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The Conditional Fault-Diameter of the *K*-ary *n*-Cube

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القطر المختل المشروط لشبكات *k*-ary *n*-cube

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خلاصة: نحصل في هذا البحث على قطر شبكات *k*-ary *n*-Cube المختلة مع افتراض توزيعات مشروطة لنقاط الخلل حيث أننا نفترض أن لكل نقطة سليمة في الشبكة ما لا يقل عن نقطة مجاورة سليمة من الخلل. نبرهن أن شبكة *k*-ary *n*-Cube التي يساوي مقدار الربط فيها 2*n* ، وتحت الشرط المذكور لتوزيع نقاط الخلل ، يمكن أن تتحمل 4*n*-3 نقطة خلل بدون قطع الاتصال بينها. كما نستنتج أن القطر المختل المشروط لشبكات *k*-ary *n*-Cube يساوي القطر في غياب أي خلل زائداً وحدتان. نستخلص من هذه النتيجة أنه إذا وجد ما لا يزيد عن 4n-3 نقطة خلل في شبكة ال*k*-ary *n*-Cube وإذا كان لكل نقطة سليمة مالا يقل عن نقطة مجاورة سليمة ، فلابد أن يكون هنالك طريق موصل خال من الأعطال في الشبكة بين كل نقطتين سليمتين. تبين في هذا البحث كيفية بناء هذا الطريق الموصل الخالي من الأعطال. بهذه النتيجة تنضم شبكة *k*-ary *n*-Cube إلى مجموعة شبكات التوصيل (التي تحتوي على شبكات hypercube وشبكات star-graph) والتي يساوي القطر المختل المشروط فيها القطر السليم زائداً وحدتان.

ABSTRACT: We obtain the conditional fault diameter of the k-ary n-cube interconnection network. It has been previously shown that under the condition of forbidden faulty sets (i.e. assuming each non-faulty node has at least one non-faulty neighbor), the *k*-ary *n*-cube, whose connectivity is 2*n*, can tolerate up to 4*n*-3 faulty nodes without becoming disconnected. We extend this result by showing that the conditional fault-diameter of the *k*-ary *n*-cube is equal to the fault-free diameter plus two. This means that if there are at most 4*n*-3 faulty nodes in the *k*-ary *n*-cube and if every non-faulty node has at least one non-faulty neighbor, then there exists a fault-free path of length at most the diameter plus two between any two non faulty nodes. We also show how to construct these fault-free paths. With this result the *k*-ary *n*-cube joins a group of interconnection networks (including the hypercube and the star-graph) whose conditional fault diameter has been shown to be only two units over the fault-free diameter.

KEYWORDS: Fault-tolerance, multiprocessor systems, interconnection architectures, *k*-ary *n*-cube, torus.

1. Introduction

The node connectivity and the fault diameter have been used as measures of the fault-tolerance of interconnection networks. These measures however do not reflect the real resilience of these networks. It is true that when the number of faulty nodes is equal to the connectivity the network may become disconnected. However, this is very unlikely to happen since only very special fault distributions of these faults cause disconnection. For instance, in the k-ary n-cube, the node-connectivity is $2n$ and $2n$ faulty nodes may cause disconnection. But the network becomes disconnected only when all the $2n$ faults are adjacent to the same node which is very improbable.

In an attempt to better quantify the fault resilience of a network, the concept of forbidden faulty sets has been introduced by Esfahanian (1989). The idea is to assume that each node has at least one non-faulty neighbor. Under this forbidden faulty set condition, the number of tolerable faulty nodes is significantly larger with a slight increase in the fault diameter. Esfahanian (1989) has proven that for the binary n-cube, whose connectivity is n, $2n$-3 nodes can fail (under the forbidden faulty set condition) without disconnecting the network. Latifi (1993) has then showed that the corresponding conditional fault diameter increases only by 2 over the fault-free diameter. In Rouskov *et al.* (1996) similar results for the star graph network have been established. Latifi *et al.* (1994) have generalized this idea by assuming that each node has at least k non-faulty neighbors. Similar results for the m-ary generalized n-cube network have also been obtained in Wu (1998).

It has been previously shown (Day, 2004) that for $k \geq 4$ and $n \geq 2$, the k-ary n-cube, whose connectivity is $2n$, can tolerate up to $4n$-3 faulty nodes without becoming disconnected. The corresponding conditional node connectivity is therefore $4n$-2. The result for the remaining small values of k and n has also been obtained in Day (2004). We extend these results in this paper by showing that the conditional fault-diameter of the k-ary n-cube is equal to $n\lfloor k/2 \rfloor + 2$. We therefore establish that the k-ary n-cube, like the hypercube and the star-graph, has conditional fault diameter equal to two plus the fault-free diameter.

This paper is organized as follows: section 2 presents some notations; section 3 obtains some preliminary results useful for the derivation of the conditional fault diameter in section 4. Section 5 concludes the paper.

2. Notations

The k-ary n-cube Q_n^k has $N = k^n$ nodes each of the form $X = x_{n-1}x_{n-2}...x_0$, $0 \leq x_i < k$, for $0 \leq i < n$. Two nodes $X = x_{n-1}x_{n-2}...\ x_0$ and $Y = y_{n-1}y_{n-2}...\ y_0$ in Q_n^k are connected if, and only if, there exists exactly one value of i, $0 \leq i < n$, such that $x_i = y_i \pm 1 \pmod k$ and $x_j = y_j$, for $i \neq j$. It is shown in Bose *et al.* (1995) that Q_n^k has degree $2n$ and diameter $n\lfloor k/2 \rfloor$. Given two nodes $X = x_{n-1}x_{n-2}...x_0$ and $Y = y_{n-1}y_{n-2}...y_0$ in Q_n^k , we denote by $d_H(X,Y)$ the Hamming distance between X and Y, i.e., the number of dimensions in which X and Y differ. For the k-ary n-cube, $d_H(X,Y)$ is not the length of a shortest path between X and Y. The length of a shortest path between X and Y is equal to the *Lee distance* (Bose *et al.* 1995) given by: $d_L(X,Y) = \Sigma_{i=0}^{n-1} w_i$, where $w_i =$ min $(|x_i\text{-}y_i|, k\text{-}|x_i\text{-}y_i|)$.

