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Flow Past A Rotating Circular Cylinder and A Rotlet Using the Finite-Difference Method

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الانسياب على اسطوانة دائرية دوارة ودوامة باستخدام طريقة الفرق المحدودة

طيفور البشير

خلاصة : يختص هذا البحث بدراسة الانسياب المتولد من دوامة في وجود اسطوانة دائرية (دوارة) باستخدام بعض التحويلات علـى المعادلات الأساسية الحاكمة للمسألة مع طريقة الفروق المحدودة. هذا وقد تم التوصل إلى حل عددي يتطابق بشكل ممتاز مع النتائج التحليلية المتاحة ، ثم تم استخدام الطريقة نفسها للحصول على نتائج جديدة لمسألة ليس لها حل تحليلي .

ABSTRACT : In this paper the flow generated by a rotlet in the presence of a circular cylinder is considered. We introduce a transformation which simplifies the equations and boundary conditions. We use the finite-difference method to obtain results in excellent agreement with all the available analytical results. Results are presented for Reynolds numbers, based on the diameter of the cylinder, in the range $0 \le \mathrm{Re} \le 20$ and the rotational parameter, α, in the range $0 \le \alpha \le 3$ and strength of rotlet, β, in the range $0 \le \beta \le 3$. The results are found to be applicable over a wide range of values of α and β. The calculated values of the drag, lift and moment coefficients and the general nature of the streamline patterns are in good agreement with analytical results . The method is then utilized to obtain new results for which no analytical solution is possible.

KEYWORDS: Slow flow, cylinder, rotlet.

A study has been made of the flow generated by rotating a circular cylinder within a uniform stream of viscous fluid in the presence of a line rotlet. With the origin of the coordinates coinciding with the centre of the cylinder, the polar coordinate system (r, ϑ) has the boundary of the cylinder at $r = a$ and the position of the rotlet at $(c^*, 3\pi/2)$, as shown in Figure 1. There are four basic parameters that occur in this problem, namely the Reynolds number, defined as $\mathrm{Re} = 2a\,U/v$, the rotational parameter $\alpha = a\omega_0/U$, the non-dimensional length $c = c^*/a$ and the non-dimensional strength of the rotlet $\beta = \Gamma/Ua$, where v is the coefficient of kinematic viscosity of the fluid, U the uperturbed main stream speed (the stream at infinity is assumed to flow parallel to the x-axis in the positive x direction), ω_0 the angular velocity of the cylinder and Γ the strength of the rotlet. At zero Reynolds number the governing equation is the biharmonic equation. In the absence of rotlet no solution of this equation, which satisfies both the boundary conditions on the cylinder and at infinity, is possible. This arises because it is necessary to maintain in the solution a term of the form $r\ln(r)\sin(\vartheta)$ in order for both the velocity components on the rotating cylinder to be satisfied. However, although a solution retaining such a term is obviously valid at points not too far away from the cylinder such a solution deteriorates as one moves further away. As such this solution fails to satisfy the boundary condition at infinity, with the exact multiple of the unwanted term at infinity remaining undetermined. This unknown constant can be established by treating the solution of the problem as the approximation to the inner flow past the circular cylinder as the Reynolds number tends to zero. Then matching with the solution as obtained from the outer region where the first approximation to the Navier Stokes equations are the Oseen equations. Full details regarding these expansions and matching procedure can be found in Proudman and Pearson (1957). Basically what this means is that the uniform flow past a circular cylinder is not a well- posed Stokes problem. In fact, at zero Reynolds number the problem is singular.

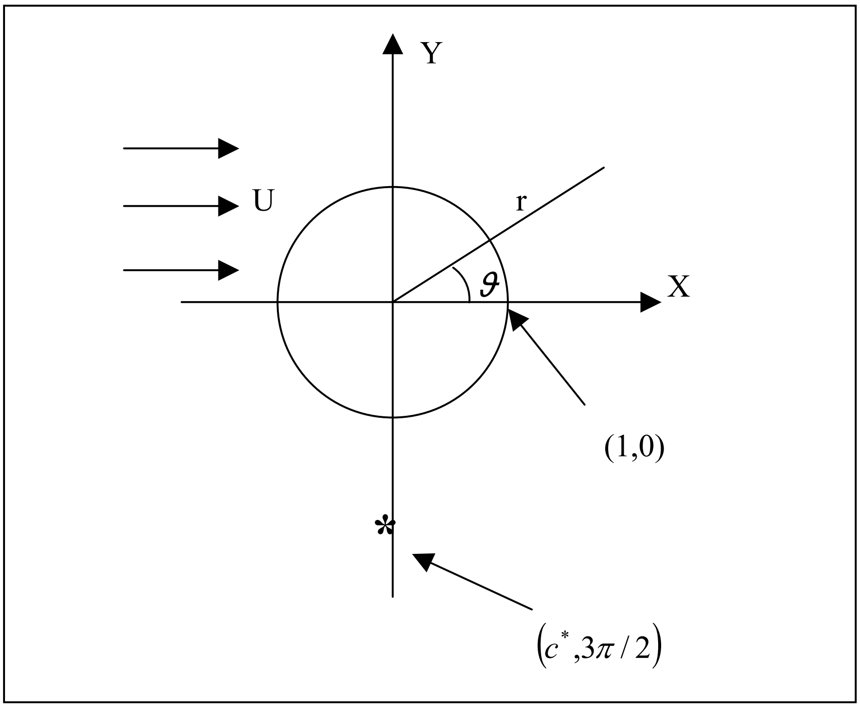

Figure 1. The geometry of the circular cylinder and the position of the rotlet

However, the introduction of the rotlet , strength $\Gamma = Uc^*$ so that $\beta = c$, into the flow field at any given distance c^* along the negative y-axis, allows a solution to be obtained. Analytical solutions to this problem have been obtained by Dorrepaal *et al* (1984). Their work examines the flows generated in a fluid by the introduction of a line singularity, such as a stokeslet or rotlet, in the presence of a circular cylinder and shows that a phenomenon analogous to the Stokes paradox exists in those flows with a uniform stream far from the cylinder. As a consequence, the uniform streaming flow past a circular cylinder, when a line stokeslet or rotlet of certain strength is present, is a well-posed problem in Stokes flow.

The solutions by Dorrepaal *et al*. (1984) employed an image type approach, plus a clever and simplistic deduction, which enabled the result to be constructed devoid of most of the analysis. However, the present work has established the same solution by using a Fourier Series approach and has confirmed this numerically by the application of a modification to the Boundary Element Method. The latter appears to provide an approach for the solution of the biharmonic equation, which requires only the position of the singularity, plus the physical values of the drag, lift and the moment on the circular cylinder to be known. In addition, it seems capable of being extended to accommodate the presence of several different bodies as well as allowing more complex shapes for which an analytical solution is impossible. It is intended to show that the presence of a rotlet in a uniform flow at non-zero Reynolds number allows an otherwise singular problem to become well-posed as the Reynolds number becomes zero.

The main aim of this paper is to solve numerically the Navier-Stokes equations for steady, two-dimensional, incompressible viscous fluid flow past a rotating circular cylinder of radius a in the presence of a rotlet of strength Γ which is located at the point $(r, \vartheta) = (c^*, 3\pi/2), c* > a$. At large distances from the cylinder it is assumed that there is a uniform flow of speed U which is parallel to the negative x-axis. Initially the strength of a rotlet is set to zero and the problem solved with

Reynolds numbers 5 and 20. The results are in very good agreement with those obtained by Dennis and Chang (1970) and Fornberg (1980). Using this as an initial estimate of the solution when a line singularity is present an iterative technique is developed in order to solve the problem when a rotlet, at $\left(c^*, 3\pi/2\right)$, of small strength is introduced into the flow. As the drag, lift and moment on the circular cylinder are the most important physical quantities, as well as being easy to measure experimentally, see Fornberg (1980), particular attention has been paid to these quantities in this work.

Having established the numerical procedure the Reynolds number is decreased towards zero whilst continuing to solve over a range of non-dimensional strengths of the rotlet, namely $0 \le \beta \le 3.5$. The question that needs to be answered is whether as Re approaches zero the value of the parameter β would tend to that unique value obtained by Dorrepaal *et al.* (1984) in their analytical solution.

Basic Equations and Boundary Conditions

The origin of the coordinate system is fixed at the centre of the circular cylinder of radius a and the positive x-axis taken in the same direction as that of the uniform flow at large distances from the cylinder. Polar coordinates (r, ϑ) are chosen such that $\vartheta = 0$ coincides with the positive x-axis,

$$x = r\cos(\vartheta) \text{ and } \ y = r\sin(\vartheta) \tag{2.1}$$

A line rotlet of strength Γ is located at the point $r = c^*, \vartheta = 3\pi/2$, where $c^* > a$. The steady flow of an incompressible fluid in a fixed two-dimensional Cartesian frame of reference can be described by the equations,

$$(\boldsymbol{u}\,.\nabla)\boldsymbol{u} = \frac{-1}{\rho}\nabla \mathrm{p} + \nu\nabla^2\boldsymbol{u}, \tag{2.2}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.3}$$

where $\boldsymbol{u}, \rho, p$ and ν are the velocity, density, pressure and the kinematics viscosity of the fluid, respectively. Applying the curl operator to the two-dimensional equation (2.2) produces

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} = \nu\nabla^2\boldsymbol{\omega}, \tag{2.4}$$

where $\boldsymbol{\omega} = \nabla \mathrm{X}\, \boldsymbol{u}$.

