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On the Solvability of a Quasilinear Parabolic Problem with Neumann Boundary Condition

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ABSTRACT

This paper establishes the existence and uniquenesss of a weak solution of a quasilinear parabolic problem in an open set whose boundary is the union of two disjoint closed surfaces. A Dirichlet condition is prescribed on the exterior boundary and a Neumann condition on the interior boundary. The existence of a solution of the parabolic problem is shown using the Faedo-Galerkin method and some a priori estimates are established to provide bounds for the solution.

Keywords: Parabolic, quasilinear, Neumann boundary condition, weak solution

INTRODUCTION

This paper presents the existence and uniqueness of a solution to a quasilinear parabolic problem on a domain with holes. More precisely, the domain consists of open sets O_0 and O_1 (representing the hole in O) with corresponding boundaries Γ_0 and Γ_1. We then form $\mathcal{O}=\mathcal{O}_0\setminus\overline{\mathcal{O}_1}$, $\mathcal{O}\in\mathbb{R}^N$ whose boundary is $\Gamma_0\cup\Gamma_1$. On Γ_0, we prescribe a Dirichlet boundary condition, and on Γ_1, a Neumann boundary condition. The parabolic system is given by

$$\begin{cases} u'(x,t)-div(A(x,u)\nabla u(x,t))=f(x,t) & in \quad \mathcal{O}\times(0,T) \\ u(x,t)=0 & on \quad \Gamma_0\times(0,T) \\ A(x,u)\nabla u(x,t)\cdot n=g(x,t) & on \quad \Gamma_1\times(0,T) \\ u(x,0)=u^0(x) & in \quad \mathcal{O}, \end{cases} \tag{1}$$

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where A is a quasilinear matrix field, n is the outward unit normal to O, and $t \in \mathbb{R}$. This problem models heat flow in a perforated cell where the temperature u is time-dependent. Here, A is a constant proper to the material, which represents the thermal conductivity of the said material, and f can be considered as the heat source. The second equation in (1) indicates that the material is insulated. For the third equation, we say that, on the inner boundary, g is the normal component of the heat flux. The fourth equation means that, at time $t = 0$, the temperature is given by u^0.

Several studies related to this problem have been previously performed. A more general problem of the elliptic case was studied by Cabbarubias and Donato (2011). The solution to a linear parabolic problem in a fixed domain was shown by Cioranescu and Donato (1999) and Cioranescu et al. (2012), while that for the quasilinear problem in a fixed domain was considered by (Zeidler, 1990). Existence results for the parabolic problem in a two-component domain with imperfect interface have been proven by Jose (2009) and for the corresponding elliptic problem by Beltran (2014tru). Estimates and bounds for solutions of certain quasilinear parabolic partial differential equations were shown by Trudinger (1968), Ikeda (1967), and Arima (1966). Moreover, bounds for solutions of some linear and quasilinear cases were presented by Cipriani (2001). As shown by Poretta (1999), existence results for nonlinear parabolic equations can be obtained by proving strong convergence of truncations of solutions.

In this study, the existence of the weak solution to problem (1) (which is posed in a perforated domain) is shown using the Faedo-Galerkin method. This method involves defining an approximate problem in a finite-dimensional space. The partial differential equation is reduced to an infinite system of ordinary differential equations. Estimates are obtained and shown to converge to the solution of the original problem. The Faedo-Galerkin method has been used for parabolic problems in fixed domains. It was used for the linear case by Cioranescu and Donato (1999) and Cioranescu et al. (2012) and for the quasilinear case in fixed domain by Zeidler (1990).

In Section 2, we present the geometric setting and assumptions for the problem. We also give the variational formulation and the approximate problem. In Section 3, we establish the existence and uniqueness of the weak solution to problem (1).

PRELIMINARIES

In this section, we discuss the notations and concepts necessary to prove the main result.

We consider the domain illustrated in Figure 1. Here, O_0 is an open subset of $\mathbb{R}^N$ with boundary and O_1 is an open subset, such that $\overline{\mathcal{O}_1} \subset \mathcal{O}_0$ with boundary $\Gamma_1 = \partial \mathcal{O}_1$. We set $\mathcal{O} := \mathcal{O}_0 \setminus \overline{\mathcal{O}_1}$. Then, $\partial \mathcal{O} = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \varnothing$.

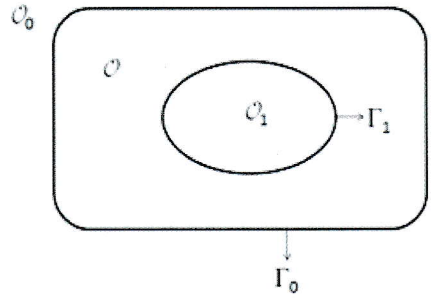

Figure 1. The perforated domain.

We will use the notion of an evolution triple to define the spaces where the solution of problem (1) exists.

Definition 2.1. Let V be a real separable and reflexive Banach space, B a real, separable Hilbert space, and V' the dual of V. If the embedding $V \subseteq B$ is continuous, that is,

$$||v||_B \leq c\, ||v||_V \; for\ some\ c\ and\ for\ all\ \ v \in V,$$

then we call $V \subseteq B \subseteq V'$ an evolution triple.

It is known that problem (1) does not always have a solution in the space of regular functions; hence, we introduce suitable spaces where weak solutions for problem (1) exist.