The k-ary n-cube Q_n^k can be partitioned into k disjoint k-ary (n-1)-cubes. Let $Q_{n-1}^{k,i}$ be the sub-graph of Q_n^k containing all Q_n^k nodes whose addresses are of the form $X = i\,x_{n-2}...\ x_0$. Clearly, $Q_{n-1}^{k,i}$

is a *k*-ary (*n*-1)-cube and the sub-graphs $Q_{n-1}^{k,0}, Q_{n-1}^{k,1}, \ldots, Q_{n-1}^{k,k-1}$ form a partition of Q_n^k. For any node $A = a_{n-1} a_{n-2} \ldots a_0$ in Q_n^k we denote by A^p, $0 \le p < k$, the node of the form $A^p = p\, a_{n-2} \ldots a_0$ obtained from A by replacing its leftmost digit a_{n-1} by p.

***Definition* 1:** The node-connectivity $C(G)$ (or point-connectivity) of a graph G is the minimum number of nodes of G whose removal results in a disconnected or trivial graph. It has been shown in Day and Al-Ayyoub (1997) that $C(Q_n^k) = 2n$.

***Definition* 2:** The fault-diameter $FD(G)$ of a graph G is the maximum distance between any two nodes of G in the presence of at most $C(G)$-1 faulty nodes. It has been shown in Day and Al-Ayyoub (1997) that $FD(Q_n^k) = n\lfloor k/2 \rfloor + 1$.

***Definition* 3:** The conditional node connectivity $CC(G)$ of a graph G is the minimum number of nodes of G whose removal results in a disconnected or trivial graph, provided that each of the remaining nodes has at least one adjacent node in G that is not removed.

***Definition* 4:** The conditional fault diameter $CFD(G)$ of a graph G is the maximum distance between any two nodes of G in the presence of at most $CC(G)$-1 faulty nodes, provided that each of the non-faulty nodes has at least one non-faulty adjacent node in G.

We use the abbreviation *FFSC* for denoting the *forbidden faulty set condition* which corresponds to the requirement that each non-faulty node must have at least one non faulty neighbor.

3. Preliminary Results

In this section we present some preliminary results that will be used in the next section for the derivation of the conditional fault diameter of the *k*-ary *n*-cube. We start by the following result which has been proven in by Day (2004):

***Theorem* 1.** The conditional node connectivity of the *k*-ary *n*-cube for $n \ge 2$ and $k \ge 4$ is: $CC(Q_n^k) = 4n-2$.

We will denote by $\pi^*_{Q_n^k}(X,Y)$ a minimum-length path between nodes X and Y in Q_n^k. The following theorem established in by Bose *et al.* (1995) about the existence of a complete set of node disjoint paths between any two nodes of the *k*-ary *n*-cube Q_n^k will be used in the next section for establishing the conditional fault diameter of Q_n^k. Additional results about node disjoint paths in the *k*-ary *n*-cube can be found in Day and Al-Ayyoub (1997).

***Theorem* 2.** Let $X = x_{n-1}x_{n-2} \ldots x_0$ and $Y = y_{n-1}y_{n-2} \ldots y_0$ be two nodes in Q_n^k of Hamming distance $h = d_H(X,Y)$ and of Lee distance $l = d_L(X,Y)$. There are a total of $2n$ node-disjoint paths between X and Y of which:

1) h paths have length l,
2) $2n-2h$ paths have length $l+2$, and
3) for each i such that $w_i > 0$, there is a path of length $l+k-2w_i$ (h paths).

In preparation for the proof of the conditional fault-diameter of the *k*-ary *n*-cube given in the next section, we present the corresponding result for the special case of the *k*-ary 2-cube (also called the *k*-torus). The *k*-torus is a wrap-around mesh which consists of *k* rows and *k* columns. Each row and each column consists of a cycle of *k* nodes. There are four node-disjoint paths connecting any two nodes of the k-torus. The connectivity of the k-torus is therefore equal to 4. Any two nodes on the same cycle (row or column) are connected by two paths along the cycle. One is called the shortest path on the cycle and the other is called the longest path. We show that under the *FFSC*, the k-torus, whose connectivity is 4, can tolerate up to 5 faulty nodes without becoming disconnected. The conditional node connectivity of the k-torus is therefore 6. We also show that the conditional fault-diameter of the k-torus is equal to its fault-free diameter plus two.

The proof of the following theorem consists of a lengthy manual construction of fault-free paths between any two non faulty nodes of the k-torus considering all possible relative locations of the source and destination nodes and those of the 5 faulty nodes. For brevity, we omit this lengthy construction here. Interested readers can find it in Touzene and Day (2005).

***Theorem* 3.** The conditional fault diameter of the *k-torus* under the *FFSC* is $2\lfloor k/2 \rfloor + 2$.

The next result states that it is always possible to construct between any two non-faulty nodes of Q_n^k a fault-free path of length at most the diameter if the number of faults does not exceed two. This result will be used in the next section to prove the conditional fault diameter of Q_n^k.

***Lemma* 1.** In Q_n^k, if $k \geq 4$, $n \geq 2$ and if there are two or less faulty nodes, then there exists between any two non-faulty nodes at least one fault-free path of length at most $n\lfloor k/2 \rfloor$.

