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Approximation Properties of de la Vallée-Poussin sums in Morrey spaces

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ABSTRACT: In this paper, we investigate the problem of the deviation of a function f from its de la Vallée-Poussin sums of Fourier series in Morrey spaces defined on the unite circle in terms of the best approximation to f. Moreover, approximation properties of de la Vallée-Poussin sums of Faber series in Morrey-Smirnov classes of analytic functions, defined on a simply connected domain bounded by a curve satisfying Dini's smoothness condition are obtained.

Keywords: de la Vallée-Poussin; Faber polynomials; modulus of smoothness; Morrey Smirnov classes.

خواص تقريب الدوال بواسطة مجاميع دي لا فالي بواسون في فضاءات موري

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الملخص: بحثنا في هذه الورقة مسألة انحراف دالة f عن مجاميع دي لا فالي بواسون لمتسلسلة فورييه للدالة f في فضاءات موري المعرفة على دائرة الواحدة في ضوء أفضل تقريب للدالة f. وعلاوة على ذلك، حصلنا على خواص تقريب الدوال من صفوف موري سميرنوف المعرفة على منطقة بسيطة الترابط (الاتصال) ومحاطة بمنحنٍ يحقق شرط ديني للملوسة بمجاميع دي لا فالي بواسون لمتسلسلة فابير.

الكلمات المفتاحية: دي فالي بواسون، كثيرات حدود فابير، معامل الملوسة، صفوف سميرنوف موري.

1. Introduction

Main approximation problems in Lebesgue spaces have been studied by several authors [1, 2]. The approximation of functions of Lebesgue spaces by partial sum of Faber-Laurent series was obtained by Israfilov [3]. These results are generalized to Muckenhoupt weighted Lebesgue's spaces [4]. Approximation properties of Faber series in weighted and non-weighted Orlicz spaces were dealt with by Jafarov and Israfilov [5-7].

The concept of Morrey space, introduced by C. Morrey [8] in 1938, has been studied intensively by various authors and plays an important role in many areas such as applied mathematics, the theory of differential equations, potential theory, and maximal and singular operator theory. Currently there are several investigations relating to the fundamental problems in this space [9-14]. Therefore, the investigation into the approximation of functions by means of Fourier trigonometric series in Morrey spaces is also important in these areas of research.

In the present paper, we investigate the problems of estimating the deviation of functions from their de la Vallée-Poussin sums in Morrey spaces. Similar results in weighted Smirnov spaces and weighted Smirnov Orlicz spaces can be found in the papers [15-17].

2. Notation and Basic definitions

Let G be a finite simply connected domain in the complex plane $\mathbb{C}$ bounded by a rectifiable Jordan curve Γ and $G^- := ext\,\Gamma$. Without loss of generality, we suppose that $0 \in G$. Further, let $\gamma_0 := \{w \in \mathbb{C} : |w| = 1\}$, $D := int\,\gamma_0$, $D^- := ext\,\gamma_0$. We denote by $w = \varphi(z)$ the conformal mapping of G^- onto domain D^- normalized by the conditions

$$\varphi(\infty) = \infty, \qquad \lim_{z\to\infty} \frac{\varphi(z)}{z} > 0,$$

and let ψ be the inverse mapping of φ.

We begin with the following definitions:

Definition 2.1 [18] For $0 \le \alpha \le 2$ and $1 \le p < \infty$, we denote by $L^{p,\alpha}(\Gamma)$ the Morrey space, as the set of locally integrable function f, with a finite norm:

$$\|f\|_{L^{p,\alpha}(\Gamma)} := \left\{ \sup_{B} \frac{1}{|B \cap \Gamma|_{\Gamma}^{1-\frac{\alpha}{2}}} \int_{B\cap\Gamma} |f(z)|^p |dz| \right\}^{\frac{1}{p}} < \infty,$$