Let

$$V = \{\varphi \in H^1(\mathcal{O}) | \varphi = 0 \ on\ \Gamma_0\} \ and\ B = L^2(\mathcal{O}) \tag{2}$$

with

$$\| \varphi \|_V = \| \nabla \varphi \|_{L^2(\mathcal{O})} \, . \tag{3}$$

Let $\{w_1, w_2, \ldots\}$ be a basis of V and $B_n = span\{w_1, w_2, \ldots\}$. Clearly, $B_n \subseteq V \subseteq B$. Also, observe that $V \subseteq B \subseteq V'$ is an evolution triple (see Remark 3.44 of Cioranescu and Donato (1999)).

We consider the space given by

$$\mathcal{W} = \{v \in L^2(0,T;V), v' \in L^2(0,T;V')\}, \tag{4}$$

equipped with the norm

$$\| v \|_{\mathcal{W}} = \| v \|_{L^2(0,T;V)} + \| v' \|_{L^2(0,T;V')} . \tag{5}$$

It is known that the space W satisfies the following (see for instance Cioranescu and Donato (1999), and Cioranescu et al. (2012)):

1. The inclusion $\mathcal{W} \subset C^0(0,T;L^2(\mathcal{O}))$ is continuous.

2. The inclusion $\mathcal{W} \subset L^2(0,T;L^2(\mathcal{O}))$ is compact.

3. For u and v in W, the following differentiation formula holds:

$$\frac{d}{dt}\int_\Omega u(x,t)v(x,t)dx = \langle u'(\cdot,t), v(\cdot,t)\rangle_{V',V} + \langle v'(\cdot,t), u(\cdot,t)\rangle_{V',V} .$$

For problem (1), we make the following assumptions:

H1. Suppose f, g are functions, such that

1. $f \in L^2(0,T;L^2(\mathcal{O}))$, and

2. $g \in L^2(0,T;L^2(\Gamma_1))$.

H2. Let $A:(x,t) \in \mathcal{O} \times \mathbb{R} \mapsto A(x,t) = a_{ij}(x,t)_{i,j=1,\dots,N} \in \mathbb{R}^{N^2}$ be a matrix field satisfying the following conditions:

1. For every $t \in \mathbb{R}$, $A(\cdot,t) \in M(\alpha,\beta,\mathcal{O})$, that is,

(i) $(A(x,t)\lambda,\lambda) \geq \alpha |\lambda|^2$, and

(ii) $|A(x,t)\lambda| \leq \beta|\lambda|$, for any $\lambda \in \mathbb{R}^N$.

2. A is a Caratheodory function, that is,

(i) $t \mapsto A(x,t)$ is continuous for every $x \in \mathrm{O}$, and

(ii) $x \mapsto A(x,t)$ is measurable for every $t \in \mathbb{R}$.

H3. There exists a function $\omega : \mathbb{R} \to \mathbb{R}$, such that

1. ω is continuous and nondecreasing, with $\omega(z) > 0 \ \forall z > 0$,

2. $| A(x,z) - A(x,z_1) | \leq \omega(| z - z_1 |)$, a.e. $x \in \mathcal{O}, z \neq z_1$, and

3. for any $y > 0, \lim_{x \to 0^+} \int_x^y \frac{dz}{\omega(z)} = +\infty$.

To obtain a weak solution to problem (1), we consider its variational formulation in the appropriate Hilbert space. To get the variational formulation, we multiply the first equation of (1) by an arbitrary test function $v(x) \in V$ and integrate over $\mathcal{O}$, and by applying Green's Theorem, one gets

$$\int_{\mathcal{O}} u'(x,t) v(x)\, dx + \int_{\mathcal{O}} A(x,u) \nabla u(x,t) \nabla v(x) dx = \int_{\mathcal{O}} f(x,t) v(x) dx + \int_{\Gamma_1} g(x,t) v(x) d\sigma . \tag{6}$$

Hence, solving (1) is equivalent to solving the problem

$$\begin{cases} Find \ u \in \mathcal{W} \ such \ that \\ \langle u'(x,t), v(x) \rangle_{V',V} + \int_{\mathcal{O}} A(x,u) \nabla u(x,t) \nabla v(x) dx = \int_{\mathcal{O}} f(x,t) v(x) dx \\ \qquad + \int_{\Gamma_1} g(x,t) v(x) d\sigma \quad \forall v \in V \ and \ t \in (0,T) \\ (x,0) = u^0(x) \ in \ \mathcal{O}, \end{cases}$$

where $\mathcal{W}$ is defined in (4).

Problem (6) is called the variational formulation of (1).

We can also show that, if $u \in C^2(\overline{\mathcal{O}})$ and u is a solution of (6), then u is a solution of (1). Indeed, we again apply Green's Theorem to (6) to obtain

$$\int_{\mathcal{O}} u'(x,t) v(x) dx + \sum_i \int_{\mathcal{O}} \frac{\partial}{\partial x_i} (A(x,u) \nabla u(x,t))_i v(x) dx - \int_{\Gamma_1} g(x,t) v(x) d\sigma = \int_{\mathcal{O}} f(x,t) v(x) dx.$$

From classical results (see Cioranescu and Donato (1999)), u satisfies (1).

