SQU Journal for Science, 2018, 23(2),137-146 DOI: http://dx.doi.org/10.24200/squjs.vol23iss2pp137-146
Sultan Qaboos University

Initial and Boundary Value Problems for Fractional Differential Equations Involving Atangana-Baleanu Derivative

Fatma S. Al-Musalhi[1], Nasser S. Al-Salti[1,2]*, and Erkinjon Karimov[2,3]

*[1]Department of Mathematics, Sultan Qaboos University, P.O. Box 36, PC 123, Al Khodh, Muscat, Sultanate of Oman; [2]FracDiff Research Group, Sultan Qaboos University, Muscat, Sultanate of Oman; [3]Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent 170, Uzbekistan, *Email: nalslti@squ.edu.om.*

Abstract: In the present work, an initial value problem involving the Atangana-Baleanu derivative is considered. An explicit solution of the given problem in integral form is obtained by using the Laplace transform. The use of the given initial value problem is illustrated by considering a boundary value problem in which the solution is expressed in the form of a series expansion using an orthogonal basis obtained by separation of variables. Some examples are also given to illustrate the obtained results.

Keywords: Atangana-Baleanu derivative, Initial-Boundary value problem, Fractional differential equation.

معادلات تفاضلية كسرية ذات شروط ابتدائية وحدودية والتي تحتوي على مشتقة أتنجانا باليانو

فاطمة المصلحي، ناصر السلطي وإيركن كريموف

الملخص: في هذه الورقة تم دراسة معادلة تفاضليه تحتوي على مشتقة كسريه لأتنجانا باليانو في ظل وجود شروط ابتدائية تارة وباستخدام شروط حدودية تارة أخرى، وقد تم أولا إيجاد حل المعادلة في ظل وجود شروط ابتدائية باستخدام تحويل لابلاس. ولقد اتضحت أهمية هذه المعادلة في إيجاد حل المعادلة التفاضلية في ظل وجود شروط ابتدائية وحدودية معا، والتي تم التعبير عن حلها في صورة متسلسلة رياضية. وفي الأخير تم عرض بعض الأمثلة لتوضيح النتائج التي توصلت لها الدراسة.

الكلمات المفتاحية: مشتقة أتنجانا باليانو ، مسألة ذات شروط إبتدائية وحدودية، معادلات تفاضلية كسرية.

1. Introduction and Preliminaries

Recently, two newly defined fractional derivatives without a singular kernel have been suggested, namely, the Caputo-Fabrizio fractional derivative [1] and the Atangana-Baleanu fractional derivative [2]. These two new derivatives have been applied to real-life problems in various areas, for example in the fields of thermal science, material sciences, groundwater modelling and mass-spring systems [2 -7], and have been considered in a number of other recent works, see for example [8-17]. The main difference between these two definitions is that the Caputo-Fabrizio derivative is based on an exponential kernel, while the Atangana-Baleanu definition uses a Mittag-leffler function as a non-local kernel. The non-locality of the kernel gives a better description of the memory within structures with different scales. These two new derivatives are defined as follows:

Definition 1.1. [2] Let $f \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1]$. The Caputo-Fabrizio fractional derivative is defined as

$$ {}^{CF}_{a}D^{\alpha}_{t} f(t) = \frac{B(\alpha)}{1-\alpha}\int_a^t f'(s) \exp\left[\frac{-\alpha}{1-\alpha}(t-s)\right]ds, \tag{1}$$

and the Atangana-Baleanu fractional derivative is given by

$$ {}^{ABC}_{a}D^{\alpha}_{t} f(t) = \frac{B(\alpha)}{1-\alpha}\int_a^t f'(s)\, E_{\alpha}\left[\frac{-\alpha}{1-\alpha}(t-s)^{\alpha}\right]ds, \tag{2}$$

where $H^1(a, b)$ is the usual Sobolev space, i.e., $H^1(a,b) = \{f \in L^2(a,b),\ f' \in L^2(a,b)\}$, $B(\alpha)$ denotes a normalization function such that $B(0) = B(1) = 1$, and

