Science Diliman (July-December 2016) 28:2, 27-53

The Real Orthogonal Group and Vahlen Matrices

Bimal Kunwor
Louisiana State University, USA

Dennis I. Merino*
Southeastern Louisiana University, USA

Edgar N. Reyes
Southeastern Louisiana University, USA

Gary A. Walls
Southeastern Louisiana University, USA

ABSTRACT

We prove the differentiability of a group homomorphism $\xi : O(4) \to PSp(2)$ from the real orthogonal group $O(4)$ into the projective symplectic group $PSp(2)$, where was constructed by Canlubo and Reyes (2012). We describe higher dimensional analogs of ξ. For $n \geq 2$, we consider a stereographic projection $\Pi : S^{n-1} \to \hat{\mathrm{R}}^{n-1}$ from the unit sphere S^{n-1} onto $\hat{\mathrm{R}}^{n-1} = \mathrm{R}^{n-1} \cup \{\infty\}$. Applying Möbius transformations and Π, we embed the real orthogonal group $O(n)$ into a projective subgroup $PSL_2(\Gamma_{n-1})$ of Vahlen matrices, where Γ_n is the Clifford group of the Clifford algebra $Cl(n)$ of dimension 2^n.

2000 Mathematic Subject Classification: 51B10, 15B33

Keywords: Quaternion, Möbius transformation, Vahlen matrix

*Corresponding Author

ISSN 0115-7809 Print / ISSN 2012-0818 Online

INTRODUCTION

Many students of mathematics are introduced to Möbius geometry by studying the linear fractional transformation $F_A(x) = (ax + b)(cx + d)^{-1}$, where x belongs to the extended complex plane $\hat{\mathrm{C}} = \mathrm{C} \cup \{\infty\}$, and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has complex entries satisfying $ad - bc = 1$. The Möbius transformation F_A is orientation preserving, and is a product of an even number of inversions about circles or reflections about lines (Beardon 1983). The field of complex numbers may be enlarged to the real algebra of quaternions, and the quaternions may be further enlarged to a real Clifford algebra, and so on. There are natural Euclidean subspaces that sit inside the real Clifford algebras (see Section 6). We adopt the same notation F_A to denote pseudo linear fractional transformations that are defined on the extended Euclidean subspaces of these real Clifford algebras. The real Clifford algebras have applications to the study of Möbius transformations on Euclidean spaces (Ahlfors 1985; Waterman 1993).

Let $n \geq 2$ be an integer. Let R^n be the n-dimensional Euclidean space. Let $\hat{\mathrm{R}}^n = \mathrm{R}^n \cup \{\infty\}$ be the extended Euclidean space. Let $GM(\hat{\mathrm{R}}^n)$ denote the group generated by inversions about (n-1)-dimensional spheres, and reflections about hyperplanes in $\hat{\mathrm{R}}^n$ (see Möbus Transformations). The elements of $GM(\hat{\mathrm{R}}^n)$ are called Möbius transformations of $\hat{\mathrm{R}}^n$, and $GM(\hat{\mathrm{R}}^n)$ is the group of Möbius transformations of $\hat{\mathrm{R}}^n$.

Let $S^{n-1} = \{x \in \mathrm{R}^n : |x| = 1\}$ be the unit sphere centered at the origin, and let $GM(S^{n-1})$ denote the subgroup of Möbius transformations of $\hat{\mathrm{R}}^n$ that leave the sphere S^{n-1} invariant. Let $\Pi : S^{n-1} \to \hat{\mathrm{R}}^{n-1}$ be a stereographic projection from the sphere S^{n-1} onto $\hat{\mathrm{R}}^{n-1}$ that satisfies $\prod(x_1,\ldots,x_n) = (1 - x_n)^{-1}(x_1,\ldots, x_{n-1})$ if $x_n \neq 1$. We consider the induced group homomorphism $\Pi^* : GM(S^{n-1}) \to GM(\hat{R}^{n-1})$ given by $\Pi^*(f) = \Pi \circ f \circ \Pi^{-1}$ for each $f \in GM(S^{n-1})$. Notice that the real orthogonal group $O(n)$ is a subgroup of $GM(S^{n-1})$. We analyze the restriction of Π^* to $O(n)$.

The special case n=4 is analyzed by Canlubo and Reyes (2012). We identify R^4 with the space of real quaternions, such that $(x_0, x_1, x_2, x_3) \in \mathrm{R}^4$ is paired with the real quaternion $x_0 + x_1 i + x_2 j + x_3 k$. The subspace R^3 is identified with the space of pure quaternions, and the unit sphere S^3 is the group of unit quaternions. For the stereographic projection, let $\prod(x_0,x_1,x_2,x_3)=(1-x_3)^{-1}(x_0,x_1,x_2)$ if $x_3 \neq 1$. In the study by Canlubo and Reyes (2012), if $f \in O(4)$, it is shown that $\Pi^*(f) = F_A$, where F_A (the same notation used earlier but in a different context) is a quaternionic linear fractional transformation that is defined on the extended space of pure quaternions, and A is some matrix in the symplectic group $Sp(2)$.