Remark 2.2. The above computations show that if the solution u is sufficiently regular, then the solution of the variational formulation (6) and the solution of problem (1) are the same, that is, the weak solution is also the solution in the classical sense.

To find the solution of problem (6), which is the weak solution of (1), we apply the Faedo-Galerkin method which is widely used in dealing with parabolic problems (see for example Cioranescu and Donato (1999) and Cioranescu et al. (2012)). We construct the approximate problem as follows.

Let $P_n : B \to B_n$ be the orthogonal projection onto B_n given by

$$P_n : v \in L^2(\mathcal{O}) \mapsto P_n(v) = \sum_{j=1}^{n} (v, w_j)_B, \, w_j \in B_n. \tag{7}$$

By Proposition 6.19 of Cioranescu et al. (2012),

$$u_n^0 := P_n(u^0) \to u^0 \quad \textit{strongly in } L^2(\mathcal{O}). \tag{8}$$

The Galerkin equations, which give the approximate problem for (6), are given by

$$\begin{cases} \textit{Find } u_n(x,t) \in B_n \textit{ such that} \\ \langle u_n'(x,t), v(x) \rangle_{V',V} + \int_{\mathcal{O}} A(x,u_n) \nabla u_n(x,t) \nabla v(x) dx = \int_{\mathcal{O}} f(x,t) v(x) dx \\ + \int_{\Gamma_1} g(x,t) v(x) d\sigma \quad \forall v \in B_n \textit{ and } t \in (0,T) \\ u_n(x,0) = u_n^0(x) \textit{ in } \mathcal{O}. \end{cases} \tag{9}$$

This is equivalent to

$$\begin{cases} \textit{Find } u_n(x,t) = \sum_{j=1}^{n} c_j^n(t) w_j(x) \in L^2(0,T;V) \textit{ such that} \\ \int_{\mathcal{O}} u_n'(x,t) w_k(x) dx + \int_{\mathcal{O}} A(x,u_n) \nabla u_n(x,t) \nabla w_k(x) dx = \int_{\mathcal{O}} f(x,t) w_k(x) dx \\ \quad + \int_{\Gamma_1} g(x,t) w_k(x) d\sigma \quad \forall k = 1,\dots,n \textit{ and for a.e. } t \in (0,T), w_k \in V, \\ u_n(x,0) = u_n^0(x) \textit{ in } \mathcal{O}. \end{cases} \tag{10}$$

which is also equivalent to the following first-order system of ordinary differential equations:

$$\begin{cases} \frac{dc_k^n}{dt}(t) + \sum_{j=1}^{n} c_j^n(t) \int_{\mathcal{O}} A(x, \sum_{j=1}^{n} c_j^n(t) w_j(x)) \nabla w_j(x) \nabla w_k(x) dx = \int_{\mathcal{O}} f(x,t) w_k(x) dx \\ + \int_{\Gamma_1} g(x,t) w_k(x) d\sigma \; \forall k = 1,2,\dots n \textit{ and for a.e. } t \in (0,T), \, w_k \in V, \\ u_k^n(0) = (u^0, w_k). \end{cases}$$

This system admits a unique solution $c_1^n, \dots c_k^n$ on $[0,T]$ (see Zeidler (1990), p.781). Thus, (10) has a unique solution $u_n(x,t) \in L^2(0,T;B_n)$ with $u_n'(x,t) \in L^2(0,T;B_{n'})$. Note that, since $B_n \subseteq V \subseteq B \subseteq V'$, it follows that $u_n \in L^2(0,T;V)$ and $u_n' \in L^2(0,T;V')$.

Remark 2.3. In the discussions that follow, we denote any arbitrary constant independent of n by C.

MAIN RESULT

This section presents the existence and uniqueness of the solution to problem (6), as well as the a priori estimates of the solution u_n and u_n' of the approximate problem (9). These estimates are shown to converge to the solution of the original problem. For uniqueness of the solution, we use the method introduced by Chipot (2009).

First, we show the a priori estimates of the solution u_n of problem (10).

Proposition 3.1. Let u_n be the solution of problem (9). Then,

$$\|u_n(x,t)\|_{L^\infty(0,T;L^2(\mathcal{O}))} + \|u_n(x,t)\|_{L^2(0,T;V)} \leq C, \tag{11}$$

where C depends on α, β, $\mathcal{O}$, and T, but is independent of n.

Proof. We fix $t \in [0,T]$, multiply (10) by $c_k^n \in C([0,T])$, and sum over k from 1 to n to get

$$\begin{aligned}&\int_{\mathcal{O}} u_n'(x,t)u_n(x,t)dx + \int_{\mathcal{O}} A(x,u_n)\nabla u_n(x,t)\nabla u_n(x,t)dx \\ &\quad = \int_{\mathcal{O}} f(x,t)u_n(x,t)dx + \int_{\Gamma_1} g(x,t)u_n(x,t)d\sigma.\end{aligned} \tag{12}$$

From assumption H2(1), we have

$$\begin{aligned}\int_{\mathcal{O}} A(x,u_n)\nabla u_n(x,t)\nabla u_n(x,t)dx &\geq \int_{\mathcal{O}} \alpha\, |\nabla u_n(x,t)|^2\, dt \\ &= \alpha\, \|\nabla u_n\|^2_{L^2(\mathcal{O})}.\end{aligned}$$