In two-dimensional motion the polar resolutes of u can be expressed in terms of the streamfunction Ψ by

$$v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \vartheta}\ , \quad v_\vartheta = -\frac{\partial \Psi}{\partial r} \tag{2.5}$$

where v_r and v_ϑ are the velocity components in the r and ϑ directions, respectively. By introducing the dimensionless variables

$$\boldsymbol{\chi}' = \boldsymbol{\chi}/a \ \ , \quad \boldsymbol{u}' = \boldsymbol{u}/\boldsymbol{U}\ , \quad \Psi' = \Psi/\boldsymbol{U}a, \text{ and } \boldsymbol{\omega}' = \boldsymbol{\omega}\, a/U, \tag{2.6}$$

then the governing equations in non-dimensional form become

$$\nabla^2\omega' = -\mathrm{Re}\frac{\partial(\Psi',\omega')}{\partial(x',y')} \tag{2.7}$$

$$\nabla^2\Psi' = -\omega', \tag{2.8}$$

where ω' and Ψ' are the non-dimensional scalar vorticity and streamfunction, respectively. For convenience the accent will from now on be ignored. It is required to solve equations (2.7) and (2.8) subject to the no-slip conditions imposed by the circular cylinder, namely

$$\Psi = 0, \quad \frac{\partial\Psi}{\partial r} = -\alpha \qquad \text{on } r = 1,\ 0 \le \vartheta < 2\pi, \tag{2.9}$$

and the boundary conditions at large distances from the cylinder

$$\frac{\partial\Psi}{\partial \boldsymbol{r}} \to -\sin(\vartheta), \quad \frac{1}{\boldsymbol{r}}\frac{\partial\Psi}{\partial\vartheta} \to \cos(\vartheta) \quad \text{as } r \to \infty,\ 0 \le \vartheta < 2\pi, \tag{2.10}$$

In the presence of a line rotlet the streamfunction behaves as

$$\Psi \cong -\beta \ln R_1 \qquad \text{as } R_1 \to 0, \tag{2.11}$$

where R_1 measures the distance from the rotlet and is thus given by

$$R_1 = \left(r^2 + c^2 + 2rc\sin(\vartheta)\right)^{1/2}, \tag{2.12}$$

where $(c, 3\pi/2)$ is the position of the rotlet. In the above definition of the streamfunction the signs appearing in the expressions in (2.5) are the opposite to those given by Dorrepaal *et al*. (1984) but follow those adopted by Fornberg (1980) since it is a comparison with their results at non-zero Reynolds number that is to be undertaken. For numerical convenience the perturbation streamfunction Ψ is introduced as

$$\psi = \Psi - y - \beta V, \tag{2.13}$$

where $V = \ln\left(r^2 + c^2 + 2rc\sin(\vartheta)\right)^{1/2}$ with $c = c^*/a$ and $\beta = \Gamma/(Ua)$ being two non-dimensional parameters. Expansion (2.13) has been taken so that $\psi \to 0$ as $r \to \infty$. If the parameter $\beta = 0$ the problem reduces to that solved by Fornberg (1980). However, with the Reynolds number equal to zero and the parameter $\beta = c$ the situation is that studied by Dorrepaal *et al*. (1984) except that the geometry in the present case corresponds to a rotation through $\pi/2$ of their flow pattern. Hence, their stream is flowing along the negative y-axis with their rotlet at $(c^*, 0)$, whereas in the present geometry the stream flows along the negative x-axis with the rotlet at $(c^*, 3\pi/2)$. Using expression (2.13) in equations (2.7) and (2.8) gives

$$\begin{aligned}\frac{\partial^2\omega}{\partial r^2} + \frac{1\partial\omega}{r\partial r} + \frac{1\partial^2\omega}{r^2\partial\vartheta^2} = &-\frac{\mathrm{Re}}{2r}\left(\frac{\partial\varphi}{\partial r}\frac{\partial\omega}{\partial\vartheta} - \frac{\partial\varphi}{\partial\vartheta}\frac{\partial\omega}{\partial r}\right) - \frac{\mathrm{Re}}{2r}\left[\frac{\partial\omega}{\partial\vartheta}\left(\sin(\vartheta) + \beta\frac{r + c\sin(\vartheta)}{r^2 + c^2 + 2rc\sin(\vartheta)}\right)\right] \\ &+ \frac{\mathrm{Re}}{2r}\left[\frac{\partial\omega}{\partial r}\left(r\cos(\vartheta) + \beta\frac{rc\cos(\vartheta)}{r^2 + c^2 + 2rc\sin(\vartheta)}\right)\right]\end{aligned} \tag{2.14}$$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1\partial\psi}{r\partial r} + \frac{1}{r^2}\frac{\partial\psi^2}{\partial\vartheta^2} = -\omega, \tag{2.15}$$

respectively. The boundary conditions (2.9) and (2.10) are then expressed in the form

$$\psi = -\sin(\vartheta) - \beta\ln\left(1 + c^2 + c\sin(\vartheta)\right)^{\frac{1}{2}} \text{ on } r = 1, 0 < \vartheta < 2\pi\,, \tag{2. 16}$$

$$\frac{\partial\psi}{\partial r} = -\alpha - \sin(\vartheta) - \beta\left(\frac{1 + c\sin(\vartheta)}{1 + c^2 + 2c\sin(\vartheta)}\right) \text{ on } r = 1, 0 < \vartheta < 2\pi\,, \tag{2.17}$$

$$\frac{\partial\varphi}{\partial r} = \frac{1\partial\varphi}{r\partial\vartheta} \to 0 \text{ as } r \to \infty, \quad 0 \le \vartheta < 2\pi\,. \tag{2.18}$$

Filon (1926) showed that in the absence of any rotlet the asymptotic form for the dimensional streamfunction at large distances from the cylinder and outside the wake region is given by

$$\psi \sim r\sin(\vartheta) + \frac{C_L\ln(r/a)}{2\pi} + \frac{C_D(\vartheta - \pi)}{2\pi}, \qquad \text{as} \qquad r \to \infty, \qquad 0 < \vartheta < 2\pi\,, \tag{2.19}$$

where $C_L = L/\left(\rho U^2 a\right)$ and $C_D = D/\left(\rho U^2 a\right)$, L and D being the lift and drag on the cylinder. Imai (1951) found higher–order terms in this streamsfunction expansion and showed how the coefficients relate to the moment on the cylinder. In the case of zero Reynolds number no solution of the equation $\nabla^4\Psi = 0$ which matches the free stream condition at infinity and satisfies the boundary condition on $r = 1$ is possible. The solution, which satisfies the no slip condition on the cylinder and tends to infinity most slowly as $r \to \infty$ is

$$\psi \sim A\sin(\vartheta)\left[r\ln(r) - r/2 + 1/(2r)\right] \tag{2.20}$$

This has been obtained by discarding the term involving r^3. The non-dimensional drag is directly related to the coefficient *A*, via the expression $4\pi A$, but the solution suffers from the defect that it does not determine the value of the constant *A*. The neglected inertial terms are of the order $\left((A)^2\ln(r)\right)/r^2$ whilst the viscous forces are of the order $A/\left(\mathrm{Re}\,r^3\right)$ and these terms are of comparable order when $A\,\mathrm{Re}\,r(\ln(r))/a \cong 0(1)$. Hence, the Stokes solution should not be expected to be valid beyond a value given by this expression. That is why the Stokes solution may be an adequate representation of the fluid flow relatively close to the cylinder but cannot represent a uniform approximation to the total velocity distribution. However, it is possible to write the Stokes solution in the form

$$\psi = A\left[\left(-r\ln(f(\mathrm{Re})) + r\ln(rf(\mathrm{Re}))\right) - r/2 + 1/(2r)\right]\sin(\vartheta), \tag{2.21}$$

where $f(\mathrm{Re})$ is an arbitrary function of Re. For $f(\mathrm{Re}) << 1$ and $rf(\mathrm{Re})$ of order unity, the dominant term is $-A(\ln(f(\mathrm{Re})))r\sin(\vartheta)$. If this is to represent the external flow, namely a uniform stream $Ur\sin(\vartheta)$, then one must set $A = -1/\ln(f(\mathrm{Re}))$. By substituting this value of A and $r = 1/f(\mathrm{Re})$ into $A\,\mathrm{Re}\,r(\ln(r))/a$, one obtains

$$\mathrm{Re}/f(\mathrm{Re}) \sim 0(1)\,. \tag{2.22}$$

So for $f(\mathrm{Re}) = \mathrm{Re}$ the Stokes solution leads to a uniform stream of order unity in that region where the Stokes equation ceases to be valid. This suggests that for small Reynolds numbers the external uniform stream condition is reached before the Stokes approximation breaks down, so that the Stokes flow represents the solution in the inner region near the cylinder but the outer flow requires the introduction on an Oseen variable and the procedure involves the matching of the inner and outer expansions. This matching process leads to the obvious linkage between the coefficients of the terms in Oseen's expansion in the outer region, closely related to Filon's expansion with those from the Stokes solution in the inner region. Hence, the appropriate value of A can be obtained, as already indicated from the solution in the outer region. Any similar term in a Stokes expansion, such as $Br\ln(r)\cos(\vartheta)$ term in the streamfunction expansion, has its coefficient similarly related to the lift. Solutions of the Stokes equation represented by the form $f(r)\sin(\vartheta)$ or $g(r)\sin(\vartheta)$ fail to produce any moment contribution, but it can easily be seen that the solution of $\nabla^4\Psi = 0$ which has a non-zero moment arises from the $\ln(r/a)$ term. Using these results, it is possible from the Stokes expansion far from the cylinder to establish both the force and the moment on the cylinder. The Stokes solution produced by Dorrepaaal *et al* (1984) is able to immediately produce the drag, lift and moment on the circular cylinder from its expansion at large distances from the cylinder; although the contribution from the singularity at the rotlet must first be removed from the coefficient of the $\ln(r/a)$ term before it represents the moment on the cylinder.