Proof. Let X and Y be any two distinct non-faulty nodes in Q_n^k. By Theorem 2, there exist between X and Y in Q_n^k a first set of $h = d_H(X,Y)$ paths each of length $d_L(X,Y)$ (which is at most $h\lfloor k/2 \rfloor$) and a second set of $2n$-$2h$ paths each of length $d_L(X,Y)+2$ (which is at most $h\lfloor k/2 \rfloor + 2$). Hence we have in total $2n$-h paths between X and Y in Q_n^k each of length at most $h\lfloor k/2 \rfloor + 2$. We distinguish the following cases:

- If $h = 1$ then $2n-h = 2n-1 \geq 3$, and $h\lfloor k/2 \rfloor + 2 = \lfloor k/2 \rfloor + 2 \leq n\lfloor k/2 \rfloor$ (since $n \geq 2$ and $k \geq 4$). Hence we have at least 3 paths each of length at most $n\lfloor k/2 \rfloor$ and at least one of these paths must be fault-free.
- If $h = 2$ and $n = 2$, the result is derived from the proof of the fault-diameter of the 2-ary n-cube (k-torus) available in Touzene and Day (2005).
- If $h = 2$ and $n \geq 3$ then $2n-h = 2n-2 \geq 4$ and $h\lfloor k/2 \rfloor + 2 = 2\lfloor k/2 \rfloor + 2 \leq n\lfloor k/2 \rfloor$ (since $n \geq 3$ and $k \geq 4$). Hence we have at least 4 paths each of length at most $n\lfloor k/2 \rfloor$, one of which must be fault-free.
- If $h \geq 3$ then at least one of the first set of h paths each of length at most $h\lfloor k/2 \rfloor \leq n\lfloor k/2 \rfloor$ is fault-free.

4. The Conditional Fault Diameter of the *k*-Ary *n*-Cube

The following result establishes a lower bound on the conditional fault-diameter of Q_n^k.

***Lemma* 2.** *CFD* $(Q_n^k) \geq n\lfloor k/2 \rfloor + 2$, for $k \geq 4$, $n \geq 2$.

Proof. Let X and Y be two nodes in Q_n^k at the maximum distance equal to the diameter $n\lfloor k/2 \rfloor$. Let Z be a neighbor of Y and let T be a neighbor of Z other than Y (see Figure 1). Assume that all $2n$-2 neighbors of Z other than Y and T are faulty and that all $2n$-1 neighbors of T other than Z are faulty. Notice that the former set of $2n$-2 faults and the latter set of $2n$ -1 faults must be disjoint since $k \geq 4$

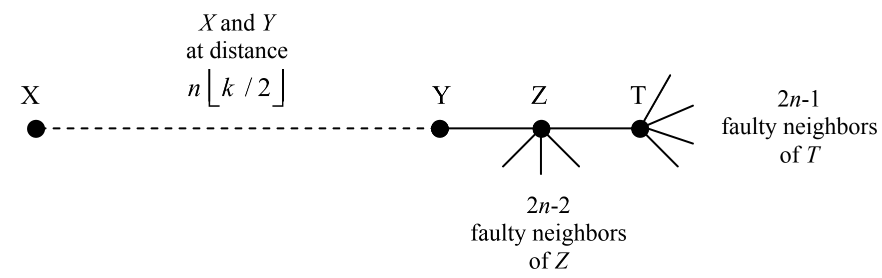

Figure 1. $n\lfloor k/2\rfloor + 2$ is a lower bound of CFD (Q_n^k).

(otherwise, if F were a faulty node appearing in both sets then $Z\,F\,T$ would form a cycle of length 3 which is impossible for $k \geq 4$). Assume there are no other faults. The total number of faults is therefore $4n$-3. A path from T to X must first go through Z then through Y then from Y to X in at least $n\lfloor k/2\rfloor$ moves and, hence, will be of length of at least $n\lfloor k/2\rfloor + 2$. Therefore, the conditional fault diameter satisfies: CFD (Q_n^k) $\geq n\lfloor k/2\rfloor + 2$. Now we establish an upper bound on the conditional fault-diameter of Q_n^k .

***Lemma* 3.** *CFD* (Q_n^k) $\leq n\lfloor k/2\rfloor + 2$, for $k \geq 4$, $n \geq 2$.

***Proof* .** We proceed by induction on n. The induction basis (n = 2) is given by Theorem 3. Now we prove the result for $n \geq 3$ assuming it is true for smaller n values. Consider two arbitrary non-faulty nodes $X = x_{n-1}x_{n-2}...x_0$ and $Y = y_{n-1}y_{n-2}...y_0$ in Q_n^k . Assume the number of faulty nodes is at most $4n$-3 and that the *FFSC* is satisfied. Our aim is to show that it is always possible to find a fault-free path between X and Y of length at most $n\lfloor k/2\rfloor + 2$. We distinguish the following cases:

Case 1. there exists m, $0 \leq m < n$, such that $x_m = y_m = i$ (assume without loss of generality $m = n$-1), then both nodes X and Y belong to the sub-graph $Q_{n-1}^{k,i}$.

Case 1.1. $Q_{n-1}^{k,i}$ has at most 4(n-1)-3 = $4n$-7 faults. By induction hypothesis there must exist a fault-free path from X to Y in $Q_{n-1}^{k,i}$ of length at most $(n-1)[k/2] + 2 \leq n[k/2] + 2$.

Case 1.2. $Q_{n-1}^{k,i}$ has more than $4n$-7 faults. This means that there are at most 3 faults outside $Q_{n-1}^{k,i}$.

Case 1.2.1. each of X and Y has all its $2n$-2 neighbors inside $Q_{n-1}^{k,i}$ faulty.

There must exist at most one faulty node outside $Q_{n-1}^{k,i}$. Therefore either $Q_{n-1}^{k,i-1}$ or $Q_{n-1}^{k,i+1}$ is fault-free (notice i-1 and i +1 are modulo k). Let X^{i-1} (respectively X^{i+1}) denote the neighbor of X in $Q_{n-1}^{k,i-1}$ (respectively $Q_{n-1}^{k,i+1}$). Similarly for Y^{i-1} and Y^{i+1} (see Figure 2).