Also, one has

$$\frac{1}{2}\frac{d}{dt}\|u_n(x,t)\|^2_{L^2(\mathcal{O})} = \int_{\mathcal{O}} u_n'(x,t)u_n(x,t)dx,$$

so that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_n(x,t)\|^2_{L^2(\mathcal{O})} + \alpha\, \|\nabla u_n(x,t)\|^2_{L^2(\mathcal{O})} \\ &\leq \int_{\mathcal{O}} u_n'(x,t)u_n(x,t)dx + \int_{\mathcal{O}} A(x,u_n)\nabla u_n(x,t)\nabla u_n(x,t)dx \\ &= \int_{\mathcal{O}} f(x,t)u_n(x,t)dx + \int_{\Gamma_1} g(x,t)u_n(x,t)d\sigma.\end{aligned}$$

Using the Cauchy-Schwarz and Poincaré inequalities, the Trace Theorem, and in view of (3), it follows that

$$\frac{d}{dt}\|u_n(x,t)\|^2_{L^2(\mathcal{O})} + 2\alpha\|u_n(x,t)\|^2_V \le C(\|f(x,t)\|_{L^2(\mathcal{O})} + \|g(x,t)\|_{L^2(\Gamma_1)})\|u_n(x,t)\|_{H^1(\mathcal{O})} \tag{13}$$

where C is a constant independent of n.

Since

$$\|u_n(x,t)\|^2_{H^1(\mathcal{O})} = \|u_n(x,t)\|^2_{L^2(\mathcal{O})} + \|\nabla u_n(x,t)\|^2_{L^2(\mathcal{O})},$$

and using the Poincaré inequality and (3), we have

$$\begin{aligned} \|u_n(x,t)\|^2_{H^1(\mathcal{O})} &= \|u_n(x,t)\|^2_{L^2(\mathcal{O})} + \|\nabla u_n(x,t)\|^2_{L^2(\mathcal{O})} \\ &\le C\,\|\nabla u_n(x,t)\|^2_{L^2(\mathcal{O})} + \|\nabla u_n(x,t)\|^2_{L^2(\mathcal{O})} \\ &= C\,\|u_n(x,t)\|^2_V + \|u_n(x,t)\|^2_V \\ &= C\,\|u_n(x,t)\|^2_V, \end{aligned} \tag{14}$$

for some C depending on $\mathcal{O}$.

Using $\eta = \frac{C}{\alpha}$ in Young's inequality $ab \le \frac{\eta}{2}a^2 + \frac{1}{2\eta}b^2$ for all $\eta \ge 0$, we get from (13),

$$\begin{aligned} &\frac{d}{dt}\|u_n(x,t)\|^2_{L^2(\mathcal{O})} + 2\alpha\|u_n(x,t)\|^2_V \\ &\le \frac{C}{\alpha}\left(\|f(x,t)\|_{L^2(\mathcal{O})} + \|g(x,t)\|_{L^2(\Gamma_1)}\right)^2 + \frac{\alpha}{C}\|u_n(x,t)\|^2_{H^1(\mathcal{O})}, \end{aligned} \tag{15}$$

where C is a constant independent of n.

Combining (14) and (15) yields

$$\frac{d}{dt}\|u_n(x,t)\|^2_{L^2(\mathcal{O})} + \alpha\,\|u_n(x,t)\|^2_V \le \frac{C}{\alpha}(\|f(x,t)\|_{L^2(\mathcal{O})} + \|g(x,t)\|_{L^2(\Gamma_1)})^2.$$

We fix $s \in [0,T]$ and integrate (15) on $(0,s)$ to obtain

$$\begin{aligned}
&\|u_n(x,s)\|^2_{L^2(\mathcal{O})} + \alpha \int_0^s \|u_n(x,t)\|^2_V \, dt \\
&\leq \|u_n(x,0)\|^2_{L^2(\mathcal{O})} + \frac{C}{\alpha}\int_0^s \left(\|f(x,t)\|_{L^2(\mathcal{O})} + \|g(x,t)\|_{L^2(\Gamma_1)} \right)^2 dt \\
&= \|u_n(x,0)\|^2_{L^2(\mathcal{O})} + \frac{C}{\alpha}\left(\int_0^s \|f(x,t)\|^2_{L^2(\mathcal{O})} dt + \int_0^s \|g(x,t)\|^2_{L^2(\Gamma_1)} dt \right) \\
&\quad + \frac{2C}{\alpha}\int_0^s (\|f(x,t)\|_{L^2(\mathcal{O})} \|g(x,t)\|_{L^2(\Gamma_1)}) dt.
\end{aligned}$$

This, together with assumption H1 and (8), implies $u_n \in L^\infty(0,T;L^2(\mathcal{O})) \cap L^2(0,T;V)$ with

$$\begin{aligned}
&\|u_n(x,t)\|_{L^\infty(0,T;L^2(\mathcal{O}))} + \|u_n(x,t)\|_{L^2(0,T;V)} \\
&\leq C(\|u_n(x,0)\|_{L^2(\mathcal{O})} + \|f(x,t)\|_{L^2(0,T;L^2(\mathcal{O}))} + \|g(x,t)\|_{L^2(0,T;L^2(\Gamma_1))} \\
&\quad + \|f(x,t)\|_{L^2(0,T;L^2(\mathcal{O}))} \|g(x,t)\|_{L^2(0,T;L^2(\Gamma_1))}).
\end{aligned} \tag{16}$$

Hence, we obtain (11).

