transformation of zero coprime system equivalence preserves the transfer function and in the sense described in Lemma 2.1, the invariant polynomials of the matrices:

- $T_i(s, z)$, $i = 1, 2$.
- $P_i(s, z)$, $i = 1, 2$.
- $[T_i(s, z) \ U_i(s, z)]$, $i = 1, 2$.
- $\left[T_i^T(s, z) \ -V_i^T(s, z) \right]^T$, $i = 1, 2$.

The following *lemma* is a direct consequence of Lemmas 2.2 and 2.3.

Lemma 2.4 The transformation of zero coprime system equivalence preserves, in the sense described in Lemma 2.2, the invariant zeros of the matrices given in Lemma 2.3.

3. Realization in 2-D Generalized State Space Form

Let $P(s, z) \in \mathbb{R}^{(r+m) \times (r+n)}[s, z]$ be a 2-D polynomial system matrix given by (2.1). First write $P(s, z)$ as

$$P(s, z) = \sum_{i=0}^p \sum_{j=0}^q P_{i,j} s^i z^j = P_{0,0} s^0 z^0 + P_{0,1} s^0 z^1 + P_{0,2} s^0 z^2 + \dots + P_{p,q} s^p z^q \quad (3.1)$$

where $P_{i,j}$, $i = 0, 1, \dots, p$ and $j = 0, 1, \dots, q$ are $(r+m) \times (r+n)$ real constant matrices. Now construct the block matrices

$$E = \begin{bmatrix} 0_{(r+n)(pq-1), (r+n)pq} \\ E_q \quad E_{q-1} \quad \dots \quad E_1 \end{bmatrix} \quad (3.2a)$$

where

$$E_j = \begin{bmatrix} P_{p,j} & P_{p-1,j} & \dots & P_{1,j} \end{bmatrix}, j = 1, 2, \dots, q. \quad (3.2b)$$

$$A_0 = \text{Diag} \left(-I_{(r+n)(pq-1)}, -P_{0,0} \right), \quad (3.3)$$

$$A_1 = \begin{bmatrix} 0_{(r+n)p(q-1),(r+n)pq} \\ 0_{(r+n)(p-1),(r+n)(pq-p+1)} I_{(r+n)(p-1)} \\ 0_{(r+m),(r+n)p(q-1)} -P_{p,0} -P_{p-1,0} \cdots -P_{1,0} \end{bmatrix}, \quad (3.4)$$

and

$$A_2 \begin{bmatrix} 0_{(r+n)p(q-1),(r+n)p} & I_{(r+n)p(q-1)} \\ 0_{(r+n)(p-1),(r+n)pq} \\ A_{2,q} & A_{2,q-1} & \cdots & A_{2,1} \end{bmatrix} \quad (3.5a)$$

where

$$A_{2,j} = \begin{bmatrix} 0_{(r+m),(r+n)(p-1)} & -P_{0,j} \end{bmatrix}, \quad j = 1, 2, \dots, q. \quad (3.5b)$$

Theorem 3.1 Let the matrices E , A_0 , A_1 and A_2 be as constructed in (3.2a), (3.3), (3.4) and (3.5a), respectively. Then the $[(r+n)pq+2m] \times [(r+n)pq+m+n]$ polynomial system matrix is in GSS form (2.5):]

$$Q(s, z) \left[\begin{array}{cc|c} szE - sA_1 - zA_2 - A_0 & -Z_m & 0 \\ -Z_n & 0 & I_n \\ \hline 0 & I_m & 0 \end{array} \right] \quad (3.6)$$

where $Z_n = \begin{bmatrix} 0_{n,[(r+n)pq+r+m-n]} & I_n \end{bmatrix}$ and $Z_m^T = \begin{bmatrix} 0_{m,[(r+n)pq-m]} & I_m \end{bmatrix}$ is related to the original system matrix $P(s, z)$ by the following :

$$S_1(s, z) P(s, z) = Q(s, z) S_2(s, z) \quad (3.7)$$

where

$$S_1(s, z) = \left[\begin{array}{cc|c} 0_{(r+n)(pq-1),r} & 0_{(r+n)(pq-1),m} \\ I_r & 0_{r,m} \\ 0_{(m+n),r} & 0_{(m+n),m} \\ \hline 0_{m,r} & I_m \end{array} \right], \quad S_2(s, z) = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_q \\ \hline -V & W \\ \hline 0_{n,r} & I_n \end{bmatrix} \quad (3.8)$$

$$Y_j = \left[s^{p-1} z^{q-j} \quad s^{p-2} z^{q-j} \quad \cdots \quad z^{q-j} \right]^T \otimes I_{r+n}, \quad (3.9)$$

where $j = 1, 2, \dots, q$ and \otimes denotes the matrix Kronecker product.