where B is an arbitrary disk centered on Γ and $|B \cap \Gamma|_{\Gamma}$ is the linear Lebesgue measure of the set $B \cap \Gamma$.
In the case of $\Gamma = \gamma_0 := \{w \in \mathbb{C}\colon\ |w| = 1\}$ we obtain the space $L^{p,\alpha}(\gamma_0)$.
Under this definition $L^{p,\alpha}(\Gamma)$ is a Banach space. If $\alpha = 2$ then the class $L^{p,2}(\Gamma)$ coincides with the class $L^{p}(\Gamma)$, and for $\alpha = 0$ the class $L^{p,0}(\Gamma)$ coincides with the class $L^{\infty}(\Gamma)$. Moreover, $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ for $0 \le \alpha_1 \le \alpha_2$. Thus, $L^{p,\alpha}(\Gamma) \subset L^{1}(\Gamma)$, $\forall \alpha \in [0,2]$.
For given $f \in L_1(\gamma_0)$, let

$$\frac{a_0}{2} + \sum_{k=0}^{\infty} a_k(f) \cos kx + b_k(f) \sin kx \tag{1}$$

be the Fourier series of f, where $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function f. Further, let

$$S_n(x,f) = \frac{a_0}{2} + \sum_{k=0}^{n} a_k(f) \cos kx + b_k(f) \sin kx$$

be the $n\ th$ partial sums of series (1).
We define the $n - th$ de la Vallée-Poussin sums of series (1) as

$$V_{n,m}(x,f) = \frac{1}{m+1} \sum_{k=n-m}^{n} S_k(x,f)\,, \quad 0 \le m \le n, \qquad m,n = 1,2,3, \dots .$$

Definition 2.2 [19] We define the r modulus of smoothness of a function $f \in L^{p,\alpha}(\gamma_0)$ for $r = 1,2,3,\dots$ by the relation

$$\omega_{p,\alpha}^{r}(f,t) := \sup_{|h|\le t} \|\Delta_h^r(f,.)\|_{L^{p,\alpha}(\gamma_0)}, \qquad t > 0,$$

where

$$\Delta_h^r(f,x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(x+kh).$$

The best approximation to $L^{p,\alpha}(\gamma_0)$ in the class $\mathcal{T}_n$ of trigonometric polynomials of degree not greater than n is defined by

$$E_n(f)_{L^{p,\alpha}(\gamma_0)} := \inf\left\{\|f - T_n\|_{L^{p,\alpha}(\gamma_0)} \colon T_n \in \mathcal{T}_n\right\}.$$

Let $T^* \in \mathcal{T}_n$ be a trigonometric polynomial such that

$$E_n(f)_{L^{p,\alpha}(\gamma_0)} := \|f - T^*\|_{L^{p,\alpha}(\gamma_0)}.$$

If $m,n \in \mathbb{N}$ such that $m \ge n$, then we get

$$E_m(f)_{L^{p,\alpha}(\gamma_0)} \le E_n(f)_{L^{p,\alpha}(\gamma_0)}. \tag{2}$$

Using the boundedness of operator $f \to S_n(.,f)$ in the Morrey spaces $L^{p,\alpha}(\gamma_0)$ we get

$$\|f - S_n(.,f)\|_{L^{p,\alpha}(\gamma_0)} \le \|f - T^*\|_{L^{p,\alpha}(\gamma_0)} + \|T^* - S_n(.,f)\|_{L^{p,\alpha}(\gamma_0)}$$

$$= E_n(f)_{L^{p,\alpha}(\gamma_0)} + \|S_n(.,f - T^*)\|_{L^{p,\alpha}(\gamma_0)} \le E_n(f)_{L^{p,\alpha}(\gamma_0)} + C\|f - T^*\|_{L^{p,\alpha}(\gamma_0)}$$

$$= (C+1)E_n(f)_{L^{p,\alpha}(\gamma_0)} = cE_n(f)_{L^{p,\alpha}(\gamma_0)},$$

where C is a positive constant and $c = C + 1$, i.e. there exists a constant c such the following relation holds

$$\|f - S_n(.,f)\|_{L^{p,\alpha}(\gamma_0)} \le cE_n(f)_{L^{p,\alpha}(\gamma_0)}. \tag{3}$$