Furthermore, let $PSp(2) = Sp(2)/\{\pm I_2\}$. Let $[A]$ be the coset of $A \in Sp(2)$ in $PSp(2)$. Canlubo and Reyes (2012) prove that the mapping $\xi : O(4) \to PSp(2)$ defined by $\xi(f) = [A]$ is a well-defined injective group homomorphism (see Quaternions and Rigid Motions). The first main result of this paper shows that is a differentiable mapping (see The Differentiability of ξ). We also evaluate a matrix representation for the differential $d\xi$ of ξ at the tangent space of the identity or Lie algebra of $O(4)$ (see The Differentiability of $d\xi$).

Let $n \geq 2$. We describe higher dimensional analogs of the homomorphism $\xi : O(4) \to PSp(2)$. The framework for our calculations is the Clifford algebra $Cl(n)$, which is generated by the usual basis $e_1, \ldots, e_n$ of R^n, such that $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ if $i \neq j$. The field of complex numbers is $Cl(1)$, and the algebra of real quaternions is $Cl(2)$.

We identify R^n with a subspace of $Cl(n)$ having the usual basis $e_1, \ldots, e_n$. Let V^{n+1} be the subspace of $Cl(n)$ that is spanned by 1, and $e_1, \ldots, e_n$. Let Γ_n be the multiplicative group generated by the nonzero vectors in V^{n+1}. We consider 2-by-2 matrices A whose nonzero entries lie in Γ_n, and the entries are subject to additional restrictions that include a nonzero real pseudo determinant for A (see Section 6). These matrices A are called Vahlen matrices, and they form a multiplicative group $GL_2(\Gamma_n)$ (Ahlfors 1985; Waterman 1993). Let $SL_2(\Gamma_n)$ be the subgroup of $GL_2(\Gamma_n)$ that consists of Vahlen matrices, where the pseudo determinants are all equal to one. If $n = 1$, $GL_2(\Gamma_1)$ consists of 2-by-2 nonsingular complex matrices whose determinants are real.

We apply the Poincaré extension of a Möbius transformation ϕ of $\hat{\mathrm{R}}^n$ (Beardon 1983). The Poincaré extension ϕ^{ext} is a Möbius transformation of $\hat{\mathrm{R}}^{n+1}$, where the restriction of ϕ^{ext} to $\hat{\mathrm{R}}^n$ is ϕ. In connection to other fields, the upper half-space of $\hat{\mathrm{R}}^{n+1}$, where $x_{n+1} > 0$, is a model for hyperbolic geometry, and ϕ^{ext} is an isometry of this geometry (Beardon 1983). In this paper, we realize ϕ^{ext} as a Möbius transformation of $\hat{\mathrm{V}}^{n+1} = V^{n+1} \cup \{\infty\}$, where the restriction of ϕ^{ext} to $\hat{\mathrm{R}}^n$ is ϕ.

We recall the stereographic projection $\Pi(x_1,\ldots,x_n)=(1-x_n)^{-1}(x_1,\ldots,x_{n-1})$, $x_n \neq 1$. We show that if $f \in O(n)$, then the Möbius transformation $\Pi^*(f)$ of $\hat{\mathrm{R}}^{n-1}$ may be rewritten as a pseudo linear fractional transformation $\Pi^*(f) = F_A$ for some $A \in SL_2(\Gamma_{n-1})$ (see discussion before Theorem 11 in Section 7). Let $PSL_2(\Gamma_{n-1}) = SL_2(\Gamma_{n-1})/\{\pm I_2\}$, and let $[A]$ denote the coset in $PSL_2(\Gamma_{n-1})$ that is defined by $[A]$. We describe our second main result. We show that there is a well-defined injective group homomorphism $\xi_{e_n} : O(n) \to PSL_2(\Gamma_{n-1})$ satisfying $\xi_{e_n}(f) = [A]$ (Theorem 11).

The mapping ξ_{e_n} is an analog in higher dimensions (but not a generalization) of the mapping ξ that was obtained in the quaternionic case by Canlubo and Reyes (2012). The reason that we do not have a generalization is because certain quaternionic matrices that define the quaternionic linear fractional transformations are not Vahlen matrices (see details after the proof of Theorem 11).

In Section 8, we evaluate the left and right eigenvalues of some Vahlen matrices. Beardon (1983) and Lawson (2010) describe related works on Möbius geometry and Clifford algebras. General references for quaternions are described by Coxeter (1946), Koecher and Remmert (1991), and Conway and Smith (2003).

QUATERNIONS AND RIGID MOTIONS

Let H be the algebra of quaternions $x = x_0 + x_1 i + x_2 j + x_3 k,$ where $x_0, x_1, x_2, x_3 \in \mathrm{R}$ and $i^2 = j^2 = k^2 = ijk = -1$. The conjugate of x is $\overline{x} = x_0 - x_1 i - x_2 j - x_3 k$, and the norm of x is $|x| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. One checks that $|x|^2 = x\overline{x}$ and $\overline{(xy)} = \overline{y}\overline{x}$. Thus, the norm is multiplicative, that is, $|xy| = |x||y|$ for every $x, y \in \mathrm{H}$. If $|x| = 1$, then we say that x is a unit quaternion. We denote the set of unit quaternions by S^3. If $x, y \in S^3$, then $xy \in S^3$. We write $\mathrm{Re}(x) = x_0$ for the real part of x. If $x_0 = 0$, then we say that x is a pure quaternion. We let H^* be the real space of all pure quaternions. If $x = x_1 i + x_2 j + x_3 k \in \mathrm{H}^*$, then one checks that $x^2 = -x_1^2 - x_2^2 - x_3^2$. Notice that H and R^4 are isomorphic real vector spaces. The usual inner-product in R^4 is

$$\mathrm{Re}(x\overline{y}) = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3. \qquad (1).$$

Notice that $x\overline{y} + y\overline{x} = 2\,\mathrm{Re}(x\overline{y})$.