Next, we show the a priori estimates of u_n', the derivative of u_n, with respect to time.

Proposition 3.2. Let u_n be the solution of (9) with u_n' its derivative with respect to time. Then,

$$\|u_n'(x,t)\|_{L^2(0,T;V')} \leq C, \tag{17}$$

where C depends on α, β, $\mathcal{O}$ and T, but is independent of n.

Proof. Let $v \in V$. From (7), we have

$$v - P_n(v) \in B_n^\perp \quad and \quad \|P_n(v)\|_V \leq \|v\|_V, \tag{18}$$

then from (9),

$$\begin{aligned}
&\langle u_n'(x,t), v(x) \rangle_{V',V} + \int_{\mathcal{O}} A(x,u_n)\nabla u_n(x,t)\nabla(P_n(v(x)))dx \\
&= (u_n'(x,t), v(x))_{L^2(\mathcal{O})} + \int_{\mathcal{O}} A(x,u_n)\nabla u_n(x,t)\nabla(P_n(v(x)))dx \\
&= (u_n'(x,t), P_n(v))_{L^2(\mathcal{O})} + \int_{\mathcal{O}} A(x,u_n)\nabla u_n(x,t)\nabla(P_n(v(x)))dx \\
&= \int_{\mathcal{O}} f(x,t)(P_n(v(x)))dx + \int_{\Gamma_1} g(x,t)P_n(v(x))dx.
\end{aligned}$$

Using assumption H2(1), the Cauchy-Schwarz and Poincaré inequalities, (3), and (18), we obtain

$$\begin{aligned} \left|\langle u_n'(x,t),v\rangle_{V',V}\right| &\le C\left(\|f(x,t)\|_{L^2(\mathcal{O})}+\|g(x,t)\|_{L^2(\Gamma_1)}\right)\|P_n(v(x))\|_V \\ &\quad+\beta\,\|u_n(x,t)\|_V\,\|P_n(v(x))\|_V \\ &\le\left[C\left(\|f(x,t)\|_{L^2(\mathcal{O})}+\|g(x,t)\|_{L^2(\Gamma_1)}\right)+\beta\,\|u_n(x,t)\|_V\right]\|v(x)\|_V, \end{aligned}$$

where C is a constant independent of n. It follows that

$$\|u_n'(x,t)\|_{V'}\le C\left(\|f(x,t)\|_{L^2(\mathcal{O})}+\|g(x,t)\|_{L^2(\Gamma_1)}\right)+\beta\,\|u_n(x,t)\|_V.$$

Integrating on $[0,T]$ with respect to t, we have

$$\begin{aligned} \|u_n'(x,t)\|_{L^2(0,T;V')} &\le C\left(\|f(x,t)\|_{L^2(0,T;L^2(\mathcal{O}))}+\|g(x,t)\|_{L^2(0,T;L^2(\Gamma_1))}\right) \\ &\quad+\beta\,\|u_n(x,t)\|_{L^2(0,T;V)}. \end{aligned} \tag{19}$$

This, together with (11), yields (17).

We now state our main result.

Theorem 3.3. Under assumptions H1-H3, problem (1) has a unique solution $u\in\mathcal{W}$. Moreover, there exists a constant C depending on $\alpha,\mathcal{O},T$, such that

$$\|u(x,t)\|_{L^\infty(0,T;L^2(\mathcal{O}))}+\|u(x,t)\|_{\mathcal{W}}\le C. \tag{20}$$

Proof. The theorem is established in four steps.

Step 1. Existence of the solution.

By the estimates obtained in Proposition 3.1 and Proposition 3.2, and using Eberlein-Simuljan Theorem, there exists a subsequence, still denoted by $\{u_n\}\in L^\infty(0,T;L^2(\mathcal{O}))$ and $u\in L^2(0,T;V)\cap L^\infty(0,T;L^2(\mathcal{O}))$, such that

$$\begin{array}{llll} (i) & u_n \rightharpoonup u & weakly* \ in & L^\infty(0,T;L^2(\mathcal{O})), \\ (ii) & u_n \rightharpoonup u & weakly \ in & L^2(0,T;V), \\ (iii) & u_n' \rightharpoonup u' & weakly \ in & L^2(0,T;V'). \end{array} \tag{21}$$

Since u_n is bounded and $\mathcal{W}$ is compact in $L^2(0,T;L^2(\mathrm{O}))$, then there exists some $U\in L^2(0,T;L^2(\mathcal{O}))$, such that

$$u_n\to U \ strongly \ in \ L^2(0,T;L^2(\mathrm{O})).$$

From (21)(ii), it follows that

$$U = u.$$

Hence,

$$u_n \to u \quad strongly\ in\ L^2(0,T;L^2(\mathcal{O})), \quad (22)$$

and so there exists a subsequence, still denoted by $\{u_n\} \in L^\infty(0,T;L^2(\mathcal{O}))$, such that

$$u_n \to u \quad strongly\ in\ L^2(\mathcal{O}).$$

Now, since A satisfies H2, we have

$$A(x,u_n) \to A(x,u) \quad strongly\ in\ L^2(\mathcal{O}),$$

so that for every $w_k \in V$,

$${}^t A(x,u_n)\nabla w_k \to {}^t A(x,u)\nabla w_k\ strongly\ in\ L^2(\mathrm{O}).$$