Since the streamfunction and vorticity equations are both elliptical in nature we should supply one condition for each of the variables rather than use directly conditions (2.16), (2.17) and (2.18). It has been reported by Fornberg (1980, 1985), that the choice of the boundary condition for the vorticity, ω, is not as sensitive as that for the streamfunction, ψ, and many authors have paid particular attention to the boundary condition for Ψ at large distances from the cylinder.

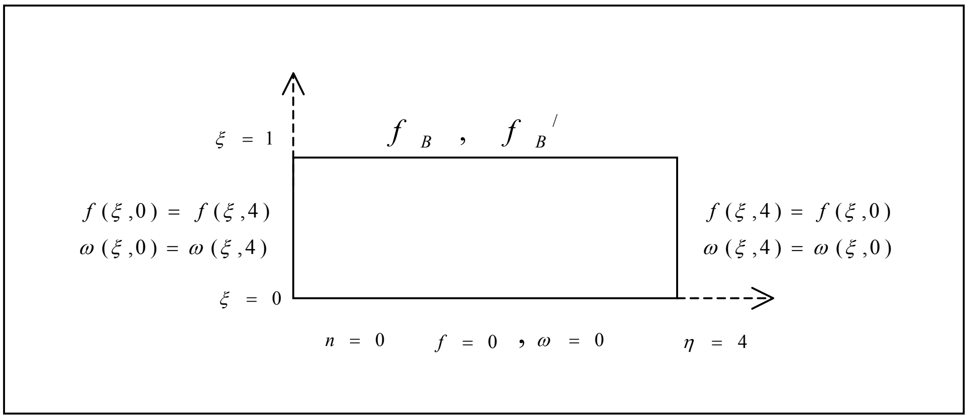

Figure 2. The geometry of the solution domain.

In this work a new numerical technique is introduced in order to avoid having to enforce the boundary condition (2.19) at large distances from the cylinder. We introduce the transformation

$$\xi = \frac{1}{r}, \quad \eta = \frac{2\vartheta}{\pi} \tag{2.23}$$

and

$$f(r,\vartheta) = \psi(r,\vartheta)/r, \tag{2.24}$$

where

$$f(\xi,\eta) = \xi\psi(\xi,\eta). \tag{2.25}$$

Thus, we have $f = 0$ on $\xi = 0$ (i.e. at $r = \infty$) and this requires no approximation to be made for ψ at large distances from the cylinder. With the transformation (2.23) the solution domain $(1 \le r < \infty,\ 0 \le \vartheta < 2\pi)$ is transformed into a finite rectangular region in the (ξ,η) plane

$(0 \le \xi < 1, \quad 0 \le \eta < 4)$, see Figure 2. Substituting expressions (2.23) and (2.24) into the governing equations (2.14) and (2.15) one obtains

$$\frac{-2\xi^2}{\pi}\frac{\partial(f,\omega)}{\partial(\xi,\eta)}+\frac{2\xi}{\pi}f\frac{\partial\omega}{\partial\eta}+\xi^2\frac{\partial\omega}{\partial\xi}\cos(\pi\eta/2)+\frac{2\xi}{\pi}\frac{\partial\omega}{\partial\eta}\sin(\pi\eta/2)$$
$$+\frac{2\beta\xi^2}{\pi}\left(\frac{1+\xi c\sin(\pi\eta/2)}{1+c^2\xi^2+2c\xi\sin(\pi\eta/2)}\right)\frac{\partial\omega}{\partial\eta}+\left(\frac{\beta\xi^4 c\cos(\pi\eta/2)}{1+c^2\xi^2+2c\xi\sin(\pi\eta/2)}\right)\frac{\partial\omega}{\partial\xi} \tag{2.26}$$
$$+\frac{2}{\mathrm{Re}}\left(\xi^4\frac{\partial^2\omega}{\partial\xi^2}+\xi^3\frac{\partial\omega}{\partial\xi}+\frac{4\xi^2}{\pi^2}\left(\frac{\partial^2\omega}{\partial\eta^2}\right)\right)=0,$$

$$\xi^3\frac{\partial^2 f}{\partial\xi^2}-\xi^2\frac{\partial f}{\partial\xi}+\frac{4\xi}{\pi^2}\frac{\partial^2 f}{\partial\eta^2}+\xi f=-\omega \tag{2.27}$$

and the boundary conditions (2.10) and (2.11) now become

$$f=-\sin(\pi\eta/2)-\beta\ln\left(1+c^2+2c\sin(\pi\eta/2)\right)^{1/2}=f_B \quad on \quad r=1, \quad 0\le\eta<4 \tag{2.28}$$

$$\frac{\partial f}{\partial\xi}=\alpha+\sin(\pi\eta/2)+\beta\left(\frac{1+c\sin(\pi\eta/2)}{1+c^2+2c\sin(\pi\eta/2)}\right)=f'_B \quad on \quad r=1, \quad 0\le\eta<4, \tag{2.29}$$

$$f=0, \quad \omega=0, \quad on\ \xi=0,\ 0\le\eta<4 \tag{2.30}$$

Further, since the solution is periodic in η we also require that

$$f(\xi,4)=f(\xi,0), \quad \omega(\xi,4)=\omega(\xi,0), \quad for \quad 0\le\xi\le1, \tag{2.31}$$

which along with all the other boundary conditions are indicated in Figure 2.

The Solution Technique

In order to obtain numerical solutions of the equations (2.26) and (2.27) subject to the boundary conditions (2.28), (2.29), (2.30) and (2.31) the region of integration $0\le\xi\le1, 0\le\eta<4$ is covered by a rectangular mesh system of size *h* and *k* in the η and ξ directions, respectively, and a modified finite-difference approximation to the differential equations (2.26) and (2.27) is employed. If the standard central-difference approximation is employed, then it is not always possible to obtain a convergent solution especially at moderate and large values of the Reynolds number. A number of schemes have been introduced with the object of improving the efficiency of the convergence of such schemes, e.g. upwind and downwind differencing as discussed by Greenspan (1968). Gosman *et al.* (1969) and Runchal *et al.* (1969). These efficiencies arise from the fact that the difference equations are associated with matrices, which are diagonally dominant and thus amenable to iterative methods of solution. However, these schemes suffer from a deficiency in that they are only of first-order accuracy since forward or backward differences are employed to approximate first derivatives rather than the more accurate second-order central-difference formulae. The formulation in which all derivatives are approximated by central differences is of second-order accuracy but the matrix associated with the difference equations may not be diagonally dominant. The iterative procedures may be slowly convergent or even divergent when this method is applied.

There are also finite-difference schemes, which are of second-order accuracy and for which the associated matrices are always diagonally dominant. These methods rely upon rather specialized forms of local approximations and yield difference equations, which involve the exponential function. These schemes were first introduced by Allen and Southwell (1955) in approximating the equation governing the vorticity during the course of finding numerical solutions for the steady two-dimensional flow past a circular cylinder. Dennis and Hudson (1978) showed that by a suitable adaptation of an alternative to the Allen and Southwell method suggested by Dennis (1960), an approximation of second-order accuracy, yielding difference equations with an associated matrix which is diagonally dominant, can be obtained. These difference equations do not involve the exponential function and can be looked upon as a rather more complicated version of the central-difference formulation. The Dennis and Hudson method contains more terms in the finite-difference equations than the usual central-difference approximation but the presence of these extra terms is very important for the associated matrix to be diagonally dominant. Numerous authors have performed several numerical experiments which confirm that the method by Dennis and Hudson succeeds where the standard central-difference formulation fails, see for example Dennis (1960) and Dennis and Hudson (1978). Thus, in this paper a modified version of the finite-difference approximation as described by Hudson and Dennis, has been used.

It is found most convenient to set up a mesh system such that the mesh size in both the ξ and η directions are $k = h = 1/N$, where N is a pre-assigned positive integer. In view of the periodic conditions (2.31), and extra line of computation $\eta = 4 + h$ for $0 \le \xi \le 1$ is introduced. Then we have $(N+1)X(M+1)$ mesh points, where $M = 4N + 1$, the mesh points (ξ_i, η_i) $(0 \le i \le N, 0 \le j \le M)$ are (ih, jh). If subscripts 0,1,2,3 and 4 denote quantities at the grid points (ξ_0, η_0), $(\xi_0, \eta_0 - h)$, $(\xi_0 + h, \eta_0)$, $(\xi_0, \eta_0 + h)$ and $(\xi_0 - h, \eta_0)$ respectively, then on using the Dennis and Hudson scheme, equations (2.26) and (2.27) may be written in the form

$$\left(\frac{\beta^{*2}}{2} + \frac{2}{\pi^2 \xi_0^2} - \frac{h^2}{4\xi_0^2}\right) f_0 = \frac{1}{\pi^2 \xi_0^2} f_1 + \left(\frac{\beta^{*2}}{4} - \frac{\beta^* h}{8\xi_0}\right) f_2 + \frac{1}{\pi^2 \xi_0^2} f_3 + \left(\frac{\beta^*}{4} + \frac{\beta^* h}{8\xi_0}\right) f_4 + \frac{h^2}{4\xi_0^3}\omega_0 \tag{3.1}$$

$$\begin{aligned}\left(\frac{\beta^{*2}}{2} + \frac{2}{\pi^2 \xi_0^2}\right)\omega_0 = &\left(\frac{1}{\pi^2 \xi_0^2} - a(\xi_0,\eta_0) - D(\xi_0,\eta_0)\right)\omega_1 + \left(\frac{\beta^{*2}}{4} + b(\xi_0,\eta_0) + E(\xi_0,\eta_0)\right)\omega_2 \\ &+ \left(\frac{1}{\pi^2 \xi_0} + a(\xi_0,\eta_0) + D(\xi_0,\eta_0)\right)\omega_3 + \left(\frac{\beta^{*2}}{4} - b(\xi_0,\eta_0) - E(\xi_0,\eta_0)\right)\omega_4\end{aligned} \tag{3.2}$$