A rigid motion on H is a bijection $f : \mathrm{H} \to \mathrm{H}$, satisfying $|f(x) - f(w)| = |x - w|$ for all $x, w \in \mathrm{H}$. Rigid motions of H have been characterized by Coxeter (1946). Let g be a rigid motion on H. If $g(0) = 0$, then g is a real linear mapping (that is, for every $s, t \in \mathrm{R}$ and every $x, w \in \mathrm{H}$, we have $g(sx + tw) = sg(x) + tg(w)$). Let G be the group of rigid motions of H that fix the origin (that is, $f(0) = 0$, and every $f \in G$ is also called an orthogonal linear map). Then, G is isomorphic to the real orthogonal matrix group $O(4)$. Let $f \in G$ be given. There exist $a, b \in S^3$ such that either $f(x) \equiv axb$ or $f(x) \equiv a\overline{x}b$. Let $c \in \mathrm{H}$ and let h be a rigid motion. If $h(0) = c$, then h may be expressed as the composition $A \circ f$ of a rigid motion $f \in G$ that fixes the origin and the translation $A(x) = x + c$.

Let $y \in S^3$ be given. For each $x \in \mathrm{H}$, let $f_y(x) = -y\bar{x}y$. Then, $f_y(y) = -y$, is a real linear mapping on H, and $f_y \in G$. Let $p \in \mathrm{H}$ be given. Notice that $p\bar{y} + y\bar{p} = 2\,\mathrm{Re}(p\bar{y}) = 0$, if and only if $p = -y\bar{p}y$. Thus, if $\mathrm{Re}(p\bar{y}) = 0$, then $f_y(p) = p$. In this case, we say that f_y is a reflection in H about the hyperplane perpendicular to y. One checks that $f_y(f_y(x)) = x$ for every $x \in \mathrm{H}$, that is f_y is its own inverse. Also, we may rewrite the reflection f_y as follows: $f_y(x) = x - 2\,\mathrm{Re}(x\bar{y})y$.

The following (cf. Theorem 9.1 of Coxeter (1946)) characterizes the rigid motions of H, which fix the origin and are products of two or four reflections.

Theorem 1. Let $p,q \in \mathrm{S}$ be unit quaternions and let α, $\beta \in \mathrm{R}$ be given. Set $a = \cos\alpha + p\sin\alpha$ and $b = \cos\beta + q\sin\beta$. Consider the rigid motion of H given by

$$B_{a,b}(x) \equiv axb, x \in \mathrm{H} \qquad (2).$$

Let $V = \{x \in \mathrm{H} : pxq = -x\}$ and $W = \{x \in \mathrm{H} : pxq = x\}$. Then, V and W are $B_{a,b}$-invariant orthogonal subspaces with respect to the inner-product (1) and $V \oplus W = \mathrm{H}$. The restriction of $B_{a,b}$ to V is a rotation through the angle $\alpha + \beta$, and the restriction of $B_{a,b}$ to W is a rotation through the angle $\alpha - \beta$. Moreover, a matrix representation of the real linear mapping $B_{a,b}$ of H is the block matrix $\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} \oplus \begin{pmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{pmatrix}$. Conversely, any orientation preserving rigid motion of H that fixes the origin has the form (2) for some basis.

Some details of the proof of Theorem 1 are given in The Differentiability of ξ. The subgroup G_0 of the orientation preserving transformations in G is

$$G_0 \equiv \{B_{a,b} : a,b \in S^3\} \qquad (3).$$

The group $SO(4) = \{A \in O(4) : Det(A) = 1\}$ is isomorphic to G_0. Let $M_2(\mathrm{H})$ be the set of all 2-by-2 matrices with entries in H. We denote by H^2 the space of column vectors $(v_1, v_2)^T$, where $v_1, v_2 \in \mathrm{H}$. Consider the symplectic scalar product in H^2 given by

$$\langle (v_1, v_2)^T, (w_1, w_2)^T \rangle = \bar{w}_1 v_1 + \bar{w}_2 v_2 \qquad (4).$$

Let $p_{11}, p_{12}, p_{21}, p_{22} \in \mathrm{H}$ be given and let $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in M_2(\mathrm{H})$. Matrix multiplication by P induces a right H-linear mapping of H^2: $P\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p_{11}v_1 + p_{12}v_2 \\ p_{21}v_1 + p_{22}v_2 \end{pmatrix}$.

Let $Sp(2)$ be the set of all $P \in M_2(\mathrm{H})$, such that $\langle Pv, Pw\rangle = \langle v, w\rangle$ for all $v, w \in \mathrm{H}^2$.