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad \operatorname{Re}(\alpha) > 0,\ z \in \mathbb{C},$$

is the Mittag-Leffler function of one parameter [18]. For properties related to these derivatives, see [2], [11], [15]. In this paper, we are concerned with solutions to initial and boundary value problems for fractional differential equations involving Atangana- Baleanu derivative. We first recall the Mittag-Leffler of two parameters

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \operatorname{Re}(\alpha) > 0,\ \operatorname{Re}(\beta) > 0,\ z \in \mathbb{C}.$$

Moreover, the Mittag-Leffler function $E_{\alpha}(\lambda t^{\alpha})$ is bounded (see [18]), i.e.

$$E_{\alpha}(\lambda t^{\alpha}) \le M, \tag{3}$$

where M denotes a positive constant.

In [2], Atangana and Baleanu considered the time fractional ordinary differential equation

$$ {}^{ABC}_{0}D^{\alpha}_{t} f(t) = u(t),$$

and, on using the Laplace transform, they found the following solution

$$f(t) = {}^{AB}_{0}I^{\alpha}_{t}\, u(t) = \frac{1-\alpha}{B(\alpha)}u(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t u(s)(t-s)^{\alpha-1}\,ds,$$

where they have defined

$$ {}^{AB}_{0}I^{\alpha}_{t}\, f(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t f(s)(t-s)^{\alpha-1}\,ds$$

to be the fractional integral associated with the fractional derivative (2).

In this paper, we consider the following initial value problem (IVP)

$$\begin{aligned} {}^{ABC}_{0}D^{\alpha}_{t}\, u(t) - \lambda u(t) &= f(t), \quad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{4}$$

where $\lambda, u_0 \in \mathbb{R}.$ The solution of this IVP is obtained here by using the Laplace transform, which can also be obtained by using the successive approximation method as illustrated in the arXiv version of this paper [19]. The use of such an IVP is illustrated by considering a boundary value problem in which the solution is expressed in the form of a series expansion using an orthogonal basis obtained by separation of variables. It is worth mentioning here that the fractional differential equation in (4), with or without initial condition, has also been considered by other authors, see for example [7], [20-22]. However, most of previous studies have either considered special forms of $f(t)$ or considered only one case of λ, namely, $\lambda \neq \frac{B(\alpha)}{1-\alpha}$ or $\lambda = \frac{B(\alpha)}{1-\alpha}$. Moreover, most of the solutions obtained previously fail to satisfy the initial condition, since the existence of a solution requires a necessary condition, namely, $f(0) = -\lambda u_0$, which was first addressed in the arXiv version of this paper [19]. The rest of the present paper is organized as follows. Section 2 is devoted to our main result, which is an explicit solution of the IVP (4) and to solving a boundary value problem utilizing the solution of the IVP (4). We conclude this paper by presenting some examples to illustrate the obtained results.

2. Main Result

2.1 Initial value problem

Here, we consider the following problem:

Find a solution $u(t) \in H^1(0,\infty)$ that satisfies the following equation

$$ {}^{ABC}_{0}D^{\alpha}_{t}\, u(t) - \lambda u(t) = f(t), \qquad t > 0, \tag{5}$$

and the initial condition

$$u(0) = u_{0,} \tag{6}$$

where $\lambda, u_0 \in \mathbb{R}.$ The solution of this initial value problem is formulated in the following theorem:

Theorem 2.1.

- If $\lambda \neq \frac{B(\alpha)}{1-\alpha}$, $f(t) \in C(0,\infty)$, $f(0) = -\lambda u_0$, then the solution of the initial value problem (5) - (6) is given by

$$\begin{aligned} u(t) = {} & \frac{B(\alpha)u_0}{B(\alpha)-\lambda(1-\alpha)} E_\alpha\left[\frac{\alpha\lambda}{B(\alpha)-\lambda(1-\alpha)} t^\alpha\right] + \frac{1-\alpha}{B(\alpha)-\lambda(1-\alpha)} f(t) \\ & + \frac{\alpha B(\alpha)}{(B(\alpha)-\lambda(1-\alpha))^2}\int_0^t f(\zeta)(t-\zeta)^{\alpha-1} E_{\alpha,\alpha}\left[\frac{\alpha\lambda}{B(\alpha)-\lambda(1-\alpha)}(t-\zeta)^\alpha\right] d\zeta. \end{aligned} \tag{7}$$