with $\beta^* = \dfrac{h}{k}$, and

$$a(\xi_0,\eta_0) = \frac{\mathrm{Re}\, h}{8\pi\xi_0^2}\left(\upsilon_0 + \frac{f_0}{\xi_0} + \frac{1}{\xi_0}\sin(\vartheta)\right) \tag{3.3}$$

$$b(\xi_0,\eta_0) = \beta\left(\frac{\mathrm{Re}\, h}{8\pi\xi_0^2} u_0 + \frac{h}{8\xi_0} + \frac{\mathrm{Re}\, h}{16\xi_0}\cos(\vartheta)\right) \tag{3.4}$$

$$D(\xi_0,\eta_0) = -\frac{\beta h \,\mathrm{Re}}{8\pi\xi_0^2}\left(\frac{1 + c\xi_0 \sin\vartheta}{1 + c^2\xi_0^2 + 2c\xi_0\sin(\vartheta)}\right) \tag{3.5}$$

$$E(\xi_0,\eta_0) = -\frac{\beta h \,\mathrm{Re}}{16}\left(\frac{c\cos(\vartheta)}{1 + c^2\xi_0^2 + 2c\xi_0\sin(\vartheta)}\right) \tag{3.6}$$

where $(u_{0,}\upsilon_0)$ are defined as

$$u_0 = \frac{\partial f(\xi_0, \eta_0)}{\partial \eta}, \quad \upsilon_0 = -\frac{\partial f(\xi_0, \eta_0)}{\partial \xi}. \tag{3.7}$$

The standard central-difference approximations may be obtained by setting the extra terms $D(\xi_0, \eta_0), E(\xi_0, \eta_0)$ to be zero.

We now briefly outline how the boundary conditions (2.28)-(2.30) can be implemented. The boundary condition for the vorticity on $\xi = 1$ can be found by using Taylor expansion for f and ω and inserting them in equation (2.27) to get second-order accurate finite differences.

Boundary conditions for ω:

$$\text{On}\, \xi = 0,\, 0 \le \eta < 4 : i = 0,\, 0 \le j < M; \qquad \omega_{0j} = 0; \tag{3.8}$$

$$\text{on } \xi = 1,\, 0 \le \eta < 4 : i = N,\, 0 \le j < M; \qquad \omega_{0j} = \omega_{nj;} \tag{3.8}$$

$$\omega_{nj} = \frac{\left[\left(1 - \frac{h^2}{2} - \frac{h^3}{6}\right) f_{nj} - f_{n-ij} - \frac{h^2}{6}\omega_{n-ij} - h\left(1 - \frac{h}{2} - \frac{h^2}{6}\right) f'_{nj} + \frac{4h^3}{6\pi^2}\left(g_{2j} - \left(1 + \frac{3}{h}\right) g_{ij}\right)\right]}{\left[\frac{h^2}{3}(1 + h)\right]} \tag{3.9}$$

with

$$f'_{nj} = \alpha - \frac{\beta\left(1 + c\sin\left(\frac{\pi}{2}\eta_j\right)\right)}{1 + c^2 + 2c\sin\left(\frac{\pi}{2}\eta_j\right)} + \frac{\beta}{2}\ln\left(1 + c^2 + 2c\sin\left(\frac{\pi}{2}\eta_j\right)\right) \tag{3.10}$$

$$g_{ij} = +\frac{\pi^2}{4}\sin\left(\frac{\pi}{2}\eta_j\right) - \frac{\beta\pi^2}{4}\left(\frac{c\sin\left(\frac{\pi}{2}\eta_j\right)\left(1 + c^2\right) + 2c^2}{\left(1 + c^2 + 2c\sin\left(\frac{\pi}{2}\eta_j\right)\right)^2}\right) \tag{3.11}$$

$$g_{2j} = -\frac{\beta\pi^2 c\left(3c + c^3 + \left(1 + 3c^2\right)\sin\left(\frac{\pi}{2}\eta_j\right) - 2c^3\cos^2\left(\frac{\pi}{2}\eta_j\right)\right)}{2\left(1 + c^2 + 2c\sin\left(\frac{\pi}{2}\eta_j\right)\right)^3} \tag{3.12}$$

$$\text{on}\, \eta = 4 + h, \quad 0 \le \xi \le 1 : 0 \le i \le N, j = M; \quad \omega_{im} = \omega_{i1}; \tag{3.13}$$

on

$$\eta = 0, \quad 0 \le \xi \le 1: \quad 0 \le i \le N, j = 0; \quad \omega_{i0} = \varpi_{im-1}; \tag{3.14}$$

Boundary conditions for f:

$$\text{On}\, \xi = 0,\, 0 \le \eta < 4 : 0 \le j < M, i = 0; \qquad f_{0j} = 0; \tag{3.15}$$

$$\text{on } \xi = 1,\, 0 \le \eta < 4 : 0 \le j < M, i = N;$$

$$f_{nj} = -\sin\left(\frac{\pi}{2}nj\right) + \frac{\beta}{2}\ln\left(1 + c^2 2c\sin\left(\frac{\pi}{2}nj\right)\right) \tag{3.16}$$

$$\text{on}\, \eta = 4 + h,\, 0 \le \xi \le 1 : 0 \le i \le N, j = M; \qquad f_{im} = f_{i1}; \tag{3.17}$$

$$\text{on}\, \eta = 0,\, 0 \le \xi \le 1 : 0 \le i \le N,\, j = 0; \qquad f_{i0} = f_{im-1}; \tag{3.18}$$

The resulting finite-difference equations (3.1) and (3.2), subject to the boundary conditions (3.8) – (3.18), were solved iteratively. From equation (2.6) the non-dimensional speed of the uniform stream at large distances from the cylinder is unity, whereas the non-dimensional strength of the rotlet at the position $(c, 3\pi/2)$, namely β, is variable. In order to compare the results with Dorrepaal *et al.* (1984) the value of *c* is fixed at the magnitude used in their calculations, namely $c = 3$.

Since the force components (drag and lift) and the moment are very sensitive to the method of solution, particular attention has been given to these quantities. If F_x and F_y are the dimensional drag and lift on the cylinder, then

$$F_x = \int_0^{2\pi} \left(\sigma_{rr}\cos(\vartheta) - \sigma_{r\vartheta}\sin(\vartheta)\right)_{r=1} r\, d\vartheta \tag{3.19}$$

$$F_y = -\int_0^{2\pi} \left(\sigma_{\vartheta r}\cos(\vartheta) + \sigma_{rr}\sin(\vartheta)\right)_{r=1} r\, d\vartheta \tag{3.20}$$

Introducing the constitutive relations

$$\sigma_{rr} = -p + 2\mu \frac{\partial \upsilon_r}{\partial r}, \tag{3.21}$$

$$\sigma_{r\vartheta} = \sigma_{\vartheta r} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{\upsilon_\vartheta}{r}\right) + \frac{1}{r}\frac{\partial \upsilon_r}{\partial \vartheta}\right), \tag{3.22}$$

$$\sigma_{\vartheta\vartheta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial \upsilon_\vartheta}{\partial \vartheta} + \frac{\upsilon_r}{r}\right), \tag{3.23}$$

into expressions (3.19) and (3.20) results in

$$F_x = \int_0^{2\pi} r\left[\frac{\partial p}{\partial \vartheta} - 2\mu \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2 \Psi}{\partial \vartheta^2} + \Psi\right) + \mu \nabla^2 \Psi\right]\sin(\vartheta)\, d\vartheta, \tag{3.24}$$

$$F_y = -\int_0^{2\pi} r\left[\frac{\partial p}{\partial \vartheta} - 2\mu \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2 \Psi}{\partial \vartheta^2} + \Psi\right) + \mu \nabla^2 \Psi\right]\cos(\vartheta)\, d\vartheta, \tag{3.25}$$

By using the $\hat{\vartheta}$ component of the Navier Stokes Equations on the surface of the cylinder where the component of velocity normal to the surface is zero, namely

$$\frac{1}{r}\left(\frac{\partial p}{\partial \vartheta}\right) = -\mu \frac{\partial}{\partial r}\left(\nabla^2 \Psi\right) \tag{3.26}$$

and substituting $\nabla^2 \Psi = -\omega$, which is equation (2.8) but in its dimensional form, the equations (3.24) and (3.25) can be written in the form

$$F_x = \mu \int_0^{2\pi} r \left[r \frac{\partial \omega}{\partial r} - \omega - \frac{2}{r} \frac{\partial}{\partial r} \left(\frac{\partial^2 \Psi}{\partial \vartheta^2} + \Psi \right) \right] \sin(\vartheta) d\vartheta, \tag{3.27}$$

$$F_y = -\mu \int_0^{2\pi} r \left[r \frac{\partial \omega}{\partial r} - \omega - \frac{2}{r} \frac{\partial}{\partial r} \left(\frac{\partial^2 \Psi}{\partial \vartheta^2} + \Psi \right) \right] \cos(\vartheta) d\vartheta. \tag{3.28}$$

The above expressions are still dimensional, but defining the lift and the drag coefficients by

$$C_D = F_x / \left(\rho U^2 a \right) \text{ and } C_L = F_y / \left(\rho U^2 a \right) \tag{3.29}$$

enables C_D and C_L to be written as

$$C_D = \frac{2}{\mathrm{Re}} \int_0^{2\pi} r \left[r \frac{\partial \omega}{\partial r} - \omega - \frac{2}{r} \frac{\partial}{\partial r} \left(\frac{\partial^2 \Psi}{\partial \vartheta^2} + \Psi \right) \right] \sin(\vartheta) d\vartheta, \tag{3.30}$$

$$C_L = -\frac{2}{\mathrm{Re}} \int_0^{2\pi} r \left[r \frac{\partial \omega}{\partial r} - \omega - \frac{2}{r} \frac{\partial}{\partial r} \left(\frac{\partial^2 \Psi}{\partial \vartheta^2} + \Psi \right) \right] \cos(\vartheta) d\vartheta, \tag{3.31}$$

where r, Ψ and ω are all non-dimensional.