Then, $Sp(2)$ is a group under usual matrix multiplication: if $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ and $Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$, then $PQ = \begin{pmatrix} p_{11}q_{11} + p_{12}q_{21} & p_{11}q_{12} + p_{12}q_{22} \\ p_{21}q_{11} + p_{22}q_{21} & p_{21}q_{12} + p_{22}q_{22} \end{pmatrix}$. The order of the multiplication of the entries in PQ matters since multiplication of quaternions is not commutative. We say that $Sp(2)$ is a symplectic group. We let P^* denote the conjugate transpose of P.

Lemma 2. Let $P \in M_2(\mathrm{H})$ be given. Then, $P \in Sp(2)$ if and only if $P^*P = I$.

Proof. Let $P \in M_2(\mathrm{H})$ be given. Let $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$. For $i, j = 1,2$, the symplectic scalar product (4) satisfies $\langle Pe_i, Pe_j\rangle = \overline{p}_{1j}p_{1i} + \overline{p}_{2j}p_{2i}$. Now, the (i, j) entry of P^*P is $\langle Pe_j, Pe_i\rangle$. If $P \in Sp(2)$, then $\langle Pe_i, Pe_j\rangle = \langle e_i, e_j\rangle$, so that $P^*P = I$. Conversely, suppose $P^*P = I$. Let $v = (v_1, v_2)^T$ and $w = (w_1, w_2)^T$ be in H^2. Since P defines a right H-linear mapping on H^2, we have $\langle Pv, Pw\rangle = \sum_{i,j}^{2}\left\langle (Pe_i)v_i, (Pe_j)w_j\right\rangle$ $= \sum_{i,j}^{2}\overline{w}_j\left\langle Pe_i, Pe_j\right\rangle v_i = \langle v, w\rangle$. Thus, $P \in Sp(2)$.

Let $\hat{\mathrm{H}}^* = \mathrm{H}^* \cup \{\infty\}$. Let $\Pi : S^3 \to \hat{\mathrm{H}}^*$ be the stereographic projection given by $\Pi(x) = (1-x_3)^{-1}(x_0 i + x_1 j + x_2 k)$, if $x = x_0 + x_1 i + x_2 j + x_3 k \neq k$ and $\Pi(k) = \infty$. Notice that Π is the restriction to S^3 of the inversion in $\hat{\mathrm{H}} = \mathrm{H} \cup \{\infty\}$ about the sphere centered at k and of radius $\sqrt{2}$ (Beardon 1983). Let $y \in S^3$ be given and let f_y be the reflection about the hyperplane in H that is perpendicular to y. Set $f_y(\infty) = \infty$. Then, $\Pi \circ f_y \circ \Pi^{-1}$ is a Möbius transformation of $\hat{\mathrm{H}}^*$. The mapping is expressed as a quaternionic linear fractional transformation as follows. Let $y = p + qi + rj + y_2 k \in S^3$, where $p, q, r, y_2 \in \mathrm{R}$. Set $y_1 = pi + qj + rk$, and consider

$$m(y) = Y \equiv \begin{pmatrix} y_1 & y_2 \\ -y_2 & \overline{y}_1 \end{pmatrix} \in M_2(\mathrm{H}) \qquad (5).$$

One checks that $Y^*Y = I$, so that Lemma 2 guarantees that $Y \in Sp(2)$. Moreover, $y_1^2 = -p^2 - q^2 - r^2$, so that $(m(y))^2 = Y^2 = -I$. We associate a quaternionic linear fractional transformation F_Y on $\hat{\mathrm{H}}^*$ by

$$F_Y(x) = (y_1 x + y_2)(-y_2 x + \overline{y}_1)^{-1}, \quad x \in \hat{\mathrm{H}}^* \qquad (6)$$

(Canlubo and Reyes 2012). Here, we have $F_Y(x) = \infty$ if $-y_2 x + \bar{y}_1 = 0$, and if $y_2 \neq 0$. Moreover, by Theorem 1.2 of Canlubo and Reyes (2012), we have

$$F_Y = \Pi \circ f_y \circ \Pi^{-1} \quad (7).$$

Let $PSp(2) \equiv Sp(2)/(\pm I)$ be a projective symplectic group. Let $[\mathrm{Y}] \in PSp(2)$ denote the coset containing Y. In The Differentiability of ξ and in the proof of Theorem 3.2 of Canlubo and Reyes (2012), the authors show that the corresponence $f_y \mapsto [Y]$ extends to a well-defined mapping

$$\xi : G \to PSp(2) \quad (8),$$

which is an injective group homomorphism. If $A \in G$, then there exist $y^{(1)}, \ldots, y^{(k)} \in S^3$, such that $A = f_{y^{(1)}} \circ \cdots \circ f_{y^{(k)}}$ (that is, A is a product of reflections). Hence, we have $\xi(A) = [m(y^{(1)}) \cdots m(y^{(k)})]$. In the next section, we shall prove that ξ is differentiable. By knowing that ξ is differentiable, we will be able to know the action of the differential $d\xi$ at the Lie algebra level (see The Differential $d\xi$).

THE DIFFERENTIABILITY OF ξ

The operator norm of a real linear endomorphism A of H is $|A|_{op} = \sup\{|Ax| : x \in S^3\}$. We derive an inequality for the orientation-preserving rigid motions $B_{a,b}$ in equation (2).