- If $\lambda = \frac{B(\alpha)}{1-\alpha}$, $f(t) \in AC^1[0,\infty)$, $f(0) = -\lambda u_0$, then the solution of the given IVP is given by

$$u(t) = -\frac{(1-\alpha)^2}{\alpha B(\alpha)}\, {}^{C}_{0}D^{\alpha}_{t} f(t) - \frac{(1-\alpha)}{B(\alpha)} f(t), \tag{8}$$

where the space $AC^1[0,\infty)$ is defined as

$$AC^1 := \left\{ f(t) = f(0) + \int_0^t g(z)\,dz, \quad g \in L_1[0,\infty) \right\}$$

and ${}^{C}_{0}D^{\alpha}_{t} f(t)$ represents the Caputo fractional derivative of order ($0 < \alpha < 1$) and is defined as [15]

$${}^{C}_{0}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-y)^{-\alpha} f(y)\,dy.$$

Proof. Applying the Laplace transform to both sides of equation (5) and using (6), we have

$$\frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha}U(s)-s^{\alpha-1}u_0}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}}-\lambda U(s)=F(s), \tag{9}$$

where $U(s)=\ell\{u(t)\}(s)$ and

$$\ell\left\{{}^{ABC}_{0}D^{\alpha}_{t}u(t)\right\}(s)=\frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha}U(s)-s^{\alpha-1}u_0}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}}.$$

Simplifying and solving for $U(s)$, we get

$$U(s)=\frac{B(\alpha)s^{\alpha-1}u_0}{s^{\alpha}(B(\alpha)-\lambda(1-\alpha))-\lambda\alpha}+\frac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}(B(\alpha)-\lambda(1-\alpha))-\lambda\alpha}F(s),\quad \text{for}\quad \lambda\neq\frac{B(\alpha)}{1-\alpha},$$

which can be rewritten as

$$\begin{aligned}U(s)&=\frac{B(\alpha)s^{\alpha-1}u_0}{(B(\alpha)-\lambda(1-\alpha))\left[s^{\alpha}-\dfrac{\alpha\lambda}{B(\alpha)-\lambda(1-\alpha)}\right]}+\frac{(1-\alpha)s^{\alpha}+\alpha}{(B(\alpha)-\lambda(1-\alpha))\left[s^{\alpha}-\dfrac{\alpha\lambda}{B(\alpha)-\lambda(1-\alpha)}\right]}F(s)\\ &=\frac{B(\alpha)s^{\alpha-1}u_0}{(B(\alpha)-\lambda(1-\alpha))\left[s^{\alpha}-\dfrac{\alpha\lambda}{B(\alpha)-\lambda(1-\alpha)}\right]}+\frac{(1-\alpha)}{(B(\alpha)-\lambda(1-\alpha))}F(s)\\ &\quad+\frac{\alpha B(\alpha)}{(B(\alpha)-\lambda(1-\alpha))^2\left[s^{\alpha}-\dfrac{\alpha\lambda}{B(\alpha)-\lambda(1-\alpha)}\right]}F(s)\end{aligned}$$

Since the Laplace transform of the Mittag-Leffler function is given by

$$\ell\left\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right\}(s)=\frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda},$$

applying the Laplace inverse will then give

$$\begin{aligned}u(t)&=\frac{B(\alpha)u_0}{B(\alpha)-\lambda(1-\alpha)}E_{\alpha,1}\left(\frac{\alpha\lambda}{B(\alpha)-\lambda(1-\alpha)}t^{\alpha}\right)+\frac{(1-\alpha)}{B(\alpha)-\lambda(1-\alpha)}f(t)\\ &\quad+\frac{\alpha B(\alpha)}{(B(\alpha)-\lambda(1-\alpha))^2}\left(t^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\alpha\lambda}{B(\alpha)-\lambda(1-\alpha)}t^{\alpha}\right)*f(t)\right),\end{aligned}$$

which is equivalent to the desired result as given in Equation (7). Note that the condition $f(0)=-\lambda u_0$ is needed to ensure that $u(0)=u_0$.