Since either $Q_{n-1}^{k,i-1}$ or $Q_{n-1}^{k,i+1}$ is fault-free, at least one of the two paths:

$$X \rightarrow X^{i-1} \parallel \pi^*_{Q_{n-1}^{k,i-1}}(X^{i-1}, Y^{i-1}) \parallel Y^{i-1} \rightarrow Y, \text{ or}$$

$$X \rightarrow X^{i+1} \parallel \pi^*_{Q_{n-1}^{k,i+1}}(X^{i+1}, Y^{i+1}) \parallel Y^{i+1} \rightarrow Y$$

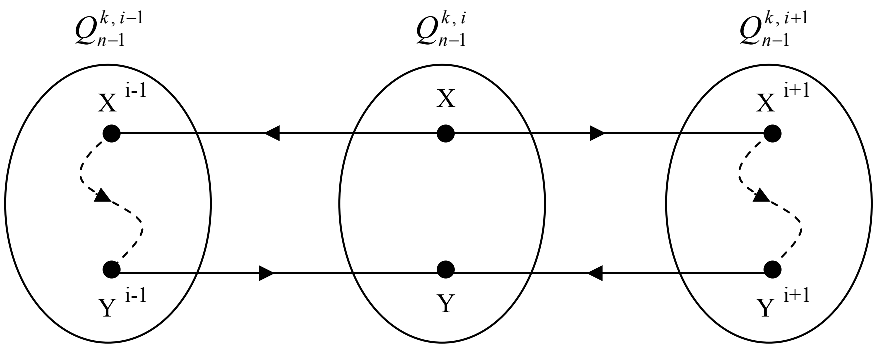

Figure 2. Path construction for Case 1.2.1.

is fault-free and is of length at most $(n-1)\lfloor k/2 \rfloor + 2$ (since it is contained in a fault-free Q_{n-1}^{k} sub-graph except for two edges). The symbol || is used to denote path concatenation.

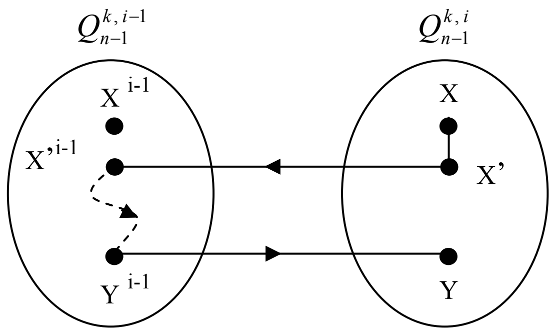

Figure 3. Path construction for Case 1.2.2.1.

Case 1.2.2. X or Y has a non-faulty neighbor inside $Q_{n-1}^{k,\,i}$.

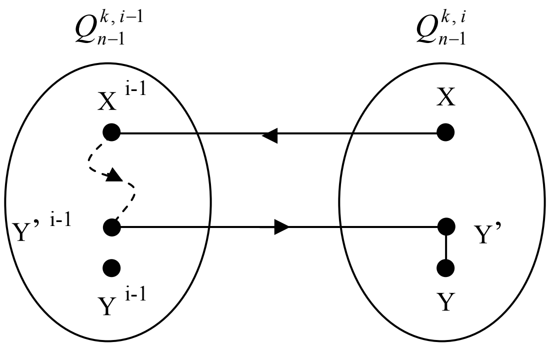

Figure 4. Path construction for Case 1.2.2.3.1.

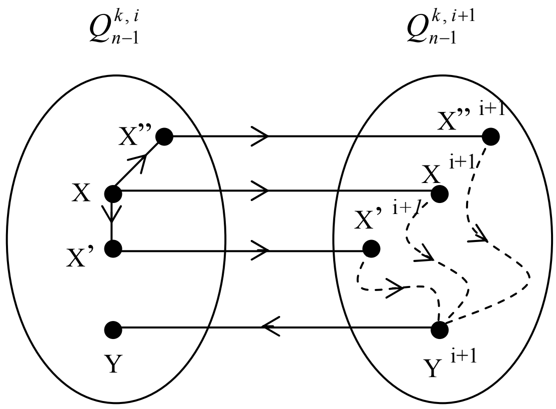

Figure 5. Path construction for Case 1.2.2.3.2.

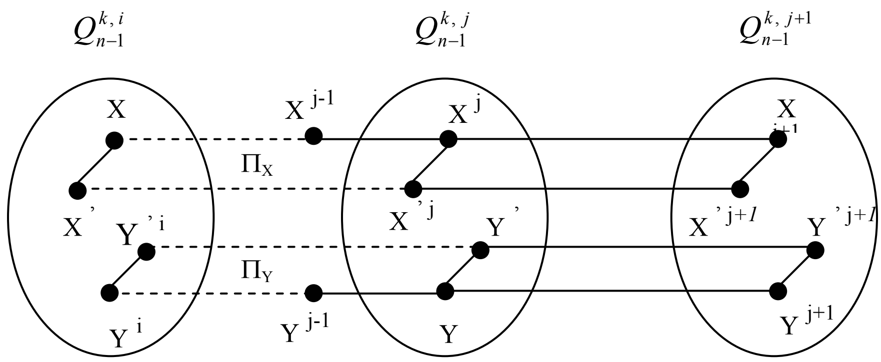

Figure 6. Path construction for Case 2.

Assume it is X which has a non-faulty neighbor X' inside $Q_{n-1}^{k,i}$ (case of Y is similar). Since there are at most three faulty nodes outside $Q_{n-1}^{k,i}$, either $Q_{n-1}^{k,i-1}$ or $Q_{n-1}^{k,i+1}$ has at most one faulty node. Assume $Q_{n-1}^{k,i-1}$ has at most one faulty node (the other case is similar).

Case 1.2.2.1. X^{i-1} is faulty (hence X^{i-1} is the only faulty node in $Q_{n-1}^{k,i-1}$).