Definition 2.3 [9] We define the Morrey-Smirnov classes $E^{p,\alpha}(G)$, $0 \le \alpha \le 2$ and $1 \le p < \infty$, of analytic functions in G as

$$E^{p,\alpha}(G) := \{f \in E^1(G) : f \in L^{p,\alpha}(\Gamma)\}.$$

If we define $\|f\|_{E^{p,\alpha}(\mathrm{G})} := \|f\|_{L^{p,\alpha}(\Gamma)}$, then $E^{p,\alpha}(G)$ becomes a Banach space.

Definition 2.4 [20] A smooth curve $\Gamma\colon \sigma(s)$ is called Dini-smooth if it satisfies the condition

$$\int_0^\delta \frac{\Omega(\sigma'(s),s)}{s} ds < \infty, \qquad \delta > 0,$$

where $\Omega(\sigma'(s),s)$ modulus of continuity of function $\sigma'(s)$. By $\mathcal{D}$ we denote the set of all Dini-smooth curves.

If $\Gamma \in \mathcal{D}$, then [21]

$$0 < c_1 \le |\psi'(w)| \le c_2 < \infty, \;\; 0 < c_3 \le |\varphi'(z)| \le c_4 < \infty \tag{4}$$

for some constants, c_1, c_2, c_3 and $, c_4$.

Hence, if $\Gamma \in \mathcal{D}$ and using (4), then by [9]

$$f \in L^{p,\alpha}(\Gamma) \Leftrightarrow f_0 := f \circ \psi \in L^{p,\alpha}(\gamma_0) \tag{5}$$

and the function $f_0^+\colon D \to \mathbb{C}$ defined by

$$f_0^+(w) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{f_0(\tau)}{\tau - w} d\tau \;,\; w \in D \tag{6}$$

is analytic in D and $f_0^+ \in E^{p,\alpha}(D)$ [9].

If $\Gamma \in \mathcal{D}$ and $r = 1,2,3, \ldots$, we define the $r-$ modulus of smoothness of $f \in L^{p,\alpha}(\Gamma)$ by the relation (see, [9])

$$\Omega^r_{\Gamma,p,\alpha}(f,t) := \omega^r_{p,\alpha}(f_0^+, \mathrm{t}), \;\; t > 0\,. \tag{7}$$

The Faber polynomials $\Phi_k(t)$ of degree k are defined by the relation [22]

$$\frac{\psi'(w)}{\psi(w) - t} = \sum_{k=0}^{\infty} \frac{\Phi_k(t)}{w^{k+1}}, \;\; t \in \mathrm{G}, \;\; w \in D^-. \tag{8}$$

If $f \in E^{p,\alpha}(G)$, then by the definition 2.3, $f \in E^1(G)$ and hence

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(s)}{s-z}ds = \frac{1}{2\pi i}\int_{\gamma_0}\frac{\psi'(w)}{\psi(w)-z}f_0(w)dw \ , z \in G.$$

From the last formula and the relation (8), for every $z \in G$ we have

$$f(z) \sim \sum_{k=0}^{\infty} a_k(f)\Phi_k(z), \ \ z \in G, \tag{9}$$

where

$$a_k(f) := \frac{1}{2\pi i}\int_{\gamma_0}\frac{f_0(w)}{w^{k+1}}dw, \qquad k = 0,1,2,\dots \ .$$

The $n-th$ de la Vallée-Poussin sums of the series (9) are defined as

$$V_{n,m}(x,f) = \frac{1}{m+1}\sum_{k=n-m}^{n} S_k(x,f), \ \ 0 \le m \le n, \qquad m,n = 1,2,3,\dots ,$$

where

$$S_n(z,f) = \sum_{k=0}^{n} a_k(f)\Phi_k(z).$$

We define the operator T as follows:

$$T: E^{p,\alpha}(D) \to E^{p,\alpha}(G)$$

$$T(f)(z) := \frac{1}{2\pi i}\int_{\gamma_0}\frac{f(w)\psi'(w)}{\psi(w)-z}dw, z \in G. \tag{10}$$

In order to prove our main results, we need the following theorems.