Note also that from (21)(ii),

$$\nabla u_n \rightharpoonup \nabla u\ weakly\ in\ [L^2(0,T;V)]^N.$$

Hence, by the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^T\int_{\mathcal{O}} A(x,u_n)\nabla u_n(x,t)\nabla w_k(x)dxdt \\
&= \lim_{n\to\infty}\int_0^T\int_{\mathcal{O}} {}^t A(x,u_n)\nabla w_k(x)\nabla u_n(x,t)dxdt \\
&= \int_0^T\int_{\mathcal{O}} {}^t A(x,u)\nabla w_k(x)\nabla u(x,t)dxdt \\
&= \int_0^T\int_{\mathcal{O}} A(x,u)\nabla u(x,t)\nabla w_k(x)dxdt.
\end{aligned} \quad (23)$$

Next, let ξ be a continuously differentiable function of t. Multiplying the approximate problem (10) by this function and integrating on $[0,T]$, we have

$$\begin{aligned}
&\int_0^T\int_{\mathcal{O}} u_n'(x,t)w_k(x)\xi(t)dxdt + \int_0^T\int_{\mathcal{O}} A(x,u_n)\nabla u_n(x,t)\nabla w_k(x)\xi(t)dxdt \\
&= \int_0^T\int_{\mathcal{O}} f(x,t)w_k(x)\xi(t)dxdt + \int_0^T\int_{\Gamma_1} g(x,t)w_k(x)\xi(t)d\sigma\,dt.
\end{aligned} \quad (24)$$

Passing to the limit as $n \to \infty$ using (21), (22) and (23), we obtain

$$\int_0^T \int_O u'(x,t) w_k(x)\xi(t)dxdt + \int_0^T \int_O A(x,u)\nabla u(x,t)\nabla w_k(x)\xi(t)dxdt$$
$$= \int_0^T \int_O f(x,t) w_k(x)\xi(t)dxdt + \int_0^T \int_{\Gamma_1} g(x,t) w_k(x)\xi(t)d\sigma\, dt.$$

Since $\{w_1, w_2, \ldots\}$ is a basis for V, then for all $v(x) \in V$,

$$\int_0^T \langle u'(x,t), \xi(t)v(x)\rangle_{V'V}\, dt + \int_0^T \int_O A(x,u)\nabla u(x,t)\nabla v(x)\xi(t)dxdt$$
$$= \int_0^T \int_O f(x,t) v(x)\xi(t)dxdt + \int_0^T \int_{\Gamma_1} g(x,t) v(x)\xi(t)d\sigma\, dt, \qquad (25)$$

which shows that u is a solution of (6).

Step 2. Satisfaction of the initial condition.

Now, we show that u satisfies the initial condition $u(x,0) = u^0(x)$. Let $\xi(0) = 1$ and $\xi(T) = 0$. Integrating the first term of (24) by parts with respect to t, we get

$$-\int_O u_n(x,0) w_k(x)dx - \int_0^T \int_O u_n(x,t)\xi'(t) w_k(x)dxdt$$
$$+ \int_0^T \int_O A(x,u_n)\nabla u_n(x,t)\nabla w_k(x)\xi(t)dxdt$$
$$= \int_0^T \int_O f(x,t) w_k(x)\xi(t)dxdt + \int_0^T \int_{\Gamma_1} g(x,t) w_k(x)\xi(t)d\sigma\, dt.$$

In view of (8) and (22), passing to the limit as $n \to \infty$ in this equation gives

$$-\int_0^T \int_O u(x,t)\xi'(t) w_k(x)dxdt - \int_O u^0(x) w_k(x)dx$$
$$+ \int_0^T \int_O A(x,u)\nabla u(x,t) w_k(x)\xi(t)dxdt$$
$$= \int_0^T \int_O f(x,t) w_k(x)\xi(t)dxdt + \int_0^T \int_{\Gamma_1} g(x,t) w_k(x)\xi(t)d\sigma\, dt.$$

Again, for all $v \in V$, we have

$$-\int_0^T \int_O u(x,t)\xi'(t) v(x)dxdt - \int_O u^0(x) v(x)dx$$
$$+ \int_0^T \int_O A(x,u)\nabla u(x,t)\nabla v(x)\xi(t)dxdt \qquad (26)$$
$$= \int_0^T \int_O f(x,t) v(x)\xi(t)dxdt + \int_0^T \int_{\Gamma_1} g(x,t) v(x)\xi(t)d\sigma\, dt.$$

Subtracting this from (25), we get

$$\int_0^T \langle u'(x,t), \xi(t)v(x)\rangle_{V',V} + \int_0^T \int_{\mathcal{O}} u(x,t)\xi'(t)v(x)dxdt + \int_{\mathcal{O}} u^0(x)v(x)dx = 0. \quad (27)$$

On the other hand, differentiating $\int_{\mathcal{O}} u(x,t)\xi(t)v(x)dx$ with respect to t, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathcal{O}} u(x,t)\xi(t)v(x)dx \\ &= \langle u'(x,t), \xi(t)v(x)\rangle_{V',V} + \langle \xi'(t)v(x), u(x,t),\rangle_{V',V} \\ &= \langle u'(x,t), \xi(t)v(x)\rangle_{V',V} + \int_{\mathcal{O}} u(x,t)\xi'(t)v(x)dx.\end{aligned}$$

Integrating this equation with respect to t over $[0, \mathrm{T}]$, we obtain

$$\int_0^T \int_{\mathcal{O}} u(x,t)\xi'(t)v(x)dx + \int_0^T \langle u'(x,t), \xi(t)v(x)\rangle_{V',V} = -\int_{\mathcal{O}} u(x,0)v(x)dx.$$

This, together with (27), gives

$$\int_{\mathcal{O}} u^0(x)v(x)dx = \int_{\mathcal{O}} u(x,0)v(x)dx$$

and hence, $u^0(x) = u(x,0)$, which implies that u satisfies (6).