Lemma 3. Let $a,b \in S^3$ and let $\delta > 0$ be given. Suppose that $\mathrm{Re}(a) = \mathrm{Re}(b)$. Then, is a product of two reflections (Coxeter 1946). If $|B_{a,b} - I|_{op} < \delta^2$, then $|a-1| < \delta$ or $|a+1| < \delta$.

Proof. If $\mathrm{Re}(a) = \mathrm{Re}(b)$ and $|a| = |b|$, then there exists $p \in S^3$, such that $ap = pb$ (Lemma 2.2 of Coxeter (1946)). Set $z = ap$, so that $a = z\bar{p}$, $b = \bar{p}z$, and

$$B_{a,b}(x) = axb = z\bar{p}x\bar{p}z = f_z \circ f_p(x). \quad (9)$$

Then, $B_{a,b}$ is a product of two reflections.

Since a reflection is its own inverse, the product of two reflections is the identity mapping, if and only if the two reflections are equal, that is, if and only if $ap = z = \pm p$. Thus, $B_{a,b} = I$, if and only if $a = b = \pm 1$, and the second claim is true if $B_{a,b} = I$.

Suppose $B_{a,b} \neq I$. The product $f_z \circ f_p$ of two reflections is a rotation in the two-dimensional subspace P spanned by p and z. The angle of rotation θ is twice the angle between z and p, where $\cos(\theta/2) = \mathrm{Re}(z\overline{p}) = \mathrm{Re}(a)$.

Consider the triangle in Figure 1 with sides p and apb. Suppose that the vertex of the triangle at the intersection of sides p and apb is the origin in H. Then the included angle between p and apb is θ, and ap bisects θ. Since $f_z \circ f_p$ is a rotation, the length of the side of the triangle opposite angle θ is the operator norm $\left|B_{a,b} - I\right|_{op}$. By the cosine law, we find $\left|B_{a,b} - I\right|_{op}^2 = 2(1-\cos\theta) = 4(1-\cos^2(\theta/2)) = [2(1-\mathrm{Re}(a))]$ $[2(1+\mathrm{Re}(a))] = \left|a-1\right|^2\left|a+1\right|^2$. Since $\left|B_{a,b} - I\right|_{op}^2 < \delta^2$, we have $\left|a-1\right| < \delta$ or $\left|a+1\right| < \delta$.

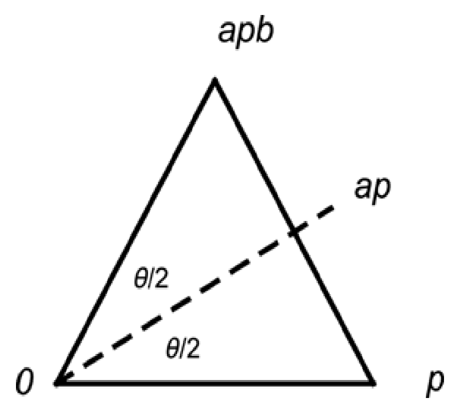

Figure 1. Side *ap* bisects the angle at *O*.

The results described in Lemma 3 are based on the assumption that $\mathrm{Re}(a) = \mathrm{Re}(b)$, where $a, b \in S^3$. However, for the proof of Theorem 4 below, we need an analysis of $B_{a,b}$, where we do not necessarily assume $\mathrm{Re}(a) = \mathrm{Re}(b)$. The analysis of $B_{a,b}$ we need can be found in the proof of Theorem 9.1 of Coxeter (1946). Coxeter shows $B_{a,b}$ may be rewritten as a product of two rotations as in (12), where possibly one of the rotations is the identity transformation. We describe Coxeter's proof of (12) because the technique used in his proof is applied again in Theorem 4. We begin with the polar form (Coxeter 1946), and we write $a = cos\alpha + p\sin\alpha$, where p is a pure unit quaternion. Since $p^2 = -1$, we have

$$e^{pt} = \cos t + p\sin t \text{, for every } t \in \mathrm{R} \tag{10}$$

Notice that $e^{p\pi/2} = p$. De Moivre's theorem ensures that

$$p^t = \left(e^{p\pi/2}\right)^t = \cos(t\pi/2) + p\sin(t\pi/2). \tag{11}$$

Thus, we obtain $a = p^{2\alpha/\pi}$. Similarly, we have $b = \cos\beta + q\sin\beta$ for some pure unit quaternion q and some $\beta \in \mathrm{R}$. Likewise, we obtain $b = q^{2\beta/\pi}$. Lemma 3 guarantees that B_{p^t, q^t} is a rotation in the real 2-dimensional subspace $V = \{x \in \mathrm{H} : pxq = -x\}$ through the angle $t\pi$. Notice also that $B_{p^u, q^{-u}}$ is a rotation in the subspace $W = \{x \in \mathrm{H} : pxq = x\}$ through the angle $u\pi$, for any $u \in \mathrm{R}$. Choose $u, t \in \mathrm{R}$, such that $(u+t)\pi = 2\alpha$ and $(t-u)\pi = 2\beta$. Then,