Theorem 2.1 [10] If $\Gamma \in \mathcal{D}$, then the operator T defined by (10) is linear, bounded, one to one and onto. Moreover $T(f_0^+) = f$ for $f \in E^{p,\alpha}(G)$.

Theorem 2.2 [9] Let $g \in E^{p,\alpha}(\mathrm{D})$ with $0 < \alpha \le 2$ and $1 < p < \infty$. Then for a given $r = 1,2,3,\dots$ the inequality

$$E_n(g)_{L^{p,\alpha}(\gamma_0)} \le c_5 \ \omega_{p,\alpha}^{r}\left(g, \frac{1}{n+1}\right), n = 1,2,3,..$$

holds with a constant $c_5 > 0$ independent of n.

3. Main Results

In this section, we present the main results.

Theorem 3.1 Let $L^{p,\alpha}(\gamma_0)$ be a Morrey space with $0 < \alpha \le 2$ and $1 < p < \infty$, then there exists a positive constant c_6 such that for any $f \in L^{p,\alpha}(\gamma_0)$, $0 \le m \le n$, $m,n = 1,2,\dots$ the inequality

$$\left\|f - V_{n,m}(.,f)\right\|_{L^{p,\alpha}(\gamma_0)} \le \frac{c_6}{m+1}\sum_{k=n-m}^{n} E_k(f)_{L^{p,\alpha}(\gamma_0)} \tag{11}$$

is true.

Proof. Let us chose the integer j such that $2^j \le m+1 \le 2^{j+1}$. Then

$$f(x) - V_{n,m}(x,f) = \frac{1}{m+1}[f(x) - S_{n-m}(x,f)]$$
$$+\frac{1}{m+1}\left\{\sum_{k=1}^{j}\sum_{i=n-m+2^{k-1}}^{n-m+2^k-1}[f(x) - S_i(x,f)]\right\} + \frac{1}{m+1}\left\{\sum_{k=n-m+2^j}^{n}[f(x) - S_k(x,f)]\right\}.$$

And from this, we get

$$\left\|f-V_{n,m}(.,f)\right\|_{L^{p,\alpha}(\gamma_0)}$$
$$\leq \frac{1}{m+1}\|f-S_{n-m}(.,f)\|_{L^{p,\alpha}(\gamma_0)}+\frac{1}{m+1}\left\{\sum_{k=1}^{j}\sum_{i=n-m+2^{k-1}}^{n-m+2^k-1}\|f-S_i(.,f)\|_{L^{p,\alpha}(\gamma_0)}\right\}$$
$$+\frac{1}{m+1}\left\{\sum_{k=n-m+2^j}^{n}\|f-S_k(.,f)\|_{L^{p,\alpha}(\gamma_0)}\right\}.$$

From the relation (3), we get

$$\left\|f-V_{n,m}(.,f)\right\|_{L^{p,\alpha}(\gamma_0)} \leq \frac{c_7}{m+1}E_{n-m}(f)_{L^{p,\alpha}(\gamma_0)}+\frac{c_8}{m+1}\left\{\sum_{k=1}^{j}\sum_{i=n-m+2^{k-1}}^{n-m+2^k-1}E_i(f)_{L^{p,\alpha}(\gamma_0)}\right\}+\frac{c_9}{m+1}\left\{\sum_{k=n-m+2^j}^{n}E_k(f)_{L^{p,\alpha}(\gamma_0)}\right\}. \tag{12}$$