Step 3. A priori estimates on u.

From (21), we have

$$\begin{aligned}\|u(x,t)\|_{L^\infty(0,T;L^2(\mathcal{O}))} &\leq \liminf_{n\to\infty} \|u_n(x,t)\|_{L^\infty(0,T;L^2(\mathcal{O}))} \\ \|u(x,t)\|_{L^2(0,T;V)} &\leq \liminf_{n\to\infty} \|u_n(x,t)\|_{L^2(0,T;V)} \\ \|u'(x,t)\|_{L^2(0,T;V')} &\leq \liminf_{n\to\infty} \|u'_n(x,t)\|_{L^2(0,T;V')}.\end{aligned}$$

Using (16) and (19), we obtain

$$\begin{aligned}&\|u(x,t)\|_{L^\infty(0,T;L^2(\mathcal{O}))} + \|u(x,t)\|_W \\ &\leq c\lim_{n\to\infty}(\|u_n(x,0)\|_{L^2(\mathcal{O})} + \|f(x,t)\|_{L^2(0,T;L^2(\mathcal{O}))} + \|g(x,t)\|_{L^2(0,T;L^2(\Gamma_1))}) \\ &\qquad + \|f(x,t)\|_{L^2(0,T;L^2(\mathcal{O}))}\|g(x,t)\|_{L^2(0,T;L^2(\Gamma_1))} \\ &\leq c(\|u_0\|_{L^2(\mathcal{O})} + \|f(x,t)\|_{L^2(0,T;L^2(\mathcal{O}))} + \|g(x,t)\|_{L^2(0,T;L^2(\Gamma_1))} \\ &\qquad + \|f(x,t)\|_{L^2(0,T;L^2(\mathcal{O}))}\|g(x,t)\|_{L^2(0,T;L^2(\Gamma_1))})\end{aligned} \quad (28)$$

which gives (20).

Step 4. Uniqueness of the solution.

We now show the uniqueness of the solution of (6). We follow the procedure introduced by Chipot (2009) (see also Cabarrubias and Donato (2011)).

Suppose u_1 and u_2 are solutions of (6). Without loss of generality, we can assume that $u_1'(x,t) - u_2'(x,t) \geq 0$ a.e. Then for every $v(x) \in V$,

$$\int_{\mathcal{O}} u_1'(x,t)v(x)dx + \int_{\mathcal{O}} A(x,u_1)\nabla u_1(x,t)\nabla v(x)dx$$
$$= \int_{\mathcal{O}} f(x,t)v(x)dx + \int_{\Gamma_1} g(x,t)v(x)d\sigma$$

and

$$\int_{\mathcal{O}} u_2'(x,t)v(x)dx + \int_{\mathcal{O}} A(x,u_2)\nabla u_2(x,t)\nabla v(x)dx$$
$$= \int_{\mathcal{O}} f(x,t)v(x)dx + \int_{\Gamma_1} g(x,t)v(x)d\sigma.$$

It follows that

$$\int_{\mathcal{O}} u_1'(x,t)v(x)dx + \int_{\mathcal{O}} A(x,u_1)\nabla u_1(x,t)\nabla v(x)dx$$
$$= \int_{\mathcal{O}} u_2'(x,t)v(x)dx + \int_{\mathcal{O}} A(x,u_2)\nabla u_2(x,t)\nabla v(x)dx,$$

which can be rewritten as

$$\int_{\mathcal{O}} [u_1'(x,t) - u_2'(x,t)]v(x)dx + \int_{\mathcal{O}} A(x,u_1)\nabla u_1(x,t)\nabla v(x)dx$$
$$= \int_{\mathcal{O}} A(x,u_2)\nabla u_2(x,t)\nabla v(x)dx. \tag{29}$$

Let $\delta > 0$. As in Chipot (2009), we define

$$F_\delta(x) = \begin{cases} \int_\delta^x \frac{dz}{\omega^2(z)} & \text{if } x \geq \delta, \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

where $\omega(z)$ is given in H3. Note that $F_\delta(x)$ is a continuous C^1 function, such that

$$|F_\delta'(x)| \leq \frac{1}{\omega^2(\delta)},$$

for all $x > \delta$. It follows from classical results (see Corollary 2.15 of Chipot (2009)) that $F_\delta(u_1 - u_2) \in V$ and

$$\nabla(F_\delta(u_1 - u_2)) = F_\delta'(u_1 - u_2)\nabla(u_1 - u_2). \tag{31}$$

We set

$$E = [u_1 - u_2 > \delta] := \{x \in \mathcal{O} \mid (u_1 - u_2)(x) > \delta\}.$$