$$B_{a,b} = B_{p^t, q^t} \circ B_{p^u, q^{-u}} \tag{12}$$

Moreover, $\mathrm{H} = V \oplus W$ is an orthogonal real direct sum; that is, if $x_1 \in V, x_2 \in W$, then $\mathrm{Re}(x_1\bar{x}_2) = -\mathrm{Re}(px_1q\bar{q}\bar{x}_2\bar{p}) = -\mathrm{Re}(px_1\bar{x}_2\bar{p}) = -\mathrm{Re}(x_1\bar{x}_2)$, and $\mathrm{Re}(x_1\bar{x}_2) = 0$. Let $p_1, z_1 \in S^3$ satisfy $z_1 = pp_1 = p_1q$. By Lemma 3, we have $B_{p^t, q^t} = f_{z_1} \circ f_{p_1}$. Then B_{p^t, q^t} is the identity mapping when restricted to W for $f_{z_1} \circ f_{p_1}(x_2) = z_1\bar{p}_1x_2\bar{p}_1z_1 = pp_1\bar{p}_1x_2\bar{p}_1z_1 = x_2\bar{q}\bar{p}_1z_1 = x_2$.

Similarly, $B_{p^u q^{-u}}$ is the identity mapping when restricted to V. Then, the restrictions of $B_{a,b}$ and B_{p^t, q^t} to V are identical, and the restrictions of $B_{a,b}$ to B_{p^t, q^t} to W are identical. Consequently, the operator norm satisfies

$$\left|B_{a,b} - I\right|_{op} = \max\left\{\left|B_{p^t, q^t} - I\right|_{op}, \left|B_{p^u, q^{-u}} - I\right|_{op}\right\} \tag{13}$$

We note a special case $\mathrm{Re}(a) = \mathrm{Re}(b)$, if and only if $B_{p^t, q^t} = I$ or $B_{p^u, q^{-u}} = I$. We are now ready to prove the differentiability of ξ.

Theorem 4. The group homomorphism $\xi : G \to PSp(2)$ is differentiable.

Proof . Since ξ is a group homomorphism, it suffices to prove continuity at the identity I (Proposition 3.12 of Brocker and Dieck (1985)). Note, the subgroup G_0 in equation (3) is isomorphic to $SO(4)$, an open subset of G that contains I. Therefore, it is without loss of generality that we restrict ξ to G_0.

Since the group multiplication in $PSp(2)$ is continuous, there are open neighborhoods V and W of $[I]$ in $PSp[2]$, such that $V^2 \equiv \{v_1 v_2 : v_1, v_2 \in V\} \subseteq W$. Consider the quotient group $S^3/\{\pm 1\}$. If $y \in S^3$, let $[y] \in S^3/\{\pm 1\}$ be the coset defined by y. The quotient topology in $S^3/\{\pm 1\}$ is equivalent to the topology defined by the metric of Becker and Kechris (1996): $d([x],[y]) = \min\{|x-y|,|x+y|\}$.

We recall the mapping $m : S^3 \to Sp(2)$, such that $m(y) = Y$ in equation (5). Then, m induces a continuous mapping $\eta : S^3/\{\pm 1\} \to PSp(2)$, such that $\eta([y]) = [Y]$. Since the continuity of group multiplication on the compact group S^3 implies uniform continuity, there exists $\delta > 0$, such that $[X][Y]^{-1} \in V$ whenever $d([x],[y]<\delta$ and $\eta([x])=[X]$.

Now, let $a,b \in S^3$ satisfy $\left|B_{a,b} - I\right|_{op} < \delta^2$. Applying equation (12), $B_{a,b}$ is a product of two rotations $B_{a,b} = B_{p^t,q^t} \circ B_{p^u,q^{-u}}$ for some $u,t \in$ R, and pure unit quaternions p,q. Furthermore, each rotation is a product of two reflections. As in equation (9), there are unit quaternions $w_1, w_2 \in S^3$ satisfying the following: $B_{p^t,q^t} = f_{p^t w_1} \circ f_{w_1}$ and $B_{p^u,q^{-u}} = f_{p^u w_2} \circ f_{w_2}$.

Notice that $\mathrm{Re}(p^t) = \mathrm{Re}(q^t)$ by equation (11), and that $\left|p^t\right| = 1 = \left|q^t\right|$. Moreover, since $\left|B_{a,b} - I\right|_{op} < \delta^2$ by identity (13), we have $\left|B_{p^t,q^t} - I\right|_{op} < \delta^2$.

Applying Lemma 3 to the above inequality we obtain $\left|p^t - 1\right| < \delta$ or $\left|p^t + 1\right| < \delta$. The metric on $S^3/\{\pm 1\}$ satisfies $d\left([p^t],[1]\right) < \delta$. Since the norm is multiplicative, we have $d\left([p^t w_1],[w_1]\right) < \delta$. Similarly, $d\left([p^u w_2],[w_2]\right) < \delta$. Consequently, $[m(p^t w_1)][m(w_1)]^{-1}, [m(p^u w_2)][m(w_2)]^{-1} \in V$.