Proof. The matrix $Q(s, z)$ in (3.6) can be represented in the form

REALIZATION IN GENERALIZED STATE SPACE FORM

$$Q(s, z) = \left[\begin{array}{cccccc|c} I_{p(r+n)} & -zI_{p(r+n)} & \cdots & 0 & 0 & 0 & 0 \\ 0 & I_{p(r+n)} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{p(r+n)} & -zI_{p(r+n)} & 0 & 0 \\ Q_q & Q_{q-1} & \cdots & Q_2 & Q_1 & -Z_m & 0 \\ 0 & 0 & \cdots & 0 & -Z_n & 0 & I_n \\ \hline 0 & 0 & \cdots & 0 & 0 & I_m & 0 \end{array} \right] \quad (3.10)$$

where

$$Q_1 = \left[\begin{array}{cccc} I_{r+n} & -sI_{r+n} & \cdots & 0 \\ 0 & I_{r+n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -sI_{r+n} \\ szP_{p,1} + sP_{p,0} & szP_{p-1,1} + sP_{p-1,0} & \cdots & szP_{1,1} + sP_{1,0} + zP_{0,1} + P_{0,0} \end{array} \right], \quad (3.11)$$

$$Q_j = \left[\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ szP_{p,j} & szP_{p-1,j} & \cdots & szP_{2,j} & szP_{1,j} + zP_{0,j} \end{array} \right], j = 2, 3, \dots, q \quad (3.12)$$

and

$$\bar{Z}_n = \begin{bmatrix} 0_{n,p(r+n)-n} & I_n \end{bmatrix}, \bar{Z}_m^T = \begin{bmatrix} 0_{m,(p-1)(r+n)+r} & I_m \end{bmatrix} \quad (3.13)$$

and the matrix $S_2(s, z)$ can be written as

$$S_2(s, z) = \begin{bmatrix} s^{p-1} z^{q-1} I_{r+n} \\ s^{p-2} z^{q-1} I_{r+n} \\ \vdots \\ z^{q-1} I_{r+n} \\ s^{p-1} z^{q-2} I_{r+n} \\ s^{p-2} z^{q-2} I_{r+n} \\ \vdots \\ z^{q-2} I_{r+n} \\ \vdots \\ \vdots \\ I_{r+n} \\ \hline -V & W \\ \hline 0 & I_n \end{bmatrix} \quad (3.14)$$

From which it can be easily verified that

$$S_1(s, z)P(s, z) = Q(s, z)S_2(s, z) \equiv \begin{bmatrix} 0_{(r+n)(pq-1),r} & 0_{(r+n)(pq-1),n} \\ T & U \\ 0_{m+n,r} & 0_{m+n,n} \\ -V & W \end{bmatrix} \quad (3.15)$$

Lemma 3.1 The matrices in (3.7), $Q(s, z)$ and $S_1(s, z)$ are zero left coprime and $P(s, z)$ and $S_2(s, z)$ are zero right coprime.

Proof. The matrix $[Q(s, z) \ S_1(s, z)]$ is given by

$$\begin{bmatrix} I_{p(r+n)} & -zI_{p(r+n)} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{p(r+n)} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{p(r+n)} & -zI_{p(r+n)} & 0 & 0 & I_r & 0 & 0 \\ Q_q & Q_{q-1} & \cdots & Q_2 & Q_1 & -\bar{Z}_m & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & -Z_n & 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & I_m & 0 & 0 & 0 & I_m \end{bmatrix} \quad (3.16)$$

It can be easily seen that the minor obtained by deleting the columns $(r+n)(pq-1)+1, \dots, (r+n)pq$ from the matrix in (3.16) is equal to ± 1 .

Similarly, the matrix $\begin{bmatrix} P(s, z) \\ S_2(s, z) \end{bmatrix}$ is given by

$$\begin{bmatrix} P(s, z) \\ Y_1 \\ Y_2 \\ \vdots \\ s^{p-1}I_{r+n} \\ s^{p-2}I_{r+n} \\ \vdots \\ I_{r+n} \\ -V \quad W \\ 0_{n,r} \quad I_n \end{bmatrix} \quad (3.17)$$

It is clear that the matrix in (3.17) contains a block identity matrix I_{r+n} , and therefore it has the highest order minor which is equal to 1.