In terms of the independent variables ζ and ϑ, C_D and C_L become

$$C_D = \frac{2}{\mathrm{Re}} \int_0^{2\pi} \left[-\frac{\partial \omega}{\partial \xi} - \omega + 2 \frac{\partial}{\partial \xi} \left(\frac{\partial^2 \Psi}{\partial \vartheta^2} + \Psi \right) \right] \sin(\vartheta) d\vartheta, \tag{3.32}$$

$$C_L = -\frac{2}{\mathrm{Re}} \int_0^{2\pi} \left[-\frac{\partial \omega}{\partial \xi} - \omega + 2 \frac{\partial}{\partial \xi} \left(\frac{\partial^2 \Psi}{\partial \vartheta^2} + \Psi \right) \right] \cos(\vartheta) d\vartheta, \tag{3.33}$$

When the boundary conditions on the cylinder are introduced, expressions (3.32) and (3.33) reduce to those given by Fornberg (1980), except for a reversal of the sign in C_L and the omission of a Factor 2 in the definition of both C_L and C_D , which are both produced by the scaling we adopted.

The moment on the cylinder

$$M = \int_0^{2\pi} r \sigma_{r\vartheta} r d\vartheta \tag{3.34}$$

becomes, in dimensional form, on substituting the appropriate constitutive equation

$$M = -\mu \int_0^{2\pi} r^2 \left[\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \vartheta^2} \right] d\vartheta \tag{3.35}$$

Introducing the moment coefficient defined by

$$C_M = M/\left(\rho U^2 a^2\right) \tag{3.36}$$

results in the non-dimensional expression

$$C_M = \frac{2}{\mathrm{Re}}\int_0^{2\pi} r^2 \left[\omega + \frac{2}{r}\frac{\partial \Psi}{\partial r} + \frac{2}{r^2}\frac{\partial^2 \Psi}{\partial \vartheta^2}\right] d\vartheta. \tag{3.37}$$

In terms of the independent variables ξ and ϑ the moment coefficient becomes

$$C_M = \frac{2}{\mathrm{Re}}\int_0^{2\pi} \left[\omega - 2\frac{\partial \Psi}{\partial \xi} + 2\frac{\partial^2 \Psi}{\partial \vartheta^2}\right] d\vartheta, \tag{3.38}$$

and the boundary conditions on the cylinder reduce this expression to

$$C_M = \frac{2}{\mathrm{Re}}\int_0^{2\pi} \left[\omega - 2\omega_0\right] d\vartheta. \tag{3.39}$$

Formulae (3.32), (3.33) and (3.39) are evaluated using Simpson's rule. Due to the need to evaluate the drag, lift and moment when $\mathrm{Re} = 0$ we will work with the non-dimensional quantities $\mathrm{Re}\,C_D/2, \mathrm{Re}\,C_L/2$ and $\mathrm{Re}\,C_M/2$ from now on, namely $F_x/(\mu U), F_y/(\mu U)$ and $M/(\mu U a)$. However, when the Reynolds number vanishes, that is the independence of the mainstream at infinity from the strength of the rotlet is no longer valid, It is necessary to redefine the above three quantities as $(F_x a)/(\mu\Gamma), (F_y a)/(\mu\Gamma)$ and $M/(\mu\Gamma)$ when discussing an iterative scheme in the following section.

The Results

Before we undertake a detailed discussion of the results it is important to stress from Dorrepaal *et al*. (1984), and from our own investigation, that at $\mathrm{Re} = 0$ it is impossible to obtain a solution for an arbitrary value of the parameter β, which is contrary to what occurs at non-zero Reynolds numbers. In the $\mathrm{Re} = 0$ situation there is only one unique value of β which will produce the uniform stream at infinity, and the difficulty numerically is to achieve some mechanism which will enable the numerical method to converge to this quantity. Since Stokes flow with a uniform flow at infinity means that body must be free of any force, one can develop an iterative numerical scheme based upon this fact to enable the unique value of β and the corresponding flow solution to be determined. Exactly how this is achieved will be discussed in detail later within this section. In order to compare the results obtained by the present numerical method with those obtained previously, the values of the Reynolds number $\mathrm{Re} = 5$ and $\mathrm{Re} = 20$, in the absence of any rotlet $(\beta = 0)$ and with the rotational parameter $\alpha = 0, 0.5, 1$ and 2, were initially investigated. Solutions are obtained for the mesh sizes $h = 1/10, 1/15, 1/20$ and $1/40$. It is found that the contour lines for the streamfunction and vorticity distribution are in good agreement with most of the previous investigators (Dennis and Chang (1970); Fornberg (1980, 1985)), at both the above Reynolds numbers, repeated extrapolation of results derived from h^2 extrapolation of the results from $h = 1/10, 1/15, 1/20$ and $1/40$ produced only a 0.1% change in the coefficients of the lift and the drag when compared with the values derived directly from using $h = 1/40$. As a consequence it is proposed not to implement extrapolation but instead to utilize a small value of *h,* namely $h = 1/40$. Values of $\mathrm{Re}/2$ times the coefficients of the lift, drag and moment are presented in table 1. The lift and drag coefficients are in good agreement with those produced by Fornberg (1980).

Table 1: The variation of the coefficients of lift, drag and moment with α for $\mathrm{Re} = 5$ and 20.

	$\mathrm{Re} = 5$			$\mathrm{Re} = 20$		
α	$\mathrm{Re}\,C_L/2$	$\mathrm{Re}\,C_D/2$	$\mathrm{Re}\,C_M/2$	$\mathrm{Re}\,C_L/2$	$\mathrm{Re}\,C_D/2$	$\mathrm{Re}\,C_M/2$
0.0	0.000	19.716	0.000	0.000	39.861	0.000
0.5	6.939	19.565	0.616	25.635	3.430	2.354
1.0	14.176	19.226	1.197	52.289	38.465	4.426
2.0	29122	17.514	2.246	114.275	32.515	7.026

Having established that the numerical procedure is producing the correct results for $\mathrm{Re} \neq 0$ and $\alpha \neq 0$, the condition $\beta = 0$ is relaxed. However, since the interest is in whether the problem is well posed as the Reynolds number tends to zero, it is proposed to set α to be zero in the remainder of the calculations to be presented in this paper. Such a restriction can easily be removed if required.

It has already been established analytically by Dorrepaal *et al.* (1984) that at $\mathrm{Re} = 0$ a solution in which there is a uniform stream of unit non-dimensional strength flowing parallel to the positive y-axis is possible provided that the rotlet is placed at the non-dimensional position $(c,0)$, where the non-dimensional strength of the rotlet is $\beta = c$. In the present situation the stream is flowing parallel to the positive x-axis and therefore the position of the rotlet has to be $(c,3\pi/2)$ in order to produce a solution. The question is whether the solutions at $\mathrm{Re} \neq 0$ for a range of values of β, namely varying non-dimensional strengths of the rotlet, can be used to predict the unique analytical value of this parameter which produces the solution at $\mathrm{Re} = 0$. The results of the solution by Dorrepaal *et al.* (1984) show that there is no force or moment on the cylinder. Hence, in the numerical solution it is possible that the value of β tends to its correct value for $\mathrm{Re} = 0$ if one considers the values of the parameter β at which the quantities such as the force or moment on the cylinder are zero as the Reynolds number tends to zero. To investigate this possibility values of $\mathrm{Re}\,C_D, \mathrm{Re}\,C_L$ and $\mathrm{Re}\,C_M$ are calculated as functions of the parameter β for $0 \leq \beta \leq 3.5$ at $\mathrm{Re} = 5,3$ and 1, with $h = 1/10, 1/20$ and $1/40$. Figures 3, 4 and 5 show the variation of $\mathrm{Re}\,C_D, \mathrm{Re}\,C_L$ and $\mathrm{Re}\,C_M$ as a function of the strength of the rotlet, β, for $\mathrm{Re} = 5,3$ and 1, respectively, when using mesh sizes 10x40, 20x80 and 40x160. It is observed from figures 4 and 5 that the variations of the lift and moment as a function of β are very non-linear. However, the drag, (see Figure 3) appears to behave almost linearly with respect to β, over the range $0 \leq \beta \leq 3.5$ for all the values of Re considered. Further, all the three values of Re considered in this paper, predict that zero drag appears to occur close to the value of $\beta(= c) = 3$. It has already been established by Dorrepaal *et al.* (1984) that only for a stream whose direction and magnitude are related to the position and strength of the rotlet will a solution be possible when $\mathrm{Re} = 0$. In this case only for the specific value of $\beta = c$ will the problem be well-posed and produce a solution for which the values for the drag, lift and moment on the cylinder are all zero. For all other values of β one will have the force and moment dependent on this parameter. It has been shown elsewhere that when $\mathrm{Re} = 0$ non-zero values of the drag and lift implies that there are terms of the type $r\ln(r)\sin(\vartheta)$ and $r\ln(r)\cos(\vartheta)$ at large values of r. Similarly non-zero values of the moment require a $\ln(r)$ term. Obviously the presence of such terms violates the boundary condition that $f = 0$ as $r \to \infty$, and it is only when all such terms are absent that one will achieve the required solution. This is why at $\mathrm{Re} = 0$, it is impossible to obtain a convergent solution for an arbitrary value of β, unlike the situation at $\mathrm{Re} \neq 0$.