Recall that, for all $y \in S^3$, we have $(m(y))^2 = -\mathrm{I}$, so that $(m(y))^{-1} = m(y)$. Since $V^2 \subseteq W$, we obtain

$$[m(p^t w_1)][m(w_1)][m(p^u w_2)][m(w_2)] \in W. \tag{14}$$

On the other hand, we also have $\xi(B_{a,b}) = \xi\left(B_{p^t,q^t} \circ B_{p^u,q^{-u}}\right) = \xi\left(f_{p^t w_1} \circ f_{w_1} \circ f_{p^u w_2} \circ f_{w_2}\right)$ $= [m(p^t w_1)][m(w_1)][m(p^u w_2)][m(w_2)]$. Thus, $\xi(B_{a,b}) \in W$ whenever $\left|B_{a,b} - I\right|_{op} < \delta^2$, and hence, ξ is continuous at the identity I.

We compute the differential of ξ.

THE DIFFERENTIAL $d\xi$

We describe the Lie algebra g_0 of the Lie group G_0 in equation (3). The underlying set of the Lie algebra g_0 is the set of tangent vectors at the identity of G_0. A standard procedure is to consider the surjective group homomorphism

$$B : S^3 \times S^3 \to G_0 \tag{15}$$

defined by $B(a, b) = B_{a,b^{-1}}$, where as before, $B_{a,b^{-1}}(x) = axb^{-1}$ for each $x \in \mathrm{H}$. It is known that the Lie algebra of S^3 is the algebra H^* of pure quaternions. Let exp: $\mathrm{H}^* \to S^3$ be the exponential mapping defined by $exp(w) = e^w$ for every $w \in \mathrm{H}^*$. For $t \in \mathrm{R}$, we have (see equation (10)) $e^{tw} = \cos(t\,|\,w\,|) + \frac{w}{|\,w\,|}\sin(t\,|\,w\,|)$ and $\left.\frac{d}{dt}\right|_{t=0} e^{tw} = w$. Now, we evaluate the differential dB of B at the identity. The commutative diagram (Figure 2) below explains the calculation of dB that follows.

Let $w_1, w_2 \in \mathrm{H}^*$ be given. The tangent vector to the differentiable curve $t = \mapsto B(e^{tw_1}, e^{tw_2})$ in $S^3 \times S^3$ at $t = 0$ is (w_1, w_2). Now, $dB(w_1, w_2)$ is the tangent vector to the curve $t = \mapsto B(e^{tw_1}, e^{tw_2})$ in G_0 at $t = 0$. Hence, the differential $dB : \mathrm{H}^* \times \mathrm{H}^* \to g_0$ is a Lie algebra homomorphism that satisfies $dB(w_1, w_2) = \left.\frac{d}{dt}\right|_{t=0} B\left(e^{tw_1}, e^{tw_2}\right)$.

In particular, $dB(w_1, w_2)$ is a real linear mapping of H. In fact, the image of $x \in \mathrm{H}$ under $dB(w_1, w_2)$ is denoted and defined by

$$dB(w_1, w_2)(x) = \left.\frac{d}{dt}\right|_{t=0} \left[e^{tw_1} x e^{-tw_2}\right] = w_1 x - x w_2 \tag{16}.$$

$$\begin{tikzcd} S^3 \times S^3 \arrow[r, "B"] & G_0 \\ H^* \times H^* \arrow[u, "exp"] \arrow[r, "dB"'] & g_0 \arrow[u, "exp"'] \end{tikzcd}$$

Figure 2. Commutative diagram for the calculation of dB

Let $gl(\mathrm{H})$ denote the Lie algebra of all real linear mappings of H into itself. Since the kernel of B is $\{\pm(1,1)\}$, the differential dB is a Lie algebra isomorphism. In particular, the Lie algebra of G_0 is $g_0 = \{dB(w_1, w_2) : w_1, w_2 \in \mathrm{H}^*\} \subset gl(\mathrm{H})$. Moreover, g_0 and $\mathrm{H}^* \times \mathrm{H}^*$ are isomorphic Lie algebras.

Since $e^{\pi w/2} = w$ whenever w is a pure unit quaternion, we introduce an extra factor in the following canonical real basis for the vector space $\mathrm{H}^* \times \mathrm{H}^*$:
$\mathrm{B} = \frac{\pi}{2}\{(i,0), (j,0), (k,0), (0,i), (0,j), (0,k)\}.$

Since dB is a real linear isomorphism, the following is a real basis for g_0:
$\mathrm{B}' = \{dB(w_1, w_2) : (w_1, w_2) \in \mathrm{B}\}.$

Let $P : Sp(2) \to PSp(2)$ be the projection mapping given by $P(Y) = [Y]$. Since $PSp(2)$ is the quotient group of $Sp(2)$ by a discrete group $\{\pm I\}$, the Lie algebras of $PSp(2)$ and $Sp(2)$ are the same. Then $PSp(2)$ and $Sp(2)$ are locally isomorphic Lie groups (by Theorem 1.11 of Helgason (1978)). Hence, there exists an open neighborhood U of I in $Sp(2)$, such that (i) $P(U)$ is an open subset of $[I]$ in $PSp(2)$, (ii) the restriction $P\big|_U : U \to P(U)$ is a diffeomorphism, and (iii) $P\big|_U$ is a local homomorphism, that is, if $x_1, x_2, x_1x_2 \in U$, then $P(x_1x_2) = P(x_1)P(x_2)$.