For $\lambda=\dfrac{B(\alpha)}{1-\alpha}$, equation (9) gives,

$$U(s)=\frac{-(1-\alpha)}{\alpha}s^{\alpha-1}u_0-\frac{(1-\alpha)^2}{\alpha B(\alpha)}\left(s^{\alpha}+\frac{\alpha}{1-\alpha}\right)F(s).$$

Applying the Laplace inverse, we obtain

$$u(t)=\frac{-(1-\alpha)}{\alpha\Gamma(1-\alpha)}t^{-\alpha}u_0-\frac{(1-\alpha)^2}{\alpha B(\alpha)}\left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)}*f'(t)\right)-\frac{(1-\alpha)^2}{\alpha B(\alpha)}\frac{t^{-\alpha}f(0)}{\Gamma(1-\alpha)}-\frac{1-\alpha}{B(\alpha)}f(t).$$

Using the condition $f(0)=-\lambda u_0,$ one can obtain the desired solution (8).

Remark 2.2. For the case $\lambda=0$ and $u(0)=0,$ we get

$$u(t)=\frac{1-\alpha}{B(\alpha)}f(t)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t f(\zeta)(t-\zeta)^{\alpha-1}d\zeta,$$

which coincides with the result obtained in [2]. Note that the obtained results for the IVP (4) are analogous to the results obtained for the case of Caputo-Fabrizio fractional derivative [8]. In the next section, we consider a direct initial-boundary value problem including a partial fractional differential equation with the Atangana-Baleanu derivative. To obtain the solution of the direct problem, we have utilized the solution of the IVP (4). It is also worth mentioning here that the condition $f(0)=-\lambda u_0$ makes it impossible to consider inverse source or inverse initial problems with the Atangana-Baleanu fractional derivative.

2.2 Initial-boundary value problem

Now, we consider a direct problem of determining $u(x,t)$ in a domain $\Omega=\{(x,t):0<x<1,\ 0<t<T\}$, such that $u(.,t)\in C^2(0,1),\ u(x,.)\in H^1(0,T)$ and satisfies the following initial-boundary value problem:

$$ {}^{ABC}_{\ \ 0}D_t^{\alpha}u(x,t)-u_{xx}(x,t)=f(x,t),\quad (x,t)\in\Omega \tag{10}$$

$$u(0,t)=0,\ \ u(1,t)=0,\qquad 0\le t\le T, \tag{11}$$

$$u(x,0)=0,\qquad 0\le x\le 1, \tag{12}$$

where $f(x,t)$ is a given function. We begin by using the separation of variables method to solve the homogeneous equation corresponding to equation (10) along with the boundary conditions (11). Thus, we obtain the following spectral problem:

$$\begin{cases} X''+\lambda X=0,\\ X(0)=0,\qquad X(1)=0.\end{cases} \tag{13}$$

which is self-adjoint and has the following eigenvalues

$$\lambda_k=(k\pi)^2,\qquad k=1,\ 2,3,\ \dots\ .$$

The corresponding eigenfunctions are

$$X_k=\sin(k\pi x),\qquad k=1,2,3,\ \dots\ . \tag{14}$$

Since the system of eigenfunctions (14) forms an orthogonal basis in $L^2(0,1)$ [23], we can then write the solution $u(x,t)$ and the given function $f(x,t)$ in the form of series expansions as follows:

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin(k\pi x), \tag{15}$$

$$f(x,t)=\sum_{k=1}^{\infty}f_k(t)\sin(k\pi x), \tag{16}$$

where $u_k(t)$ are the unknown coefficients to be found, and the coefficients $f_k(t)$ are given by $f_k(t) = 2\int_0^1 f(x,t)\sin(k\pi x)\,dx.$ Substituting (15) and (16) into (10) and (12), we obtain the following fractional differential equation

$$ {}_0^{ABC}D_t^{\alpha} u_k(t) + k^2\pi^2 u_k(t) = f_k(t), $$

along with the following condition

$$ u_k(0) = 0. $$

Whereupon using Theorem 2.1, the solution is given by

$$ u_k(t) = \frac{1-\alpha}{B(\alpha)+k^2\pi^2(1-\alpha)} f_k(t) + \frac{\alpha B(\alpha)}{(B(\alpha)+k^2\pi^2(1-\alpha))^2} \int_0^t f_k(t)(t-\zeta)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{-\alpha k^2\pi^2}{B(\alpha)+k^2\pi^2(1-\alpha)}(t-\zeta)^{\alpha}\right) d\zeta, $$

with $f_k(0) = 0,$ which is achieved by assuming $f(x,0) = 0.$ Thus, the solution $u(x,t)$ can now be written as

$$ u(x,t) = \sum_{k=1}^{\infty}\left(\frac{1-\alpha}{B(\alpha)+k^2\pi^2(1-\alpha)} f_k(t) + \frac{\alpha B(\alpha)}{(B(\alpha)+k^2\pi^2(1-\alpha))^2} \int_0^t f_k(\zeta)(t-\zeta)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{-\alpha k^2\pi^2}{B(\alpha)+k^2\pi^2(1-\alpha)}(t-\zeta)^{\alpha}\right) d\zeta\right)\sin(k\pi x). $$

In order to complete the proof of the existence of solution, we need to show the uniform convergence of the series representations of $u(x,t),\ u_x(x,t),\ u_{xx}(x,t),\ {}_0^{ABC}D_t^{\alpha} u(x,t).$

Since the Mittag-Leffler function appearing in the expression of $u(x, t)$ is bounded, the uniform convergence of the series representation of $u(x,t)$ is ensured by assuming $f(x,.) \in C(0,T).$ Now, the series representation of $u_{xx}(x,t)$ is given by

$$ u_{xx}(x,t) = -\sum_{k=1}^{\infty}\left(\frac{k^2\pi^2(1-\alpha)}{B(\alpha)+k^2\pi^2(1-\alpha)} f_k(t) + \frac{k^2\pi^2\alpha B(\alpha)}{(B(\alpha)+k^2\pi^2(1-\alpha))^2} \int_0^t f_k(\zeta)(t-\zeta)^{\alpha-1} E_{\alpha,\alpha}\left(\frac{-\alpha k^2\pi^2}{B(\alpha)+k^2\pi^2(1-\alpha)}(t-\zeta)^{\alpha}\right) d\zeta\right)\sin(k\pi x) $$

$$ = \sum_{k=1}^{\infty} f_k(t)\sin(k\pi x) + \sum_{k=1}^{\infty}\frac{B(\alpha)}{B(\alpha)+k^2\pi^2(1-\alpha)}\sin(k\pi x)\int_0^t f_k'(\zeta)\, E_{\alpha,1}\left(\frac{-\alpha k^2\pi^2}{B(\alpha)+k^2\pi^2(1-\alpha)}(t-\zeta)^{\alpha}\right) d\zeta. $$

Assuming $f_t(x,t)$ is integrable, it is clear that the second term of the above series converges uniformly. For the convergence of the first term, we assume $f(0,t) = f(1,t) = 0$ and use integration by parts to get

$$ \left|\sum_{k=1}^{\infty} f_k(t)\sin(k\pi x)\right| = \left|\sum_{k=1}^{\infty}\frac{1}{k\pi} f_{1k}(t)\sin(k\pi x)\right| \le \sum_{k=1}^{\infty}\frac{1}{k\pi}\left|f_{1k}(t)\right|, $$

where

$$f_{1k}(t) = 2\int_0^1 f_x(x,t)\cos(k\pi x)\,dx.$$

Using the inequality $ab \le \frac{1}{2}(a^2+b^2)$ and the Bessel's inequality for trigonometric series, we then have the following estimate

$$\left|\sum_{k=1}^{\infty} f_k(t)\sin(k\pi x)\right| \le \sum_{k=1}^{\infty}\frac{1}{2}\left(\frac{1}{k^2\pi^2}+\left|f_{1k}\right|^2\right) \le \sum_{k=1}^{\infty}\frac{1}{2k^2\pi^2} + \frac{1}{2}\|f_x(x,t)\|^2_{L^2(0,1)}.$$

Therefore, the expression of $u_{xx}(x,t)$ is uniformly convergent. The uniform convergence of ${}^{ABC}_{0}D_t^{\alpha}\,u(x,t),$ can be proved in a similar way.