Theorem 3.2 If $P(s, z)$ is an arbitrary $(r+m) \times (r+n)$ polynomial system matrix over $\mathbb{R}[s, z]$ given by (2.1) and $Q(s, z)$ is the corresponding $[(r+n)pq+2m] \times [(r+n)pq+m+n]$ 2-D system matrix in GSS form (3.6), then $P(s, z)$ and $Q(s, z)$ are zero coprime system equivalent.

Proof. The result follows immediately from Theorem 3.1 and Lemma 3.1.

4. Example

Consider the system matrix $P(s, z) \in \mathbb{R}^{2 \times 2}[s, z]$ given by

$$P(s, z) \equiv \left[\begin{array}{c|c} t(s, z) & u(s, z) \\ \hline -v(s, z) & w(s, z) \end{array} \right] \quad (4.1)$$

where

$$t(s, z) = (z^2 + 1)s^2 - (2z^2 - z - 3)s + z^2 - 4z + 1, \quad (4.2a)$$

$$u(s, z) = (z^2 - z)s^2 - (z^2 - 2)s + z^2 - z, \quad (4.2b)$$

$$v(s, z) = -(z + 2)s^2 + (z^2 - z)s + 4z + 1, \quad (4.2c)$$

$$w(s, z) = (2z^2 - z)s^2 + 5zs + z^2 - z + 3 \quad (4.2d)$$

Here $r = m = n = 1$ and $p = q = 2$.

Using Maple, the transfer function of the system matrix $P(s, z)$ is given by :

$$\begin{aligned}
G^{[P]}(s, z) = & \left[(z^2 + 1)s^2 - (2z^2 - z - 3)s + z^2 - 4z + 1 \right]^{-1} \\
& \times \left[-(z^3 + z^2 - 2z)s^4 + (z^4 - z^3 + 3z^2 - 2z - 4)s^3 \right. \\
& \left. - (z^4 - 4z^3 + 2z)s^2 + (z^4 - 6z^3 + 13z + 2)s + 4z^3 - 2z^2 - 2z + 3 \right],
\end{aligned} \tag{4.3}$$

the invariant polynomials of $P(s, z)$ are computed as :

$$\begin{aligned}
\phi_1^{[P]} &= 1 \\
\phi_2^{[P]} &= (2z^4 - 2z^3 + z^2 + z)s^4 + (-3z^4 + 8z^3 + 8z^2 - 4)s^3 \\
&+ (2z^4 - 16z^3 + 13z^2 + 12z + 3)s^2 \\
&+ (-z^4 + 2z^3 - 24z^2 + 13z + 11)s \\
&+ z^4 - z^3 + 5z^2 - 14z + 3,
\end{aligned} \tag{4.4}$$

The reduced Grobner bases of the ideals generated by the minors of the matrices in Lemma 2.2 associated with $P(s, z)$ are given by :

Matrix	Order of i minors	Ideal generated by the $i \times i$ minors
$\begin{bmatrix} t(s, z) & u(s, z) \end{bmatrix}$	1	$\langle z^8 - 2z^7 + 11z^6 - 32z^5 + 39z^4 - 2z^3 - 19z^2 - 4z + 4, 16s + 3z^7 - 6z^6 + 35z^5 - 98z^4 + 137z^3 - 50z^2 - 27z + 10 \rangle$
$\begin{bmatrix} t(s, z) \\ -v(s, z) \end{bmatrix}$	1	$\langle z^8 + 17z^6 - 35z^5 + 22z^4 + 103z^3 - 29z^2 - 114z - 9, 179880s + 1879z^7 - 1563z^6 + 37934z^5 - 93203z^4 + 202249z^3 - 68516z^2 - 12719z + 102237 \rangle$
$P(s, z)$	1	$\langle 1 \rangle$

REALIZATION IN GENERALIZED STATE SPACE FORM

$P(s, z)$	2	$\begin{aligned} & \langle (2z^4 - 2z^3 + z^2 + z)s^4 \\ & + (-3z^4 + 8z^3 + 8z^2 - 4)s^3 \\ & + (2z^4 - 16z^3 + 13z^2 + 12z + 3)s^2 \\ & + (-z^4 + 2z^3 - 24z^2 + 13z + 11)s \\ & + z^4 - z^3 + 5z^2 - 14z + 3 \rangle \end{aligned}$	(4.5)
-----------	---	--	-------