From (12) and using (2), we get

$$\left\|f-V_{n,m}(.,f)\right\|_{L^{p,\alpha}(\gamma_0)} \leq \frac{c_{10}}{m+1}\left\{E_{n-m}(f)_{X,\omega}+\sum_{k=1}^{j}2^{k-1}E_{n-m+2^{k-1}}(f)_{L^{p,\alpha}(\gamma_0)}\right\}+c_{11}\frac{1}{m+1}\left(m-2^j+1\right)E_{n-m+2^j}(f)_{L^{p,\alpha}(\gamma_0)}. \tag{13}$$

On the other hand, we have

$$\sum_{k=1}^{j}2^{k-1}E_{n-m+2^{k-1}}(f)_{L^{p,\alpha}(\gamma_0)} \leq E_{n-m+1}(f)_{L^{p,\alpha}(\gamma_0)}+2\sum_{k=2}^{j}\sum_{i=n-m+2^{k-2}}^{n-m+2^{k-1}-1}E_i(f)_{L^{p,\alpha}(\gamma_0)} \leq c_{12}\sum_{k=n-m}^{n-m+2^{j-1}}E_k(f)_{L^{p,\alpha}(\gamma_0)}. \tag{14}$$

Since $2^j \leq m+1 < 2^{j+1}$, we get $2^j > m-2^j+1$. Hence

$$\left(m-2^j+1\right)E_{n-m+2^j}(f)_{L^{p,\alpha}(\gamma_0)} \leq \sum_{k=n-m}^{n-m+2^j-1}E_k(f)_{L^{p,\alpha}(\gamma_0)}. \tag{15}$$

From (13), (14) and (15) we obtain

$$\left\|f-V_{n,m}(.,f)\right\|_{L^{p,\alpha}(\gamma_0)}$$
$$\leq \frac{c_{13}}{m+1}\left\{E_{n-m}(f)_{L^{p,\alpha}(\gamma_0)}+\sum_{k=n-m}^{n-m+2^{j-1}}E_k(f)_{L^{p,\alpha}(\gamma_0)}+\sum_{k=n-m}^{n-m+2^j-1}E_k(f)_{L^{p,\alpha}(\gamma_0)}\right\}$$
$$\leq \frac{c_6}{m+1}\sum_{k=n-m}^{n}E_k(f)_{L^{p,\alpha}(\gamma_0)}$$

and the inequality (11) is true.

Corollary 3.1 Let $L^{p,\alpha}(\gamma_0)$ be a Morrey space with $0<\alpha\leq 2$ and $1<p<\infty$, then there exists a positive constant c_{14} such that for any $f\in L^{p,\alpha}(\gamma_0)$, $0\leq m\leq n$, $m,n=1,2,\ldots$ the inequality

$$\left\|f-V_{n,m}(.,f)\right\|_{L^{p,\alpha}(\gamma_0)} \leq \frac{c_{14}}{m+1}\sum_{k=n-m}^{n}\omega_{p,\alpha}^{r}\left(f,\frac{1}{k+1}\right) \tag{16}$$

is true.

Proof. From Theorem 3.1 we have

$$\left\|f-V_{n,m}(.,f)\right\|_{L^{p,\alpha}(\gamma_0)} \leq \frac{c_6}{m+1}\sum_{k=n-m}^{n}E_k(f)_{L^{p,\alpha}(\gamma_0)}$$

and from Theorem 2.2 we get

$$E_n(f)_{L^{p,\alpha}(\gamma_0)} \leq c_5\,\omega_{p,\alpha}^{r}\left(f,\frac{1}{n+1}\right),\ n=1,2,3,\ldots\ .$$

We reach

$$\left\|f - V_{n,m}(.,f)\right\|_{L^{p,\alpha}(\gamma_0)} \le \frac{c_{14}}{m+1}\sum_{k=n-m}^{n} \omega_{p,\alpha}^{r}\left(f,\frac{1}{k+1}\right), n = 1,2,\dots .$$