The uniqueness of the solution can be obtained using the completeness properties of the system $\{\sin(k\pi x)\}$. The main result for the direct problem can be summarized in the following theorem:

Theorem 2.3. *Assume* $f(.,t)\in C(0,1),\ f(x,.)\in C(0,T)$ *such that* $f(x,0)=0,\ f(0,t)=f(1,t)=0,$

$f_t(x,t)\in L^1(0,T),$ *and* $f_x(x,t)\in L^2(0,1),$ *then the problem (10) - (12) has a unique solution* $u(x,t)$ *given by*

$$u(x,t)=\sum_{k=1}^{\infty}\Bigg(\frac{1-\alpha}{B(\alpha)+k^2\pi^2(1-\alpha)}f_k(t)+\frac{\alpha B(\alpha)}{(B(\alpha)+k^2\pi^2(1-\alpha))^2}$$
$$\int_0^t f_k(\zeta)(t-\zeta)^{\alpha-1}E_{\alpha,\alpha}\left(\frac{-\alpha k^2\pi^2}{B(\alpha)+k^2\pi^2(1-\alpha)}(t-\zeta)^{\alpha}\right)d\zeta\Bigg)\sin(k\pi x).$$

where,

$$f_k(t)=2\int_0^1 f(x,t)\sin(k\pi x)\,dx.$$

3. Illustrative Examples

In this section, we give some examples to illustrate our results presented in Theorems (2.1) and (2.3), choosing $B(\alpha)=1.$

Example 1**.** Here, we present solutions of three different initial value problems by considering different choices of $f(t),\lambda$ and u_0.

- The constant function $u(t)=1$ solves the following IVP: $\begin{cases} {}^{ABC}_{0}D_t^{\alpha}u(t)+u(t)=1, \\ u(0)=1. \end{cases}$
- For $f(t)=t^{\nu-1},(\nu>1)$, the solution of IVP (5)-(6) with $\lambda=1$ and $u_0=0$ is given by

$$u(t)=\frac{1-\alpha}{\alpha}t^{\nu}+\frac{\Gamma(\nu)}{\alpha}t^{\alpha+\nu-1}E_{\nu,\nu+\alpha}(t^{\alpha});$$

- The solution of the following IVP:

$$\begin{cases} {}^{ABC}_{0}D_t^{\alpha}u(t)+u(t)=E_{\alpha}(\gamma t^{\alpha}),\ \gamma\in\mathbb{R}^{-} \\ u(0)=1. \end{cases}$$

is given by:

$$u(t) = \frac{\alpha + \gamma(1-\alpha)}{\alpha + \gamma(2-\alpha)} E_\alpha(\gamma t^\alpha) + \frac{\gamma}{\alpha + \gamma(2-\alpha)} E_\alpha\left(\frac{-\alpha}{2-\alpha} t^\alpha\right).$$

Example 2**.** Consider the boundary value problem (10) − (12) with $f(x,t) = t\sin(\pi x)$. Using Theorem 2.3, the solution $u(x,t)$ is given by

$$u(x,t) = \left(\frac{1-\alpha}{1+\pi^2(1-\alpha)} t + \frac{\alpha}{(1+\pi^2(1-\alpha))^2} t^{\alpha+1} E_{\alpha,\alpha+2}\left(-\frac{\alpha\pi^2}{1+\pi^2(1-\alpha)} t^\alpha\right)\right)\sin(\pi x).$$

The obtained solution is illustrated in Figures (1 − 2) for $T = 1$. Figure 1 demonstrates the solution profile at different times for a fixed value of the order α of the fractional derivative. The effect of the nonhomogeneous term, particularly its *x*-dependence, is clearly seen in the solution profile. Moreover, the solution increases with time and reaches its maximum when $t = T = 1$. The effect of the order α of the fractional derivative is shown in Figure 2 at a fixed time. It shows that as α increases, the solution decreases and reaches its minimum when α = 1, which represents the classical derivative case.

