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Inequalities Concerning the Growth of Polynomials

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ABSTRACT: In this paper we consider a polynomial $P(z)$ having no zeros in the disk $|z|<1$. We investigate the dependence of $max_{|z|=R>1}|P(z)|$ on $max_{|z|=1}|P(z)|$ and obtain a refinement of a famous result due to Rivilin ([5], [7]). Our results not only generalize some polynomial inequalities but also refine a result by Aziz [1].

Keywords: Growth of polynomials, Maximum modulus, Inequalities.

علاقة المتراجحات بتزايد الدوال

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الملخص: نفترض في هذه الورقة الدالة $P(z)$ التي ليس لها أصفار في القرص $|z|<1$. سنبحث عن علاقة المقدار $max_{|z|=R>1}|P(z)|$ على $max_{|z|=1}|P(z)|$ والحصول على تنظيم لنتيجة معروفة تعود إلى أنكيني وريفيلين. إن نتائجنا ليست تعميما لبعض متراجحات الدوال فحسب ولكنها أيضا تنظيما لنتيجة عزيز [1].

الكلمات المفتاحية: تزايد الدوال ، نموذج الحد الأقصى، المتراجحات.

1. Introduction

Let $P(z)$ be a polynomial of degree n. Then ([5] or [6], p. 347), for a fixed $R>1$, we have

$$Max_{|z|=R}|P(z)|\le R^n Max_{|z|=1}|P(z)|. \tag{1}$$

Equality in (1) holds for the polynomial $P(z)=\alpha z^n$.

It was shown by Rivilin ([5] , [7]) that if $P(z)$ is a polynomial of degree n having no zeros on $|z|<1$, then (1) can be replaced by

$$Max_{|z|=R}|P(z)|\le\left(\frac{R^n+1}{2}\right)Max_{|z|=1}|P(z)|. \tag{2}$$

Inequality (2) is sharp and equality holds for $P(z)=\alpha+\beta z^n,\ |\alpha|=|\beta|$.

Aziz [1] has further improved and generalized inequality (2) by proving the following result:

Theorem A. If $P(z)$ is a polynomial of degree n which does not vanish in the disk $|z|<k$ where $k\ge 1$, then

$$Max_{|z|=R}|P(z)|\le\left(\frac{R^n+1}{2}\right)Max_{|z|=1}|P(z)|-\left(\frac{R^n-1}{2}\right)Min_{|z|=1}|P(z)|. \tag{3}$$

The result is best possible and equality holds for the polynomial $P(z)=\alpha z^n+\beta k^n,\ |\alpha|=|\beta|=1, k\ge 1$.

As a generalization of inequality (2), Aziz [1] conjectured the following results.

Conjectured Results. If $P(z)$ is a polynomial of degree n which does not vanish in the disk $|z|< k$, then

$$Max_{|z|=r}|P(z)| \geq \frac{r^n + k^n}{1+k^n} Max_{|z|=1}|P(z)|, k^2 < r < 1, k < 1 \tag{4}$$

and

$$Max_{|z|=R}|P(z)| \leq \frac{R^n + k^n}{1+k^n} Max_{|z|=1}|P(z)|, R > k^2, k > 1. \tag{5}$$

In an attempt to answer inequality (4), Dewan and Hans [4] proved the following partial results.

Theorem B. If $P(z)$ is a polynomial of degree n, which does not vanish in $|z|< k, k<1,$ then for $0<k<r<\lambda \leq 1,$

$$M(p,r) \geq \frac{r^n + k^n}{\lambda^n + k^n} M(p,\lambda),$$

provided $|p'(z)|$ and $|q'(z)|$ attain the maximum at the same point on $|z|=1$, where

$$q(z) = z^n \overline{P(\frac{1}{\bar{z}})} \text{ and } Max_{|z|=r}|P(z)| = M(p,r), M(p,\lambda) = Max_{|z|=\lambda}|P(z)|.$$

The result is best possible and equality holds for $p(z) = z^n + k^n$.

Theorem C. If $P(z)$ is a polynomial of degree n, which does not vanish in $|z|< k, k<1,$ then for $0<k<r<1$

$$M(p,r) \geq \left(\frac{r^n + k^n}{1+k^n}\right) M(p,1) + \left(\frac{1-r^n}{1+k^n}\right) m(p,k) \tag{6}$$

provided $|p'(z)|$ and $|q'(z)|$ attain the maximum at the same point on $|z|=1$, where

$$q(z) = z^n \overline{P(\frac{1}{\bar{z}})} \quad \text{and} \quad m(p,k) = Min_{|z|=k}|P(z)|.$$

The result is best possible and equality in (6) holds for $P(z) = z^n + k^n$.

In this paper we shall first present the following interesting refinement of Theorem A.