Theorem 3.2 Let G be a simply connected domain in the complex plane, bounded by a curve $\Gamma \in \mathcal{D}$. If $f \in E^{p,\alpha}(G)$ with $0 < \alpha \le 2$ and $1 < p < \infty$, then for every $0 \le m \le n,\ n,m \in \mathbb{N}$ the estimate

$$\|f - V_n(.,f)\|_{L^{p,\alpha}(\Gamma)} \le c_{15}\sum_{k=n-m}^{n} \Omega_{\Gamma,p,\alpha}^{r}\left(f,\frac{1}{k+1}\right)$$

holds, where c_{15} is a positive constant.

Proof. Since $f \in E^{p,\alpha}(G)$ and Γ is a Dini – smooth curve, then the boundary function of f belongs to $L^{p,\alpha}(\Gamma)$ and from the relation (5) we get $f_0 \in L^{p,\alpha}(\gamma_0)$, and the function f_0^+ which defined by (6) belongs to $E^{p,\alpha}(D)$. Since $E^{p,\alpha}(D) \subset E^1(D)$, we obtain $f_0^+ \in E^1(D)$ which has the following Taylor expansion

$$f_0^+(w) = \sum_{k=0}^{\infty} a_k(f_0^+)w^k,\ \ w \in D. \tag{17}$$

Let $\{c_k\}$ be the Fourier coefficients of the boundary function of f_0^+, then by [23] we get $c_k = a_k(f_0^+)$ for $k \ge 0$ and $c_k = 0$ for $k < 0$, and then by substitution in (17) we obtain

$$f_0^+(w) = \sum_{k=0}^{\infty} c_k w^k,\ \ w \in D.$$

Note that for the function $f \in E^{p,\alpha}(G)$ the following Faber series holds

$$f(z) \sim \sum_{k=0}^{\infty} a_k(f)\Phi_k(z),\ \ z \in G,$$

where $a_k(f), k = 0,1,2,\dots$ are the Taylor coefficients of the function f_0^+, and by Theorem 2.1 we obtain

$$T\left(\sum_{k=0}^{n} a_k(f_0^+)w^k\right) = \sum_{k=0}^{n} a_k(f)\Phi_k(z)$$

and

$$T\left(V_{n,m}(\mathrm{w},f_0^+)\right) = V_{n,m}(z,f), 0 \le m \le n, \qquad n,m = 0,1,2,\dots .$$

Hence, using the boundedness of operator T defined by (10) and the relation (11), we reach

$$\left\|f - V_{n,m}(.,f)\right\|_{L^{p,\alpha}(\Gamma)} = \left\|T(f_0^+) - T\left(V_{n,m}(.,f_0^+)\right)\right\|_{L^{p,\alpha}(\Gamma)} \le c_{16}\left\|f_0^+ - V_{n,m}(.,f_0^+)\right\|_{L^{p,\alpha}(\gamma_0)}$$
$$\le \frac{c_{17}}{m+1}\sum_{k=n-m}^{n} E_k(f_0^+)_{L^{p,\alpha}(\gamma_0)}.$$

Using the Theorem 2.2 we get

$$\left\|f - V_{n,m}(.,f)\right\|_{L^{p,\alpha}(\Gamma)} \le \frac{c_{15}}{m+1}\sum_{k=n-m}^{n} \omega_{p,\alpha}^{r}\left(f_0^+,\frac{1}{k+1}\right).$$

And by the relation (7) we reach

$$\left\|f - V_{n,m}(.,f)\right\|_{L^{p,\alpha}(\Gamma)} \le \frac{c_{15}}{m+1}\sum_{k=n-m}^{n} \Omega_{\Gamma,p,\alpha}^{r}\left(f,\frac{1}{k+1}\right).$$

Consequently, we have proved the Theorem 3.2.

4. Conclusion

A method was developed to estimate the deviation of functions from their de la Vallée-Poussin sums in Morrey spaces in terms of the best approximation.

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