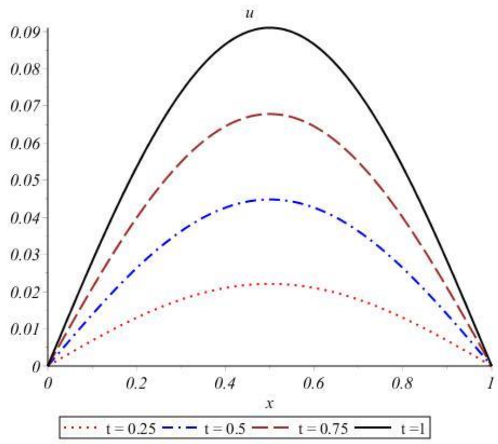

Figure 1. Graphs of $u(x, t)$ at different times for $\alpha = 0.5$.

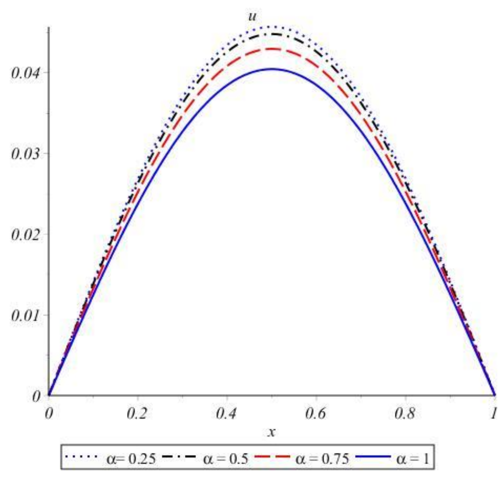

Figure 2. Graphs of $u(x, t)$ at $t = 0.5$ for different values of α.

4. Conclusion

In this paper, we have considered an initial value problem (IVP) for a nonhomogeneous linear fractional differential equation with the Atangana-Baleanu derivative. An explicit solution of the IVP in integral form has been obtained using the Laplace transform. The solution can be also obtained using the successive approximation method as shown in the arXiv version of this paper. A condition relating the initial data and the initial value of the nonhomogeneous term, namely, $f(0) = -\lambda u_0$, has been found to be necessary for the existence of a solution of the IVP. On the other hand, this condition makes it impossible to consider inverse source or inverse initial problems with the Atangana-Baleanu fractional derivative. We have also considered an initial-boundary value problem for a nonhomogeneous linear partial fractional differential equation with the Atangana-Baleanu derivative and a given time and space dependent nonhomogeneous term. A regular classical solution of this direct problem was obtained in the form of a series expansion using an orthogonal basis, which is a set of eigenfunctions of a self-adjoint spectral problem obtained by considering the corresponding homogeneous equation and using the method of separation of variables. The uniform convergence of the series solution was obtained by imposing certain conditions on the given nonhomogeneous term. The uniqueness of the solution can be obtained using the completeness properties of the orthogonal basis used. Finally, a number of examples were presented to illustrate the obtained results.

Acknowledgment

This work is funded by the Research Council (TRC), Oman under the open research grant number ORG/SQU/CBS/13/030.