Let $X'^{\,i-1}$ be the neighbor of X' located in $Q_{n-1}^{k,i-1}$ (see Figure 3). By Lemma 1, there exists at least one fault-free path $\pi(X'^{\,i-1}, Y^{i-1})$ of length at most $(n-1)\lfloor k/2 \rfloor$ between $X'^{\,i-1}$ and Y^{i-1} in $Q_{n-1}^{k,i-1}$. We have, therefore, a fault-free path: $X \to X' \to X'^{\,i-1} \parallel \pi(X'^{\,i-1}, Y^{i-1}) \parallel Y^{i-1} \to Y$ of length at most $(n-1)\lfloor k/2 \rfloor + 3$ which is less than or equal to $n\lfloor k/2 \rfloor + 2$ since $k \geq 4$.

Case 1.2.2.2. X^{i-1} is non-faulty and Y^{i-1} is non-faulty.

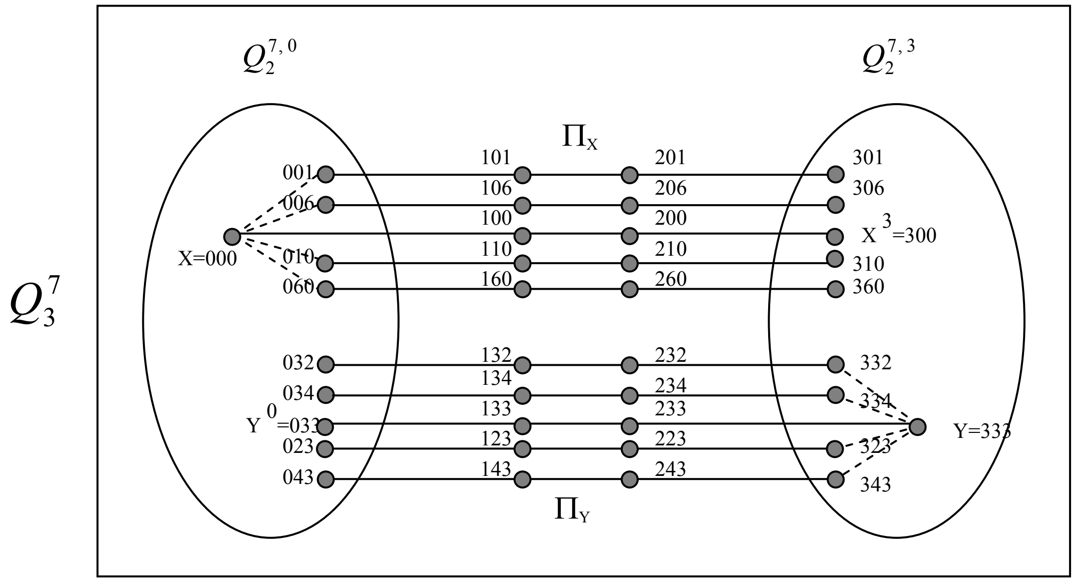

Figure 7. An Example of the $4n$-2 node-disjoint paths in $\Pi_X \cup \Pi_Y$ for $n = 3$ and $k = 7$.

Since there is at most one fault in $Q_{n-1}^{k,i-1}$, Lemma 1 gives a path $\pi(X^{i\text{-}1}, Y^{i\text{-}1})$ of length at most $(n-1)\lfloor k/2 \rfloor$ in $Q_{n-1}^{k,i-1}$ yielding the path: $X \rightarrow X^{i\text{-}1} \parallel \pi(X^{i\text{-}1}, Y^{i\text{-}1}) \parallel Y^{i\text{-}1} \rightarrow Y$ of length at most $(n-1)\lfloor k/2 \rfloor + 2 \leq n\lfloor k/2 \rfloor$ (since $k \geq 4$).

Case 1.2.2.3. $X^{i\text{-}1}$ non-faulty and $Y^{i\text{-}1}$ faulty (the only fault in $Q_{n-1}^{k,i-1}$).

Case 1.2.2.3.1. Y has a non faulty neighbor Y' in $Q_{n-1}^{k,i}$.

Let $Y'^{\,i\text{-}1}$ be the neighbor of Y' located in $Q_{n-1}^{k,i-1}$ (Figure 4). By Lemma 1, there exists at least one fault-free path $\pi(X^{i\text{-}1}, Y'^{\,i\text{-}1})$ of length at most $(n-1)\lfloor k/2 \rfloor$ from $X^{i\text{-}1}$ to $Y'^{\,i\text{-}1}$ in $Q_{n-1}^{k,i-1}$. Therefore the path: $X \rightarrow X^{i\text{-}1} \parallel \pi(X^{i\text{-}1}, Y'^{\,i\text{-}1}) \parallel Y'^{\,i\text{-}1} \rightarrow Y^{i\text{-}1} \rightarrow Y$ is fault-free and is of length at most $(n-1)\lfloor k/2 \rfloor + 3 \leq n\lfloor k/2 \rfloor + 2$ (since $k \geq 4$).

Case 1.2.2.3.2. all neighbors of Y in $Q_{n-1}^{k,i}$ are faulty.

In this case Y^{i+1} must be non-faulty (by the *FFSC*). We know that $Q_{n-1}^{k,i+1}$ has at most 2 faulty nodes. If it has only one faulty node then it is possible to enter $Q_{n-1}^{k,i+1}$ from X going directly from X to X^{i+1} or from X to X' then to $X'^{\,i+1}$ requiring in both cases at most two moves (see Figure 5). We can then use Lemma 1 in $Q_{n-1}^{k,i+1}$ to obtain a fault-free path of length at most $(n-1)\lfloor k/2 \rfloor$ from X^{i+1} or from $X'^{\,i+1}$ to Y^{i+1} and then make a final move from Y^{i+1} to Y. Hence we have a fault-free path from X to Y of length at most $(n-1)\lfloor k/2 \rfloor + 3$ which is at most $n\lfloor k/2 \rfloor + 2$ since $k \geq 4$.