Writing $P(s, z)$ in the form (3.1),

$$\begin{aligned}
 P(s, z) = & \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}}_{P_{0,0}} s^0 z^0 + \underbrace{\begin{bmatrix} -4 & 1 \\ -4 & -1 \end{bmatrix}}_{P_{0,1}} s^0 z^1 + \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{P_{0,2}} s^0 z^2 \\
 & + \underbrace{\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}}_{P_{1,0}} s^1 z^0 + \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix}}_{P_{1,1}} s^1 z^1 + \underbrace{\begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}}_{P_{1,2}} s^1 z^2 \\
 & + \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}}_{P_{2,0}} s^2 z^0 + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}}_{P_{2,1}} s^2 z^1 + \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{P_{2,2}} s^2 z^2
 \end{aligned} \tag{4.6}$$

and constructing the 10×10 polynomial system matrix in GSS form $Q(s, z)$ corresponding to (3.6) gives

$$Q(s, z) \equiv \begin{bmatrix} T(s, z) & U(s, z) \\ -V(s, z) & W(s, z) \end{bmatrix} \equiv \left[\begin{array}{ccc|c} I_4 & -zI_4 & 0 & 0 \\ Q_2 & Q_1 & -Z_1 & 0 \\ 0 & -Z_1^T & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \end{array} \right] \tag{4.7}$$

where

$$\begin{aligned}
 Q_1 & \equiv \begin{bmatrix} I_2 & -sI_2 \\ szP_{2,1} + sP_{2,0} & szP_{1,1} + sP_{1,0} + zP_{0,1} + P_{0,0} \end{bmatrix} \\
 & = \left[\begin{array}{cc|cc} 1 & 0 & -s & 0 \\ 0 & 1 & 0 & -s \\ \hline s & -sz & sz + 3s - 4z + 1 & 2s - z \\ sz + 2s & -sz & sz - 4z - 1 & 5sz - z + 3 \end{array} \right]
 \end{aligned} \tag{4.8}$$

$$Q_2 \equiv \begin{bmatrix} 0_{2,2} & 0_{2,2} \\ szP_{2,2} & szP_{1,2} + zP_{0,2} \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline sz & sz & -2sz & -sz + z \\ 0 & 2sz & -sz & z \end{array} \right] \quad (4.9)$$

and $Z_1 = \bar{Z}_1$ is the fourth column of I_4 .

The matrices E , A_0 , A_1 and A_2 corresponding to (3.2a, 3.3, 3.4 and 3.5a) are given by

$$E \equiv \begin{bmatrix} 0_{6,4} & 0_{6,4} \\ E_2 & E_1 \end{bmatrix} \equiv \left[\begin{array}{cc|cc} 0_{6,2} & 0_{6,2} & 0_{6,2} & 0_{6,2} \\ P_{2,2} & P_{1,2} & P_{2,1} & P_{1,1} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 1 & -2 & -1 & 0 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 1 & -1 & 1 & 5 \end{array} \right], \quad (4.10)$$

$$A_0 \equiv \begin{bmatrix} -I_6 & 0 \\ 0 & -P_{0,0} \end{bmatrix} = \left[\begin{array}{cccccc|cc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 \end{array} \right], \quad (4.11)$$

$$A_1 \equiv \begin{bmatrix} 0_{4,4} & 0_{4,2} & 0_{4,2} \\ 0_{2,4} & 0_{2,2} & I_2 \\ 0_{2,4} & -P_{2,0} & -P_{1,0} \end{bmatrix} = \left[\begin{array}{cccc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \end{array} \right], \quad (4.12)$$

REALIZATION IN GENERALIZED STATE SPACE FORM

$$\begin{aligned}
 A_2 &\equiv \begin{bmatrix} 0_{4,4} & I_4 \\ 0_{2,4} & 0_{2,4} \\ A_{2,2} & A_{2,1} \end{bmatrix} \equiv \left[\begin{array}{cc|cc} 0_{2,2} & 0_{2,2} & I_2 & 0_{2,2} \\ 0_{2,2} & 0_{2,2} & 0_{2,2} & I_2 \\ \hline 0_{2,2} & 0_{2,2} & 0_{2,2} & 0_{2,2} \\ 0_{2,2} & -P_{0,2} & 0_{2,2} & -P_{0,1} \end{array} \right] \\
 &= \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & -1 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & 1 \end{array} \right] \tag{4.13}
 \end{aligned}$$