Theorem 1. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ...+ a_1 z + a_0,$$

be a polynomial of degree n which does not vanish in $|z|< k, k \geq 1$. Then for $R>1,$

$$Max_{|z|=R}|P(z)| \leq \frac{\rho}{\rho+1}\left(R^n+1\right) Max_{|z|=1}|P(z)| - \frac{1}{\rho+1}\left(R^n-\rho\right) Min_{|z|=1}|P(z)| \tag{7}$$

where

$$\rho = \frac{|a_0| + R(|a_n|+m)}{|a_0|R + (|a_n|+m)} \tag{8}$$

and $m = Min_{|z|=1}|P(z)|.$

The result is sharp and equality in (7) holds for $P(z) = \frac{\alpha + \beta z^n}{2}, |\alpha| = |\beta| = 1.$

Remark 1.1. Here we have replaced k by ρ simply not to confuse it with the region for which $P(z)$ does not vanish. Now

$$\rho=\frac{R(|a_n|+m)+|a_0|}{R|a_0|+|a_n|+m}<1$$

It is easy to verify the above inquality for $\rho<1$ if

$$\frac{|a_0|}{|a_n|+m}>1.$$

To show it holds, let $m=Min_{|z|=1}|P(z)|$ then $m\le|P(z)|$ for $|z|=1,$ so that $m|\alpha z^n|<|P(z)|$ where α is any real or complex number with $|\alpha|<1.$ Since $P(z)$ does not vanish in $|z|<1$ the polynomial

$$F(z)=P(z)+\alpha m z^n=(a_n+\alpha m)z^n+.....+a_0$$

does not vanish in $|z|<1.$ Therefore, $\left|\frac{a_0}{a_n+\alpha m}\right|>1$ or $\frac{|a_0|}{|a_n+\alpha m|}>1$, for every α with $|\alpha|<1.$ Choosing argument of α such that

$$|a_n+\alpha m|=|a_n|+|\alpha|m$$

we get

$$|a_0|>|a_n|+|\alpha|m,|\alpha|<1.$$

Letting $|\alpha|\to 1$ it follows that

$$|a_0|\ge|a_n|+m.$$

Now it is easy to verify that for $\rho<1$,

$$\frac{\rho}{\rho+1}<\frac{1}{2}\quad \text{SO}\quad (R^n+1)\frac{\rho}{\rho+1}<\frac{R^n+1}{2}$$

and

$$\frac{R^n-1}{2}=\frac{R^n}{2}-\frac{1}{2}<\frac{R^n}{\rho+1}-\frac{\rho}{\rho+1}$$

which is true. This shows Theorem 1 is an improvement of Theorem A.

As an application of Theorem 1, we next establish the following result which, in a way, is similar to inequality (6).

Theorem 2. If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n which does not vanish in $|z|\le 1$, then for $0\le r<1$, we have

$$\frac{\rho}{\rho+1}\frac{(r^n+1)}{r^n}Max_{|z|=r}|P(z)|-\frac{1}{\rho+1}\frac{(1-\rho r^n)}{r^n}Min_{|z|=r}|P(z)|\ge Max_{|z|=1}|P(z)| \qquad (9)$$

and
$$\rho=\frac{|a_0|+R(|a_n|+m)}{|a_0|R+(|a_n|+m)}.$$

The result is best possible and equality in (9) holds for the polynomial $P(z)=\alpha z^n+\beta$, where $|\alpha|=|\beta|=1.$

Lemmas

For the proof of Theorem 1, we need the following Lemmas. The first Lemma is due to Dubinin [3, Theorem 5].

Lemma 1. If

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0,$$

is a polynomial of degree n which does not vanish in $|z|<1$, then for every $R\ge 1$

$$|P(Rz)|=\frac{|a_0|+R|a_n|}{R|a_0|+|a_n|}|Q(Rz)|,|z|=1 \qquad (10)$$

where

$$Q(z) = z^n \overline{p(1/\bar{z})}.$$

Equality is attained for the polynomial $P(z)$ whose zeros lie on the unit circle $|z|=1$.
Our next lemma is due to Aziz and Mohammad [2].

Lemma 2. If $P(z)$ is a polynomial of degree n, then for all $R \geq 1$ and $0 \leq \theta < 2\pi$

$$|P(Re^{i\theta})| + |Q(Re^{i\theta})| \leq (R^n + 1) Max_{|z|=1} |P(z)|$$

where

$$Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}.$$

2. Proofs of Theorems

Proof of Theorem 1. Let $m = Min_{|z|=1} |P(z)|$. Then $m \leq |P(z)|$ for $|z|=1$ so that $m|\alpha z^n| < |P(z)|$ for $|z|=1$, where α is any real or complex number with $|\alpha|<1$.
Since the polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots\ldots + a_1 z + a_0,$$

does not vanish in $|z|<1$, an application of Rouches Theorem shows that the polynomial $P(z) + \alpha m z^n$ does not vanish in $|z|<1$, so that the polynomial

$$\begin{aligned} F(z) &= P(z) + \alpha m z^n \\ &= (a_n + \alpha m) z^n + a_{n-1} z^{n-1} + \ldots. + a_1 z + a_0, \end{aligned}$$

does not vanish in $|z|<1$ for every α, $|\alpha|<1$.
Let

$$G(z) = z^n \overline{F(\frac{1}{\bar{z}})} = z^n \overline{P(\frac{1}{\bar{z}})} + \bar{\alpha} m = Q(z) + \bar{\alpha} m$$

and

$$Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}.$$

Using Lemma 1, it follows that

$$|F(z)| \leq \frac{|a_0| + R|a_n + \alpha m|}{R|a_0| + |a_n + \alpha m|} |G(z)|, \qquad \text{for } |z| \geq 1.$$