If, however, $Q_{n-1}^{k,i+1}$ has two faulty nodes, then X must have a second non-faulty neighbor X'' inside $Q_{n-1}^{k,i}$ (see Figure 5), otherwise the total number of faulty nodes would include 2n-2 faulty neighbors of Y inside $Q_{n-1}^{k,i}$, 2n-3 faulty neighbors of X inside $Q_{n-1}^{k,i}$, two faults inside $Q_{n-1}^{k,i+1}$ and one fault (namely Y^{i-1}) inside $Q_{n-1}^{k,i-1}$ for a total of 4n-2 which exceeds 4n-3. Let $X''^{\,i+1}$ denote the neighbor of X'' located in $Q_{n-1}^{k,i+1}$. At least one of the three nodes X^{i+1}, $X'^{\,i+1}$ or $X''^{\,i+1}$ must be non-faulty since there are only two faults in $Q_{n-1}^{k,i+1}$. Let $X^{*\,i+1}$ denote this non-faulty node. So it is possible to enter $Q_{n-1}^{k,i+1}$ starting from X in at most two moves. We can then use Lemma 1 in $Q_{n-1}^{k,i+1}$ to obtain a fault-free path of length at most $(n-1)\lfloor k/2 \rfloor$ from $X^{*\,i+1}$ to Y^{i+1} and then make a final move from Y^{i+1} to Y. Hence we have in total a fault-free path from X to Y of length at most $(n-1)\lfloor k/2 \rfloor + 3$ which is at most $n\lfloor k/2 \rfloor + 2$ since $k \geq 4$.

Case 2. $x_m \neq y_m$ for all m, $0 \leq m < n$.

In this case, X and Y differ in all n dimensions. Let $i = x_{n-1}$ and $j = y_{n-1}$ ($i \neq j$). We therefore have $X \in Q_{n-1}^{k,i}$ and $Y \in Q_{n-1}^{k,j}$. Hence X and Y are of the form: $X = i\, x_{n-2}\, x_{n-3} \ldots x_0$ and $Y = j\, y_{n-2}\, y_{n-3} \ldots y_0$. Assume without loss of generality that i, i+1, … j is the shortest path between i and j on the cycle 0, 1, 2, … k-1. Let Π_x be the set of 2n-1 minimum-distance paths each joining either X to X^j or any of the 2n-2 neighbors X' of X in $Q_{n-1}^{k,i}$ to its isomorphic node X'^{j} in $Q_{n-1}^{k,j}$ (see Figure 6). The notations $\pi_x(X, X^j)$ and $\pi_x(X', X'^{j})$ will be used to denote these 2n-1 paths of Π_x. Similarly, let Π_y be the set of 2n-1 minimum-distance paths each joining either Y to Y^i or any of the 2n-2 neighbors Y' of Y in $Q_{n-1}^{k,j}$ to its isomorphic node Y'^{i} in $Q_{n-1}^{k,i}$. The notations $\pi_y(Y^i, Y)$ and $\pi_y(Y'^{i}, Y')$ will be used to denote these 2n-1 paths of Π_y. The 4n-2 paths in $\Pi_x \cup \Pi_y$ are all minimum-distance paths. Each is of length equal to $d_L(X, X^j) = d_L(Y^i, Y)$ which is at most $\lfloor k/2 \rfloor$ and they are all mutually node-disjoint. Figure 7 illustrates the 4n-2 paths of $\Pi_x \cup \Pi_y$ for n = 3, k = 7, X = 000, and Y = 333. These 4n-2 paths are mutually disjoint and therefore at least one of them must be completely fault-free (since the total number of faults is at most 4n-3). Let π be this fault-free path and assume that $\pi \in \Pi_x$ (the case $\pi \in \Pi_y$ is symmetric). Let X^* denote the end node of X^* located in $Q_{n-1}^{k,j}$ (i.e. X^* is either X^j or one of the 2n-2 nodes denoted X'^{j}). From the previous discussion we have: $|\pi| = d_L(X, X^j) \leq \lfloor k/2 \rfloor$.

Notice also that at most one of the sub-graphs $Q_{n-1}^{k,l}$, $0 \leq l < k$, can possibly contain more than 4n-7 faulty nodes otherwise the number of faults would exceed 4n-3 since $n \geq 3$ (by assumption of the induction step).

Case 2.1. $Q_{n-1}^{k,j}$ has at most 4n-7 faults.

We can build a path $\pi \parallel \pi(X^*,Y)$ of length at most $n\lfloor k/2\rfloor+2$, where $\pi(X^*,Y)$ is a fault-free path between X^* and Y in $Q_{n-1}^{k,j}$ of length at most $(n-1)\lfloor k/2\rfloor+2$ obtained using the induction hypothesis in $Q_{n-1}^{k,j}$.

Case 2.2. If $Q_{n-1}^{k,j}$ has more than 4*n*-7 faults (hence at most 3 faults outside $Q_{n-1}^{k,j}$).

Case 2.2.1. If $\pi_y(Y^i, Y)$ is fault-free.

In this case we can build a path: $\pi(X, Y^i) \parallel \pi_y(Y^i, Y)$ of length at most $n\lfloor k/2\rfloor+2$, where the path $\pi(X, Y^i)$ is a fault-free path of length at most $(n-1)\lfloor k/2\rfloor+2$ between X and Y^i in $Q_{n-1}^{k,i}$ obtained using the induction hypothesis in $Q_{n-1}^{k,i}$. Notice here that the induction hypothesis applies in $Q_{n-1}^{k,i}$ since the number of faults outside $Q_{n-1}^{k,j}$ (hence inside $Q_{n-1}^{k,i}$) is at most 3 which is less than 4(*n*-1)-2 (since $n \geq 3$, by induction step assumption). Furthermore, the *FFSC* is satisfied in $Q_{n-1}^{k,i}$ because it would take at least 2*n*-2 faults inside $Q_{n-1}^{k,i}$ to make all the neighbors of one of its nodes faulty which is not possible since 2*n*-2 > 3 (because $n \geq 3$) and there are at most 3 faults in $Q_{n-1}^{k,i}$.