However, for $\mathrm{Re} = 0$ one can think of the lift, drag and moment as functions of β and only when these quantities acquire the value of zero will β achieve its required value, namely $\beta = c$, and produce a

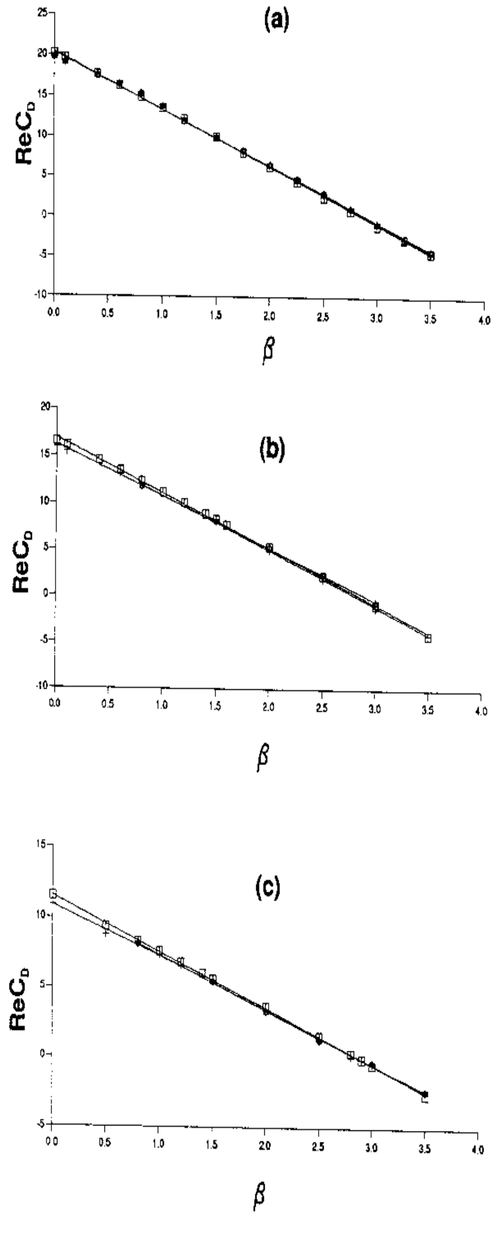

Figure 3. The variation of $\mathrm{Re}\,C_D$ as a function of β ◇◇◇ 10×40,+++20×80,***40×160.

convergent solution. Hence, the technique adopted was that of a Newton-Raphson iteration on $C_D(\beta)$. The choice of C_D rather than C_L or C_M being based upon the linearity of this expression with respect to β, compared with the rather complex behavior of the latter two quantities when considered as functions of this variable. The governing partial differential equation in finite-difference form, namely equation (3.1) and (3.2), were solved for two values of β, say β_1 and β_2, the number of iterations being terminated after a prescribed number since due to the boundary condition posed at $\xi = 0$ convergent solutions are impossible. The resulting two non-zero values of $C_D(\beta_1)$ and $C_D(\beta_2)$ are the values used in the Newton-Raphson method in order to establish a new value of β, say β_3 at which $C_D(\beta_3)$ is estimated to be zero.

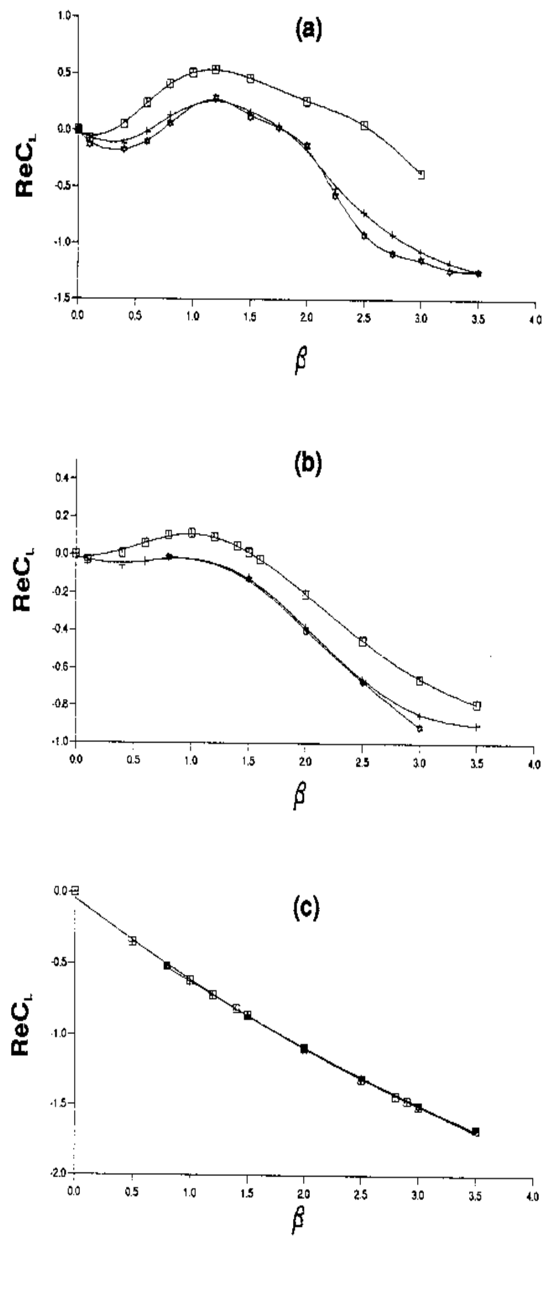

Figure 4. The variation of $\mathrm{Re}\, C_L$ as a function of β ◇◇◇ 10×40,+ + +20×80,* * *40×160.
$(a)\mathrm{Re}=5$, $(b)\mathrm{Re}=3$, $(c)\mathrm{Re}=1$.

Resolving for this new value of β, and using $C_D(\beta_3)$ together $C_D(\beta_1)$ or $C_D(\beta_2)$, whichever is closest to zero, to produce from the Newton-Raphson solution a new value of β. This process is continued until the value of $C_D(\beta_3)$ reaches zero to within the required tolerance needed. The value of β automatically results in the correct values of C_L and C_M. The whole iterative procedure could equally well have been applied to setting either the values of $C_L(\beta)$ or $C_M(\beta)$ to zero. The resulting values of β achieved by this process were 3.0004, 3.005 and 3.0010 for the mesh seizes 40x160, 20x80 and 10x40 respectively,

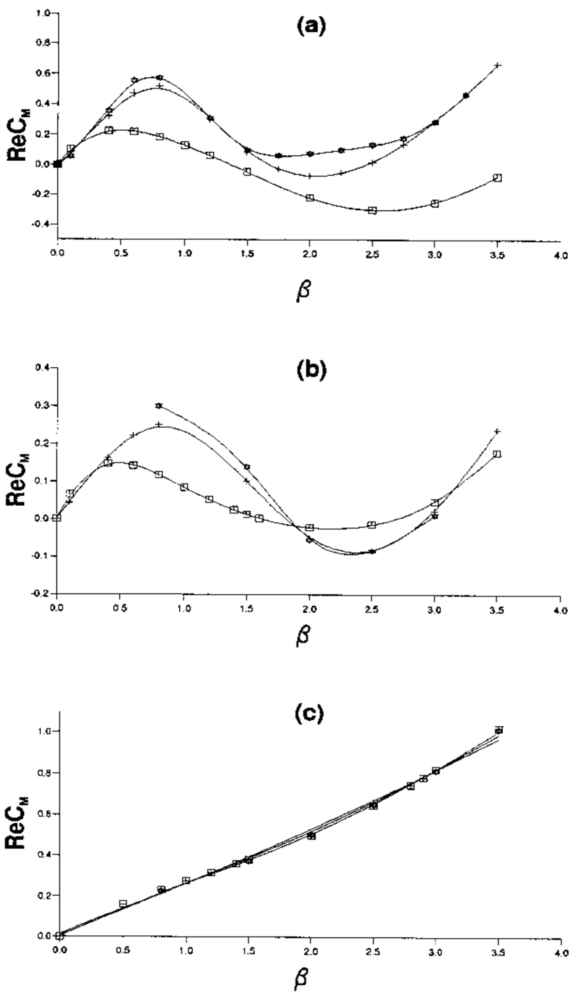

Figure 5. The variation of $\mathrm{Re}\, C_M$ as a function of β $\diamond\diamond\diamond$ $10\times 40, +++ 20\times 80, *** 40\times 160.$
$(a)\mathrm{Re}=5,$ $(b)\mathrm{Re}=3,$ $(c)\mathrm{Re}=1.$

compared with the exact analytical value of 3. The streamline and vorticity patterns produced from the numerical computation and shown in Figures 6 (a) and 7 (a) are identical with those obtained from the analytical solution. Figure 6 shows the streamline pattern obtained for $\beta = 3$ and $\mathrm{Re} = 0,1,3$ and 5 for fine mesh size 40x160. In these figures the effect of changing the Reynolds number appears to be slight, but if we compare this figure with Figure 10 the effect of the Reynolds number becomes more clear.

Figure 7 shows the vorticity pattern for $\beta = 3$ and $\mathrm{Re} = 0,1,3$ and 5 for 40x160. The effect of changing the Reynolds number is unclear, since the Reynolds number is small and the strength of the rotlet is strong.

Figure 8 shows the streamline pattern for $\beta = 1.5$ and $\mathrm{Re} = 0,1,3$ and 5. The region around the rotlet appears to be unaffected by the small change in the Reynolds number but compared with Figure 7, where the strength of the rotlet is double the present value, the decreasing strength of the rotlet causes a shrinkage of the region for the same streamline values.