This implies

$$|P(z) + \alpha m z^n| \leq \frac{|a_0| + R|a_n + \alpha m|}{R|a_0| + |a_n + \alpha m|} |Q(z) + \bar{\alpha} m|.$$

We now show that for $|\alpha|<1$ and $R>1$,

$$\frac{|a_0| + R|a_n + \alpha m|}{R|a_0| + |a_n + \alpha m|} < \frac{|a_0| + R(|a_n| + m)}{R|a_0| + (|a_n| + m)} \tag{11}$$

Inequality (11) holds if

$$\begin{aligned} &|a_0|^2 R + R^2 |a_0| (|a_n| + \alpha m) + |a_0| (|a_n| + \alpha m) + R(a_n + \alpha m)(|a_n| + m) \\ &\geq |a_0|^2 R + R^2 |a_0| |a_n + m| + |a_0| (|a_n| + \alpha m) + R|a_n + \alpha m| (|a_n| + m), \end{aligned}$$

which after a simple calculation, yields

$$R^2 |a_0| \{(|a_n| + \alpha m) - |a_n + m|\} \geq |a_0| \{(|a_n| + \alpha m) - (|a_n| + m)\}.$$

This implies $R \geq 1$, which is true. Hence (11) is established.
Taking in particular $z = Re^{i\theta}$, where $R>1$ and $0 \leq \theta < 2\pi$, we get

$$|P(Re^{i\theta}) + \alpha m R^n e^{in\theta}| \leq \rho |Q(Re^{i\theta}) + \bar{\alpha} m| \tag{12}$$

for every α with $|\alpha|<1$. Choosing the argument of α in (12) such that

$$|P(Re^{i\theta}) + \alpha m R^n e^{in\theta}| = |P(Re^{i\theta})| + |\alpha| m R^n,$$

we get

$$|P(Re^{i\theta})| + |\alpha| R^n m \leq \rho |Q(Re^{i\theta})| + \rho |\alpha| m.$$

This gives

$$|P(Re^{i\theta})|+|\alpha|m(R^n-\rho)\le\rho|Q(Re^{i\theta})|,\qquad 0\le\theta<2\pi. \tag{13}$$

Letting $|\alpha|\to 1$ in (13), we get

$$|P(Re^{i\theta})|+(R^n-\rho)m\le\rho|Q(Re^{i\theta})|,\qquad 0\le\theta<2\pi.$$

Adding $\rho|P(Re^{i\theta})|$ on both sides it follows that

$$(\rho+1)|P(Re^{i\theta})|+(R^n-\rho)m\le\rho\{|P(Re^{i\theta})|+|Q(Re^{i\theta})|\},$$

for all $\theta,\ 0\le\theta<2\pi$.

This gives, with the help of Lemma 2, that

$$(\rho+1)|P(Re^{i\theta})|+(R^n-\rho)m\le\rho(R^n+1)Max_{|z|=1}|P(z)| \tag{14}$$

for all $\theta,\ 0\le\theta<2\pi$.

From (14), it follows that

$$|P(Re^{i\theta})|\le\frac{\rho}{\rho+1}(R^n+1)Max_{|z|=1}|P(z)|-\frac{1}{\rho+1}(R^n-\rho)m$$

for all $\theta,\ 0\le\theta<2\pi$, which is equivalent to the desired result. □

Proof of Theorem 2. All the zeros of $P(z)$ lie in $|z|\ge 1$; therefore for $0<r\le 1$, the polynomial $P(rz)$ has all the zeros in $|z|\ge\frac{1}{r}>1$. Applying Theorem 1 to the polynomial $P(rz)$, we obtain

$$Max_{|z|=1}|P(rz)|\le\frac{\rho}{\rho+1}(R^n+1)Max_{|z|=1}|P(rz)|-\frac{1}{\rho+1}(R^n-1)m.$$

Equivalently,

$$Max_{|z|=1}|P(Rz)|\le\frac{\rho}{\rho+1}(R^n+1)Max_{|z|=r}|P(z)|-\frac{1}{\rho+1}(R^n-\rho)Min_{|z|=r}|P(z)|.$$

Taking $R=\frac{1}{r}$, then for $0<r\le 1$, we obtain

$$\frac{\rho}{\rho+1}\frac{(r^n+1)}{r^n}Max_{|z|=r}|P(z)|-\frac{1}{\rho+1}\frac{(1-\rho r^n)}{r^n}Min_{|z|=r}|P(z)|\ge Max_{|z|=1}|P(z)|,$$

which proves Theorem 2. □

3. Conclusion

We generalize some polynomial inequalities and refine a previous result on the dependence of $max_{|z|=R>1}|P(z)|$ on $max_{|z|=1}|P(z)|$, where $P(z)$ is a polynomial having no zeros in the disk $|z|<1$.

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