Taking $v = F_\delta(u_1 - u_2)$ in (29) as test function and subtracting

$$\int_O A(x,u_1)\nabla u_2 \nabla F_\delta(u_1 - u_2)dx$$

on both sides of (29), we get

$$\begin{aligned}&\int_O (u_1' - u_2')F_\delta(u_1 - u_2)dx + \int_O A(x,u_1)\nabla(u_1 - u_2)\nabla F_\delta(u_1 - u_2)dx \\ &= -\int_O [A(x,u_1) - A(x,u_2)]\nabla u_2 \nabla F_\delta(u_1 - u_2)dx.\end{aligned} \tag{32}$$

From our assumption on $u_1' - u_2'$ and F_δ, we have the first term of (32) positive so that

$$\begin{aligned}&\int_O A(x,u_1)\nabla(u_1 - u_2)\nabla F_\delta(u_1 - u_2)dx \\ &\le -\int_O [A(x,u_1) - A(x,u_2)]\nabla u_2 \nabla F_\delta(u_1 - u_2)dx.\end{aligned}$$

This, together with (30) and (31), yields

$$\begin{aligned}&\int_E A(x,u_1)\frac{1}{\omega^2(u_1 - u_2)}\nabla(u_1 - u_2)\nabla(u_1 - u_2)dx \\ &\le -\int_E [A(x,u_1) - A(x,u_2)]\frac{1}{\omega^2(u_1 - u_2)}\nabla u_2 \nabla(u_1 - u_2)dx.\end{aligned}$$

We follow similar computations as in Cabarrubias and Donato (2011).

Using H2(1), H3(2), and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\alpha \int_E \frac{|\nabla(u_1 - u_2)|^2}{\omega^2(u_1 - u_2)}dx \\ &\le \int_E A(x,u_1)\frac{1}{\omega^2(u_1 - u_2)}\nabla(u_1 - u_2)\nabla(u_1 - u_2)dx \\ &\le -\int_E [A(x,u_1) - A(x,u_2)]\frac{1}{\omega^2(u_1 - u_2)}\nabla u_2 \nabla(u_1 - u_2)dx. \\ &\le \int_E \frac{|A(x,u_1) - A(x,u_2)||\nabla u_2||\nabla(u_1 - u_2)|}{\omega^2(u_1 - u_2)}dx \\ &\le \int_E |\nabla u_2|\frac{|\nabla(u_1 - u_2)|}{\omega(u_1 - u_2)}dx \\ &\le \left(\int_E |\nabla u_2|^2\, dx\right)^{\frac{1}{2}}\left(\int_E \left|\frac{\nabla(u_1 - u_2)}{\omega(u_1 - u_2)}\right|^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

It follows that

$$\alpha \left\| \frac{\nabla (u_1 - u_2)}{\omega (u_1 - u_2)} \right\|^2_{L^2(E)} \leq \| \nabla u_2 \|_{L^2(E)} \left\| \frac{\nabla (u_1 - u_2)}{\omega (u_1 - u_2)} \right\|_{L^2(E)},$$

which implies

$$\left\| \frac{\nabla (u_1 - u_2)}{\omega (u_1 - u_2)} \right\|_{L^2(E)} \leq c \|\nabla u_2\|_{L^2(E)} \leq c \|\nabla u_2\|_{L^2(\mathcal{O})} \leq C, \tag{33}$$

for some c and C independent of δ. Next, we set

$$G_\delta(y) = \begin{cases} \int_\delta^y \frac{ds}{\omega(s)} & \text{if } y \geq \delta, \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

Again, from classical results (see Corollary 2.15 of Chipot (2009)), $G_\delta(u_1 - u_2) \in V$ and

$$\nabla (G_\delta (u_1 - u_2)) = G'_\delta (u_1 - u_2) \nabla (u_1 - u_2). \tag{35}$$

Using this in (33) yields

$$\|G_\delta (u_1 - u_2)\|_V = \|\nabla (G_\delta (u_1 - u_2))\|_{L^2(O)} \leq C.$$

Hence, there exists a sequence $\{\delta_k\}$, which converges to 0 as $k \to +\infty$ and $G \in V$, such that up to a subsequence,

$$\begin{aligned} G_{\delta_k}(u_1 - u_2) &\rightharpoonup G && \text{weakly in} && V, \\ G_{\delta_k}(u_1 - u_2) &\to G && \text{strongly in} && L^2(\mathcal{O}), \\ G_{\delta_k}(u_1 - u_2) &\to G && \text{a.e. in} && \mathcal{O}. \end{aligned}$$

It follows that, as $k \to +\infty$,

$$\lim_{\delta_k \to 0} G_{\delta_k}(u_1 - u_2)(x) < +\infty, \quad \text{a.e. in } \mathcal{O}.$$

But the definition in (34) implies that

$$\lim_{\delta_k \to 0} G_{\delta_k}(u_1 - u_2)(x) = \lim_{\delta_k \to 0} \int_{\xi_k}^{(u_1 - u_2)(x)} \frac{ds}{\omega(s)} = +\infty \quad \text{a.e. in } E.$$

This implies that $m(E) = 0$, that is,

$$u_1 - u_2 \leq 0.$$

Reversing the roles of u_1 and u_2, we get $u_1 = u_2$.

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