Figure 9 shows the vorticity pattern for $\beta = 1.5$ and $\mathrm{Re} = 0,1,3$ and 5. Whilst the three figures are virtually indistinguishable, the decrease in the strength of the rotlet to half its earlier value, see Figure 7, has produced a more symmetric pattern of vorticity between the rotlet and the cylinder whilst at the same time

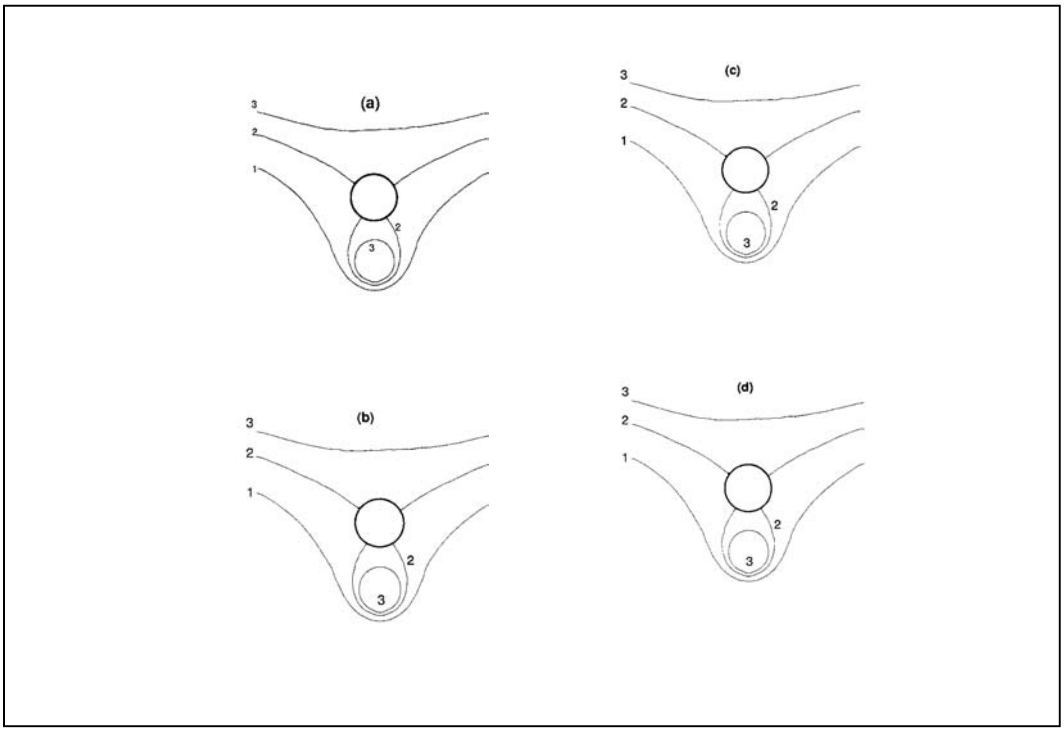

Figure 6 (a,b,c, d). The numerically obtained streamlines, Ψ , for $\beta = 3$ with a mesh size 40×160 . The streamlines labelled 1, 2 and 3 correspond to $\Psi = -1, 0 \; and \; 0.7$, respectively.
$(a)\mathrm{Re} = 0, \; (b)\mathrm{Re} = 1, (c)\mathrm{Re} = 3, \; and \; (d)\mathrm{Re} = 5.$

the vortex lines of the same magnitude have been constrained in this region closer to the surface of the cylinder. This is contrary to what appears to take place above the cylinder with the vortex lines of the same magnitude showing a departure from the surface.

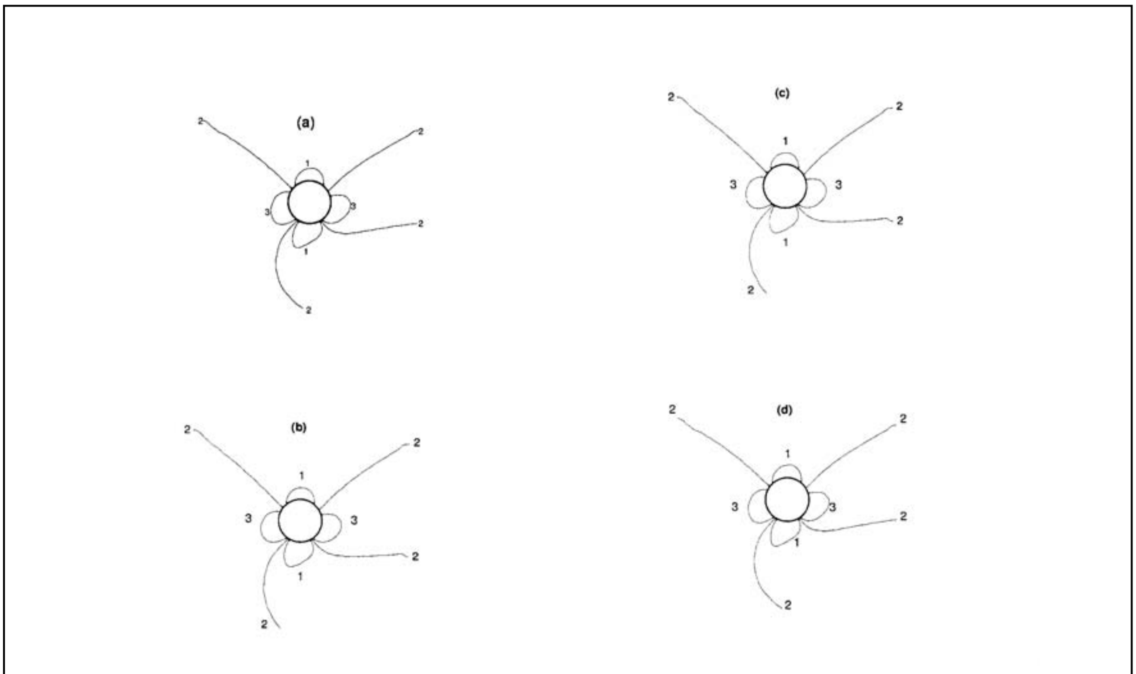

Figure 7(a, b, c, d). The numerically obtained vorticity, ω , for $\beta = 3$ with a mesh size 40×160 . The streamlines labelled 1, 2 and 3 correspond to $\omega = -0.4, 0 \; and \; 0.4$, respectively.
$(a)\mathrm{Re} = 0, \; (b)\mathrm{Re} = 1, (c)\mathrm{Re} = 3, \; and \; (d)\mathrm{Re} = 5.$

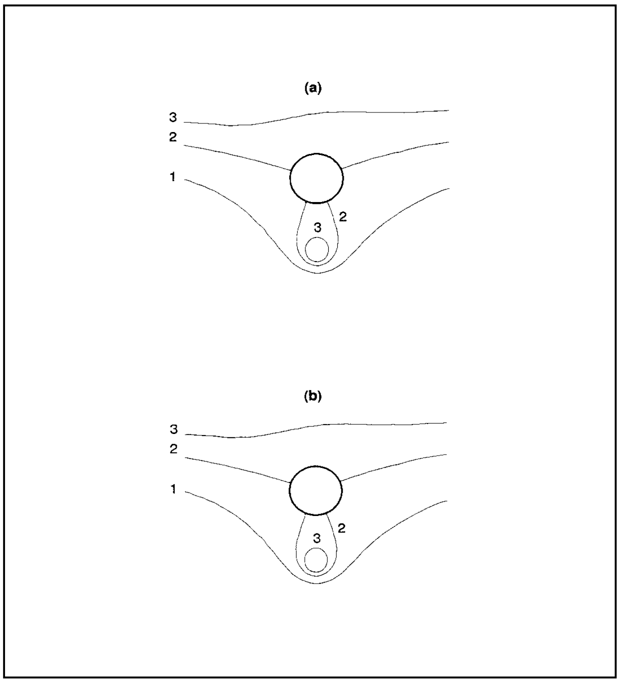

Figure 8 (a, b). The numerically obtained streamlines, Ψ, for $\beta = 1.5$ with a mesh size 40×160. The streamlines labelled 1, 2 and 3 correspond to $\Psi = -1, 0$ *and* 0.7, respectively.

Figure 10 shows the effect of the strength of the rotlet and how the results tend to those obtained by Fornberg (1980). Figure 10 (f) is identical with the figure obtained by Fornberg (1980). Further, close to the body, the flow (when β exceeds unity) appears to be dominated by the rotlet, even at a Reynolds number of 20. The dominance of the rotlet is seen by the symmetrical nature of