Case 2.2.2. $\pi_y(Y^i, Y)$ is faulty.

Case 2.2.2.1. X^{*j+1} is not faulty and Y^{j+1} is not faulty.

If $d_L(X,Y) < n\lfloor k/2\rfloor$, we can build a path: $\pi \parallel X^* \to X^{*j+1} \parallel \pi(X^{*j+1}, Y^{j+1}) \parallel Y^{j+1} \to Y$ of length at most $n\lfloor k/2\rfloor+2$, where $\pi(X^{*j+1}, Y^{j+1})$ is a fault-free path between X^{*j+1} and Y^{j+1} of length at most $(n-1)\lfloor k/2\rfloor$ obtained using Lemma 1 in $Q_{n-1}^{k,j+1}$. Notice that there are at most three faults outside $Q_{n-1}^{k,j}$ and one of them is located on $\pi_y(Y^i, Y)$, therefore there are at most two faults inside $Q_{n-1}^{k,j+1}$ and hence Lemma 1 applies in $Q_{n-1}^{k,j+1}$.

If, however, $d_L(X,Y) = n\lfloor k/2\rfloor$, let X_1' and X_2' be neighbors of X in $Q_{n-1}^{k,i}$ corresponding to minimum distance moves from X to Y along two different dimensions (these neighbors exist since $n \geq 3$). Consider the 3 paths joining X to X^{j+1}, X_1' to $X_1'^{j+1}$, and X_2' to $X_2'^{j+1}$, corresponding to the sequence of moves $i \to i\text{-}1 \to i\text{-}2 \to \ldots \to j+1$ in dimension *n*-1. These paths are node-disjoint and each is of length at most $\lfloor k/2\rfloor+1$. Therefore, it is possible to enter the sub-graph $Q_{n-1}^{k,j+1}$ from X in at most $\lfloor k/2\rfloor+2$ moves (in fact if we had started with the edge $X \to X^{i-1}$ then we would enter the sub-graph $Q_{n-1}^{k,j+1}$ from X in at most $\lfloor k/2\rfloor+1$ moves). Since in addition $Q_{n-1}^{k,j+1}$ has at most two faults, Lemma 1 can be used to go from one of the nodes X^{j+1}, $X_1'^{j+1}$ or $X_2'^{j+1}$ to Y^{j+1} in at most $(n-1)\lfloor k/2\rfloor$ moves (in fact in at most $(n-1\lfloor k/2\rfloor-1$ moves if we had started with edge $X \to X_1'$ or edge $X \to X_2'$ since the initial move is a minimum distance move) and then from Y^{j+1} to Y yielding an overall path from X to Y of length at most $n\lfloor k/2\rfloor+2$.

Case 2.2.2.2. Both X^{*j+1} and Y^{j+1} are faulty.

Case 2.2.2.2.1. Y^{j-1} is not faulty.

In this case, the only faulty nodes outside $Q_{n-1}^{k,j}$ are X^{*j+1}, Y^{j+1} and one node other than Y^{j-1} on $\pi_y(Y^i, Y)$. In this case we can build a path: π-1 || $\pi(X^{j-1}, Y^{j-1})$ || $Y^{j-1} \rightarrow Y$, of length at most $n\lfloor k/2\rfloor + 2$, where $\pi(X^{j-1}, Y^{j-1})$ is a minimum distance fault-free path between X^{j-1} and Y^{j-1} in $Q_{n-1}^{k,j-1}$ (notice that there are no faults in $Q_{n-1}^{k,j-1}$ in this case) and π-1 denotes the path obtained from π by removing its last $X^{j-1} \rightarrow X^*$ edge.

Case 2.2.2.2.2. Y^{j-1} is faulty.

The only faulty nodes outside $Q_{n-1}^{k,j}$ are X^{*j+1}, Y^{j+1} and Y^{j-1}. In this case, *Y* must have at least one neighbor *Y'* inside $Q_{n-1}^{k,j}$ which is not faulty (by *FFSC*). We can, therefore, build a fault-free path: $\pi(X, Y'^{i})$ || $\pi_y(Y'^{i}, Y')$ || $Y' \rightarrow Y$ of length at most $n\lfloor k/2\rfloor + 2$ where $\pi(X, Y'^{i})$ is a minimum distance fault-free path between *X* and Y'^{i} in $Q_{n-1}^{k,i}$ (notice that there are no faults in $Q_{n-1}^{k,i}$ in this case).

Case 2.2.2.3. X^{*j+1} is not faulty and Y^{j+1} is faulty. Here Y^{j+1} is faulty and we have at least one fault located on $\pi_y(Y^i, Y)$ (condition of all sub cases under case **2.2.2**). Therefore, there is at most one other fault outside $Q_{n-1}^{k,j}$. Furthermore, node *Y* must have at least one non faulty neighbor (by *FFSC*). We distinguish the following two sub-cases:

Case 2.2.2.3.1. Some neighbor *Y'* of *Y* inside $Q_{n-1}^{k,j}$ is not faulty. Hence, at least one of the following two paths must be fault-free:

$\pi(X, Y'^{i}) \| \pi_y(Y'^{i}, Y') \| Y' \rightarrow Y$, (here $|\pi(X, Y'^{i})| \le (n-1)\lfloor k/2\rfloor$ (by Lemma 1)

$\pi \| X^* \rightarrow X^{*j+1} \| \pi(X^{*j+1}, Y'^{j+1}) \| Y'^{j+1} \rightarrow Y' \rightarrow Y$ (all moves are minimum distance moves except for the two moves $X^* \rightarrow X^{*j+1}$ and $Y'^{j+1} \rightarrow Y'$).