References

1. Caputo, M. and Fabrizio, M. A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation and Applications*, 2015, **2**, 73-85.
2. Atangana, A. and Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *Thermal Science*, 2016, **20**, 763-769.
3. Algahtani, R.T. Atangana-Baleanu derivative with fractional order applied to the model of groundwater within an unconfined aquifer. *Journal of Nonlinear Sciences and Applications*, 2016, **9**, 3647-3654.
4. Alkahtani, B.S.T. Chuas circuit model with Atangana-Baleanu derivative with fractional order. *Chaos, Solitons and Fractals*. 2016, **89**, 547-551.
5. Alkahtani, B.S.T. and Atangana, A. Controlling the wave movement on the surface of shallow water with the Caputo-Fabrizio derivative with fractional order. *Chaos, Solitons and Fractals*, 2016, **89**, 539-546.
6. Al-Salti, N., Karimov, E. and Sadarangani, K. On a differential equation with Caputo-Fabrizio fractional derivative of order $1 < \beta \leq 2$ and application to mass-spring-damper system. *Progress in Fractional Differentiation and Applications*, 2016, **2**, 257-263.
7. Gómez-Aguilar, J.F., Atangana, A. and Morales-Delgado, V.F. Elecrtical circuits RC, LC, and RL described by Atangana-Baleanu fractional derivatives. *International Journal of Circuit Theory and Applications*, 2017, 1514–1533.
8. Al-Salti, N., Karimov, E. and Kerbal, S. Boundary-value problem for fractional heat equation involving Caputo-Fabrizio derivative. *New Trend in Mathematical Sciences*, 2016, **4**, 79-89.
9. Atangana, A. Non-validity of index law in fractional calculus: A fractional differential operator with Markovian and non-Markovian properties, *Physica A*, 2018, **505**, 688-706.
10. Atangana, A. and Gómez -Aguilar, J.F. Decolonisation of fractional calculus rules: Breaking commutativity and associativity to capture more natural phenomena, *The European Physical Journal Plus*, 2018, **133**, 1-22.
11. Atangana, A. and Koca, I. Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order. *Chaos, Solitons and Fractals*, 2016, **89**, 446-454.
12. Caputo, M. and Fabrizio, M. Applications of new time and spatial fractional derivatives with exponential kernels. *Progress in Fractional Differentiation and Applications*, 2016, **1**, 1-11.
13. Djida, J.D., Atangana, A. and Area, I. Numerical computation of a fractional derivative with non-local and non-singular kernel. *Mathematical Modelling of Natural Phenomena*, 2017, **12**, 4-13.
14. Jefain, O. and Algahtani, J. Comparing the Atangana-Baleanu and Caputo-Fabrizio derivative with fractional order: Allen Cahn model. *Chaos, Solitons and Fractals*, 2016, **89**, 552-559.
15. Losada, J. and Nieto, J. Properties of a new fractional derivative without singular kernel. *Progress in Fractional Differentiation and Applications*, 2015, **2**, 87-92.
16. Sheikh, N.A., Ali, F., Saqib, M., Khan, I., Jan, S.A.A., Alshomrani, A.S. and Alghamdi, M.S. Comparison and analysis of the Atangana-Baleanu and Caputo-Fabrizio fractional derivatives for generalized Casson fluid model with heat generation and chemical reaction. *Results in Physics*, 2017, **7**, 789-800.
17. Tateishi, A.A., Haroldo V. Ribeiro, H.V. and Lenzi, E. K. The Role of Fractional Time-Derivative Operators on Anomalous Diffusion, *Frontiers in Physics*, 2017, **5**, 1-9.

18. Podlubny, I. Fractional differential equations. Academic Press Inc., San Diego, CA, 1999.
19. Al-Musalhi, F., Al-Salti, N. and Karimov, E., Karimov, Initial and boundary value problems for fractional differential equations involving Atangana-Baleanu derivative, 2017, arxiv: 1706.00740.
20. Al-Refai, M. Comparison principles for differential equations involving Caputo fractional derivative with Mittag-Leffler non-singular kernel, *Electronic Journal of Differential Equations*, 2018,2018, 1-10.
21. Baleanu, D. and Fernandez, A. On some new properties of fractional derivatives with Mittag-Leffler kernel. *Communications in Nonlinear Science and Numerical Simulation*, 2018,**59**, 444-462.
22. Djida, J.D., Mophou, G.M and Area, I. Optimal control of diffusion equation with fractional time derivative with nonlocal and nonsingular Mittag-Leffler kernel, *Journal of Optimization Theory and Applications* (2018). https://doi.org/10.1007/s10957-018-1305-6.
23. Moiseev, E.I. The basis property of systems of sines and cosines. *Doklady Akademii Nauk SSSR*, 1984,**275**, 794–798.

Received 23 March 2018
Accepted 27 May 2018