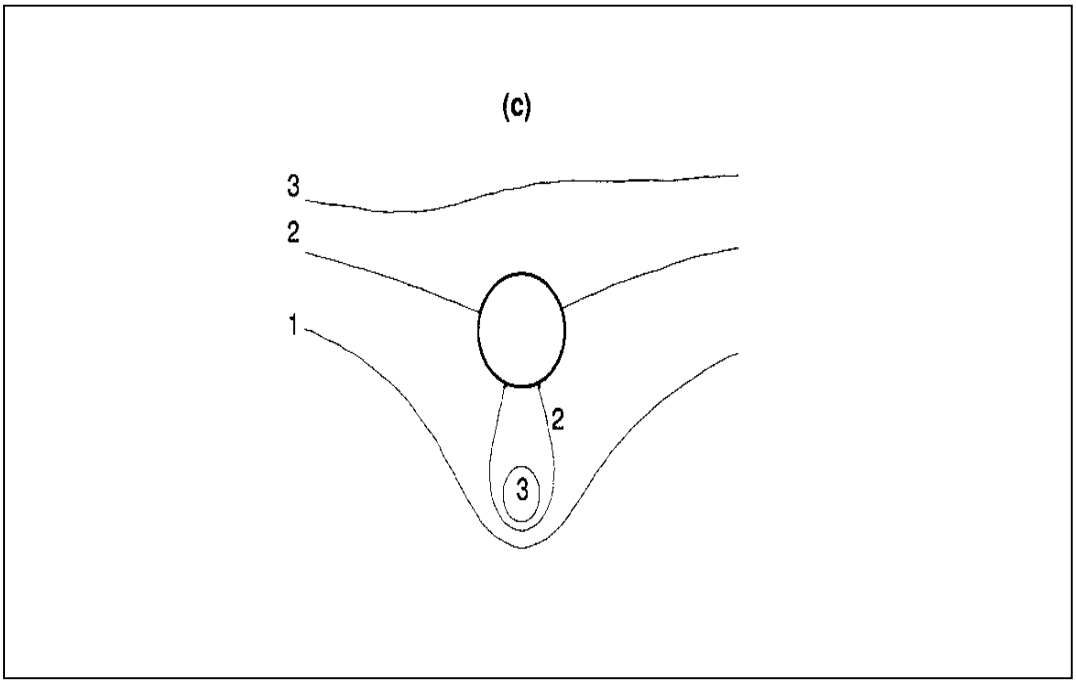

Figure 8 (c). The numerically obtained streamlines, Ψ, for $\beta = 1.5$ with a mesh size 40×160. The streamlines labelled 1, 2 and 3 correspond to $\Psi = -1, 0$ *and* 0.7, respectively.
$(a)\mathrm{Re} = 1,\ (b)\mathrm{Re} = 3, (c)\mathrm{Re} = 5.$

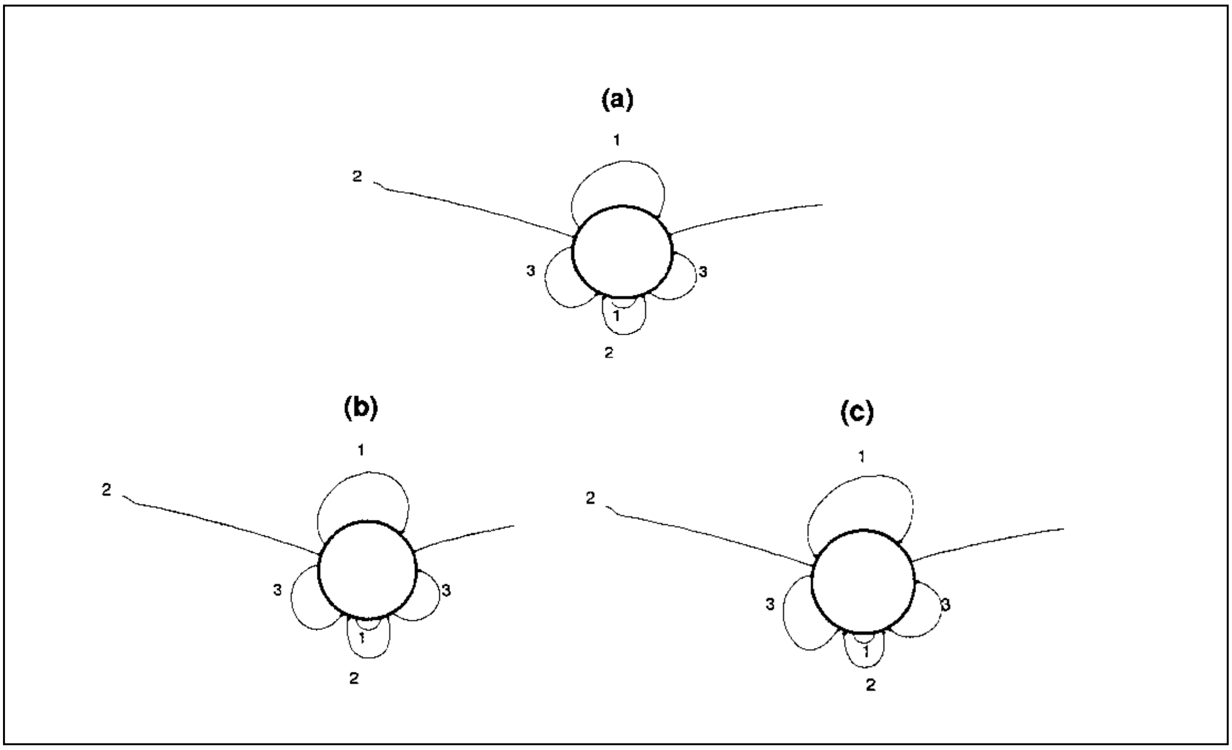

Figure 9 (a, b, c). The numerically obtained vorticity, ω, for $\beta = 1.5$ with a mesh size 40×160. The streamlines labelled 1, 2 and 3 correspond to $\omega = -0.4,\ 0\ \mathit{and}\ 0.4$, respectively.

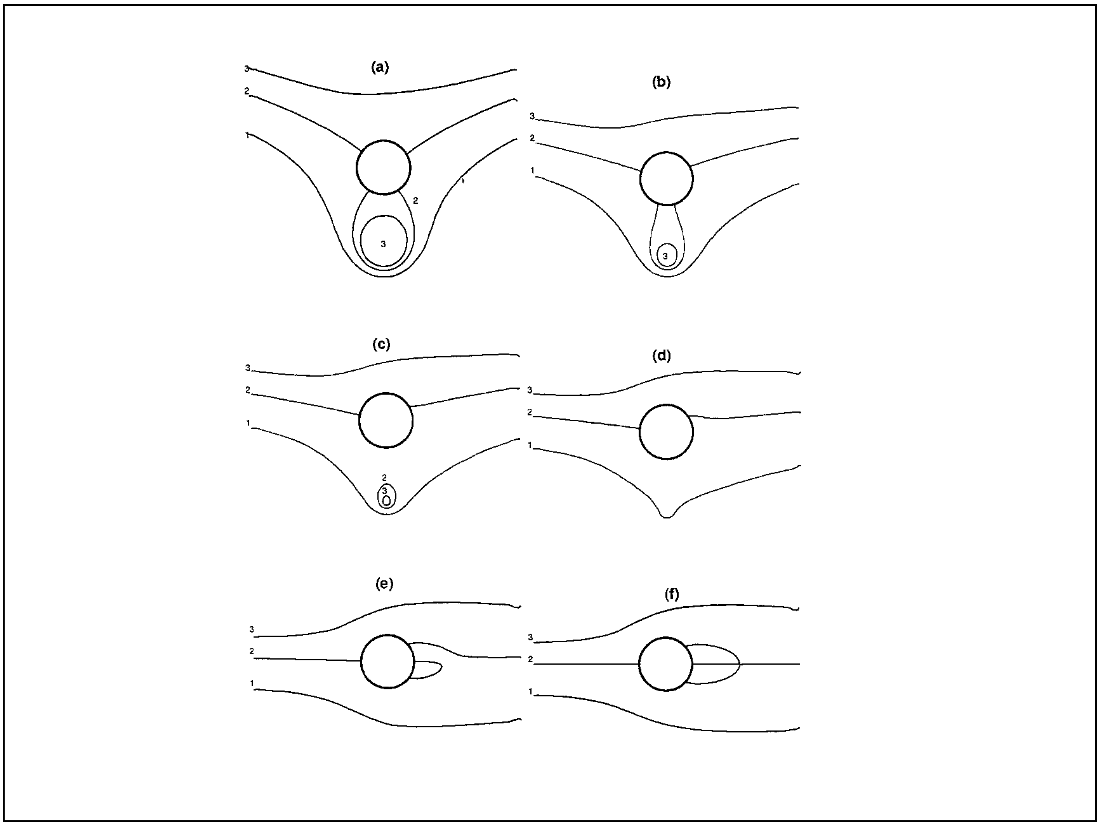

Figure 10 (a, b, c, d, e, f). The numerically obtained streamlines, Ψ, for $\mathrm{Re} = 20$ with a mesh size 40×160. The streamlines labelled 1, 2 and 3 correspond to $\Psi = -1,\ 0\ \mathit{and}\ 0.7$, respectively.
$(a)\beta = 3,\ (b)\beta = 1.5,\ (c)\beta = 1,\ (d)\beta = 0.5,\ (e)\beta = 0.1,\ \mathit{and}\ (f)\beta = 0.$

the flow (when β exceeds unity) appears to be dominated by the rotlet, even at a Reynolds number of 20. The dominance of the rotlet is seen by the symmetrical nature of the flow about the x-axis and y-axis in the vicinity of the body. As the strength of the rotlet decreases, the symmetrical flow pattern diminishes. However, it is only when the value of β falls below 0.5 that one sees the closed streamline behind the cylinder and even when the value of β reaches 0.1 this closed region behind the cylinder is only present in the region $y < 0$ due to the rotation from the rotlet. Obviously as β tends to zero the closed streamline region will move to its symmetrical pattern about the x - axis.

Conclusions

Numerical solutions of the full, steady, two-dimensional Navier-stokes equations have been obtained for the fluid flow past a circular cylinder for Reynolds number $\mathrm{Re} = 1, 3, 5$ and 20 and rotational parameter $0 \le \alpha \le 2$ when a rotlet of any given strength is placed at various positions outside the cylinder. Although numerical solutions can be found for any value of Reynolds number greater than zero it is well known that when the Reynolds number is zero that a solution only exists provided there is a relationship between the strength of the rotlet and its location. Further, when the Reynolds number is zero, the analytical solutions are such that the drag, lift and moment are all identically zero. Therefore in this paper we have concentrated on obtaining numerical results with the drag, lift and moment zero and developed a technique for extrapolating the results obtained at small values of the Reynolds number to predict the solutions for zero Reynolds number. Using this technique it is found that the numerical results are in excellent agreement with all the available theoretical results. We conclude that the technique developed in this paper may be used with confidence to predict solutions for flows at zero Reynolds numbers for situations where there are no analytical solutions.

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