The length of each of these two paths is clearly at most $n\lfloor k/2\rfloor + 2$.

Case 2.2.2.3.2. The neighbor Y^{j-1} of *Y* is not faulty.

Let Y'^{j-1} and Y''^{j-1} denote two neighbors of Y^{j-1} inside $Q_{n-1}^{k,j-1}$. Obviously Y'^{j-1} and Y''^{j-1} cannot be on the path $\pi_y(Y, Y^i)$ since Y^{j-1} is the only node on this path that is located in $Q_{n-1}^{k,j-1}$ (this is justified by the fact that the paths in Π_y are minimum distance paths). Therefore, at least one of the following two paths must be fault-free and is of length at most $n\lfloor k/2\rfloor + 2$:

$\pi(X, Y'^{i}) \| \pi_y(Y'^{i}, Y')\text{-1} \| Y'^{j-1} \rightarrow Y^{j-1} \rightarrow Y$

$\pi(X, Y''^{i}) \| \pi_y(Y''^{i}, Y'')\text{-1} \| Y''^{j-1} \rightarrow Y^{j-1} \rightarrow Y$

where $\pi(X, Y'^{i})$ and $\pi(X, Y''^{i})$ are paths of length at most $(n-1)[k/2]$ obtained using Lemma 1 in $Q_{n-1}^{k,i}$.

Case 2.2.2.4. X^{*j+1} is faulty and Y^{j+1} is not faulty.

If Y has a non faulty neighbor other than Y^{j+1} then the same sub-cases **2.2.2.3.1** and **2.2.2.3.2** apply for this case too. If, however, all the 2*n*-1 neighbors of Y other than Y^{j+1} are faulty, then for at least one of the 2*n*-2 neighbors $X^{*'}$ of X^* inside $Q_{n-1}^{k,j}$ we must have both $X^{*'}$ and its neighbor $X^{*'j+1}$ in $Q_{n-1}^{k,j+1}$ non-faulty. Otherwise, there will be at least 2*n*-1 (neighbors of *Y*) plus 2*n*-2 (neighbors $X^{*'}$ of X^* in $Q_{n-1}^{k,j}$ or their corresponding $X^{*'j+1}$ in $Q_{n-1}^{k,j+1}$) plus one (X^{*j+1}) faults for a total of 4*n*-2 faults which exceeds the number of faults 4*n*-3. In this case the path:

$$\pi \,\|\, X^* \rightarrow X^{*'} \rightarrow X^{*'j+1} \,\|\, \pi(X^{*'j+1}, Y^{j+1}) \,\|\, Y^{j+1} \rightarrow Y$$

is fault-free, where $\pi(X^{*'j+1}, Y^{j+1})$ is a path of length at most $(n-1)\lfloor k/2 \rfloor$ obtained applying Lemma 1 in $Q_{n-1}^{k,j+1}$. The overall length would not exceed $n\lfloor k/2 \rfloor + 2$ if the length of π is strictly less than $\lfloor k/2 \rfloor$. If however the length of π is at its maximum value $\lfloor k/2 \rfloor$ (i.e. *X* and *Y* are diametrically opposite along dimension *n*-1) then a different path is needed. An alternative path can be built going first from *X* to a non faulty neighbor $Y^{'i}$ of Y^i following edges of a minimum distance path from *X* to Y^i inside $Q_{n-1}^{k,i}$ (this is possible making use of Theorem 2 inside $Q_{n-1}^{k,i}$ and remembering that *X* and *Y* differ in all dimensions and that $Q_{n-1}^{k,i}$ contains at most one faulty node). In fact it is possible to find a non faulty neighbor $Y^{'i}$ of Y^i such that $Y^{'j+1}$ is also not faulty since we are left only with one fault outside $Q_{n-1}^{k,j}$ other than X^{*j+1} and Y^{j-1}. The path can then continue going from $Y^{'i}$ to $Y^{'j+1}$ correcting the digit at dimension *n*-1 along the opposite direction of that followed along $\pi_y(Y^i, Y)$ (i.e. in the direction $i \rightarrow i\text{-}1 \rightarrow i\text{-}2 \ \ldots \rightarrow j\text{+}1$). Going in this direction along dimension *n*-1 requires at most one extra move beyond the minimum distance since *X* and *Y* are diametrically opposite along dimension *n*-1. The alternative path can be completed by the two moves from $Y^{'j+1}$ to Y^{j+1} and then from Y^{j+1} to Y. The overall length of this alternative path would not exceed $n\lfloor k/2 \rfloor + 2$ since all moves are along minimum distance paths except possibly for the one extra move on the path from $Y^{'i}$ to $Y^{'j+1}$ (the opposite direction path) and the last move from Y^{j+1} to *Y*.

***Theorem* 4.** The conditional fault-diameter of the *k*-ary *n*-cube is equal to $n\lfloor k/2 \rfloor + 2$ for $k \geq 4$ and $n \geq 2$.

Proof. Derived from combining the results of Lemma 2 and Lemma 3.

5. Conclusion

We have contributed to the study of the fault-tolerance of the *k*-ary *n*-cube interconnection network by establishing its conditional fault-diameter under the *FFSC* (i.e., assuming that each non-faulty processor has at least one non-faulty neighbor). We have shown that under this condition and for $k \geq 4$ and $n \geq 2$, the conditional fault-diameter of the *k*-ary *n*-cube is $n\lfloor k/2 \rfloor + 2$. This means that if there are less than 4*n*-2 faults in the *k*-ary *n*-cube and if every non faulty node has at least one non-faulty neighbor, then there is a fault-free path of length at most $n\lfloor k/2 \rfloor + 2$ between any two non-faulty nodes. We have shown how to construct these fault-free paths. With this result the *k*-ary *n*-cube joins a group of interconnection networks (including the hypercube and the star-graph) whose conditional fault-diameter have been proved to be only two units over the fault-free diameter.

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