By virtue of Theorem 3.1 and Theorem 3.2, the polynomial matrix $P(s,z)$ in (4.1) and the corresponding system matrix in GSS form $Q(s,z)$ in (4.7) are related by the zero coprime system equivalence transformation $S_1(s,z)P(s,z) = Q(s,z)S_2(s,z)$, where

$$S_1(s, z) = \left[\begin{array}{c|c} 0_{6,1} & 0_{6,1} \\ 1 & 0 \\ \hline 0_{2,1} & 0_{2,1} \\ 0 & 1 \end{array} \right], \tag{4.14}$$

and

$$S_2(s, z) = \left[\begin{array}{c|c} sz & 0 \\ 0 & sz \\ z & 0 \\ 0 & z \\ s & 0 \\ 0 & s \\ 1 & 0 \\ 0 & 1 \\ \hline (z+2)s^2 - (z^2+z)s - 4z - 1 & (2z^2 - z)s^2 + 5sz + z^2 - z + 3 \\ \hline 0 & 1 \end{array} \right], \tag{4.15}$$

In fact it can be easily verified that

$$S_1(s, z)P(s, z) = Q(s, z)S_2(s, z) \equiv \begin{bmatrix} 0_{6,1} & 0_{6,1} \\ t(s, z) & u(s, z) \\ 0_{2,1} & 0_{2,1} \\ -v(s, z) & w(s, z) \end{bmatrix} \quad (4.16)$$

where $t(s, z)$, $u(s, z)$, $v(s, z)$ and $w(s, z)$ are given by (4.2a), (4.2b), (4.2c) and (4.2d), respectively.

The matrices $Q(s, z)$, $S_1(s, z)$ are zero left coprime and the matrices $P(s, z)$, $S_2(s, z)$ are zero right coprime since the matrices

$$[Q(s, z) \ S_1(s, z)]$$

and

$$\begin{bmatrix} P(s, z) \\ S_2(s, z) \end{bmatrix}$$

have respectively a 10×10 and a 2×2 minor which is equal to 1. The transfer function of the system matrix $Q(s, z)$ is given by :

$$\begin{aligned} G^{[Q]}(s, z) &= \left[(z^2 + 1)s^2 - (2z^2 - z - 3)s + z^2 - 4z + 1 \right]^{-1} \\ &\quad \times \left[-(z^3 + z^2 - 2z)s^4 + (z^4 - z^3 + 3z^2 - 2z - 4)s^3 \right. \\ &\quad \left. - (z^4 - 4z^3 + 2z)s^2 + (z^4 - 6z^3 + 13z + 2)s + 4z^3 - 2z^2 - 2z + 3 \right] \\ &= G^{[P]}(s, z) \end{aligned} \quad (4.17)$$

and the invariant polynomials of $Q(s, z)$ are :

$$\begin{aligned} \phi_1^{[Q]} &= \phi_2^{[Q]} = \phi_3^{[Q]} = \phi_4^{[Q]} = \phi_5^{[Q]} = \phi_6^{[Q]} = \phi_7^{[Q]} = \phi_8^{[Q]} = \phi_9^{[Q]} = 1 \\ &= \phi_1^{[P]}, \\ \phi_{10}^{[Q]} &= (2z^4 - 2z^3 + z^2 + z)s^4 + (-3z^4 + 8z^3 + 8z^2 - 4)s^3 \\ &\quad + (2z^4 - 16z^3 + 13z^2 + 12z + 3)s^2 \\ &\quad + z^4 - z^3 + 5z^2 - 14z + 3 \\ &= \phi_2^{[P]} \end{aligned} \quad (4.18)$$

REALIZATION IN GENERALIZED STATE SPACE FORM

The reduced Grobner bases of the ideals generated by the minors of the matrices in Lemma 2.2 associated with $Q(s,z)$ are given by:

Matrix	Order i of minors	Ideal generated by the $i \times i$ minors
$[T(s,z) \ U(s,z)]$	$i = 1, \dots, 8$	$\langle 1 \rangle$
	9	$\langle z^8 - 2z^7 + 11z^6 - 32z^5 + 39z^4$ $- 2z^3 - 19z^2 - 4z + 4,$ $16s + 3z^7 - 6z^6 + 35z^5 - 98z^4$ $+ 137z^3 - 50z^2 - 27z + 10 \rangle$
$\begin{bmatrix} T(s,z) \\ -V(s,z) \end{bmatrix}$	$i = 1, \dots, 8$	$\langle 1 \rangle$
	9	$\langle z^8 + 17z^6 - 35z^5 + 22z^4$ $+ 103z^3 - 29z^2 - 114z - 9,$ $179880s + 1879z^7 - 1563z^6$ $+ 37934z^5 - 93203z^4 + 202249z^3$ $- 68516z^2 - 12719z + 102237 \rangle$
$Q(s,z)$	$i = 1, \dots, 9$	$\langle 1 \rangle$
$Q(s,z)$	10	$\langle (2z^4 2z^3 + z^2 + z)s^4$ $+ (-3z^4 + 8z^3 + 8z^2 - 4)s^3$ $+ (2z^4 - 16z^3 + 13z^2 + 12z + 3)s^2$ $+ (-z^4 + 2z^3 - 24z^2 + 13z + 11)s$ $+ z^4 - z^3 + 5z^2 - 14z + 3 \rangle$

(4.19)

which is in accord with Lemmas 2.3 and 2.2.

5. Conclusions

In this paper, a new direct reduction procedure to obtain a realization in GSS form for an arbitrary 2-D polynomial system matrix has been presented. The numerical operations required to determine the GSS form are simple. The exact connection between the original system matrix with its corresponding GSS form has been set out and shown to be *zero coprime system equivalence*. The zero structure of the original polynomial system matrix is preserved making it possible to analyze the polynomial system matrix in terms of its associated GSS form. The resulting 2-D system matrix may be larger in size than the one obtained by the algorithm used by Pugh *et al.* (1998), however, the method presented in this paper has the advantage of providing a priori both the final 2-D system

matrix in GSS form and the transformation relating it to the original polynomial system matrix. To reduce the size of the resulting system matrix while preserving its GSS form, a constant zero coprime system equivalence transformation may be used.

6. Acknowledgment

The authors wish to express their thanks to Sultan Qaboos University (Oman) for their support in carrying out this research work and to the anonymous reviewers for their helpful comments.

7. References

- ATTASI, S. 1973. Systemes lineaires a deux indices, *Technical Report 31*, IRIA. FORNASINI, E. and MARCHESINI, G. 1976. State space realization theory of two-dimensional filters, *IEEE Trans. Autom. Control AC-21*, **4**: 484-492.
- FUHRMANN, P.A. 1977. On strict system equivalence and similarity, *Int. J. Control* **25(1)**: 5–10.
- GIVONE, D.D. and ROESSER, R.P. 1972. Multidimensional linear iterative circuits- General properties, *IEEE Trans. Computers AC-21*, **10**: 1067-1073.
- JOHNSON, D.S. 1993. Coprimeness in multidimensional system theory and symbolic computation, PhD thesis, Loughborough University of Technology, UK.
- KACZOREK, T. 1988. The singular general model of 2-D systems and its solution, *IEEE Trans. Autom. Control* **33**, **11**: 1060-1061.
- LEVY, B.C. 1981. 2-D polynomial and rational matrices and their applications for the modelling of 2-D dynamical systems, PhD thesis, Stanford University, USA.
- PUGH, A.C. and EL-NABRAWY, E.M.O. 2003. Zero structures of n-D systems, Proceedings Mediterranean Conference on Control and Automation, Rhodes (Greece).
- PUGH, A.C., MCINERNEY, S.J., M. HOU and HAYTON, G.E. 1996. A transformation for 2-D systems and its invariants, in Proceedings of the 35th IEEE conference on decision and control, Kobe (Japan).
- PUGH, A.C., MCINERNEY, S.J., BOUDELLIOUA, M.S., JOHNSON, D.S. and HAYTON, G.E. 1998. A transformation for 2-D linear systems and a generalization of a theorem of Rosenbrock, *Int. J. Control*, **71(3)**: 491-503.
- PUGH, A.C., MCINERNEY, S.J., BOUDELLIOUA, M.S. and HAYTON, G.E. 1998. Matrix pencil equivalents of a general 2-D polynomial matrix, *Int. J. Control*, **71(6)**: 1027-1050.
- SONTAG, E.D. 1980. On generalized inverses of polynomial and other matrices, *IEEE Trans. Autom. Control AC-25(3): 514-517.*
- YOULA, D.C. and GNAVI, G. 1979. Notes on n-dimensional system theory, *IEEE Trans. Circuits and Systems CAS-26(2): 105-111.*
- ZAK, S.H. 1984. On state-space models for systems described by partial differential equations, in Proceedings of the 33rd IEEE conference on decision and control, Las Vegas (USA).
- ZERZ, E. 1996. Primeness of multivariate polynomial matrices, *Systems and Control Letters* **29**: 139-145.

Received 8 September 2004

Accepted 20 July 2005