Let $P\big|_U^{-1}$ denote the inverse function of $P\big|_U$. Let $\xi : G \to PSp(2)$ be the differentiable homomorphism in Theorem 4. Then, $\xi^{-1}(P(U))$ is an open subset of G. Let $\psi : \xi^{-1}(P(U)) \to Sp(2)$ be the differentiable mapping defined by $\psi(x) = (P\big|_U^{-1} \circ \xi)(x)$, where $x \in \xi^{-1}(P(U))$. We evaluate the differential $d\psi$ of ψ at the elements of B'. The Lie algebra $\Im$ of $Sp(2)$ consists of 2-by-2 quaternionic matrices u satisfying $u + u^* = 0$. For instance, by equations (16) and (11), we find

$$d\psi\left(dB\left(\tfrac{\pi}{2} i,0\right)\right) = d\psi\left(\left.\frac{d}{dt}\right|_{t=0} B_{i^t,1}\right) = \left.\frac{d}{dt}\right|_{t=0} \psi\left(B_{i^t,1}\right) \in \Im.$$

Applying equation (12), we express $B_{i^t,1}$ as a product of two rotations. Each rotation is a product of reflections (see equation (9)). Then, $B_{i^t,1} = B_{i^{t/2},i^{t/2}} \circ B_{i^{t/2},i^{-t/2}} = f_{i^{t/2}} \circ f_1 \circ f_{i^{t/2}k} \circ f_k \in \xi^{-1}(P(U))$ for sufficiently small values of t. Since $-I \notin U$, we find $B_{i^t,1} = P\big|_U^{-1}\left(\left[B_{i^t,1}\right]\right)$ for small values of t. Since ψ is a local homomorphism, we obtain

$$d\psi\left(dB\left(\frac{\pi}{2} i,0\right)\right) = \left.\frac{d}{dt}\right|_{t=0} \left(m(i^{t/2})m(1)m(i^{t/2}k)m(k)\right). \tag{17}$$

However, by applying equation (5) we find $m(i^{t/2}) = \begin{pmatrix} i\cos\frac{\pi t}{4} + j\sin\frac{\pi t}{4} & 0 \\ 0 & -i\cos\frac{\pi t}{4} - j\sin\frac{\pi t}{4} \end{pmatrix}$

and $m(i^{t/2}k) = \begin{pmatrix} -k\sin\frac{\pi t}{4} & \cos\frac{\pi t}{4} \\ -\cos\frac{\pi t}{4} & k\sin\frac{\pi t}{4} \end{pmatrix}$

Applying the product rule to equation (17), we obtain $d\psi\left(dB\left(\frac{\pi}{2}i,0\right)\right) = \frac{\pi}{4}k\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Furthermore, by the chain rule, $d\psi = d\left(P\big|_U^{-1}\right) \circ d\xi$. Thus, we obtain and use the indicated notation: $d\xi\left(dB\left(\frac{\pi}{2}i,0\right)\right) = dP\left(\frac{\pi}{4}k\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) \equiv f_3$.

Similarly, by evaluating $d\psi$ at the other elements of B', we find the following:

(i) $d\xi\left(dB\left(\frac{\pi}{2}j,0\right)\right) = -dP\left(\frac{\pi}{4}j\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) \equiv -f_2$,

(ii) $d\xi\left(dB\left(\frac{\pi}{2}k,0\right)\right) = dP\left(\frac{\pi}{4}i\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) \equiv f_1$,

(iii) $d\xi\left(dB\left(0,\frac{\pi}{2}i\right)\right) = dP\left(\frac{\pi}{4}k\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right) \equiv f_6$,

(iv) $d\xi\left(dB\left(0,\frac{\pi}{2}j\right)\right) = dP\left(\frac{\pi}{4}j\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right) \equiv f_5$, and

(v) $d\xi\left(dB\left(0,\frac{\pi}{2}k\right)\right) = dP\left(\frac{\pi}{4}i\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right) \equiv f_4$.

Using the ordered bases B', and $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ for the domain and range of $d\xi$, respectively, we obtain a matrix representation for the differential $d\xi$.

Theorem 5. If we realize $d\xi$ as a real linear isomorphism onto its image, then a matrix representation for $d\xi$ is the following block matrix $matrix(d\xi) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Moreover, the eigenvalues of $d\xi$ are $\{\pm 1, \pm 1, \pm 1\}$, counting multiplicities.

MÖBIUS TRANSFORMATIONS

In order to describe the higher dimensional analogues of ξ , as described in (8), we describe additional elementary prerequisites. We determine the kernel of the group homomorphism Π in (20) that is induced by the stereographic projection $\Pi : S^{n-1} \to \hat{\mathrm{R}}^{n-1}$, as defined in this section. The kernel described in Lemma 6 is needed to complete the proof of our second main result, namely, Theorem 11 in Embedding the Orthogonal Group $O(n)$.

We review general facts about Möbius transformations (Beardon 1983). Let $n \geq 2$ be a given integer. Let $e_i \in \mathrm{R}^n$ be the unit vector with 1 in the ith position and 0 elsewhere, so that $E = \{e_1, \ldots, e_n\}$ is the standard basis for R^n. Let $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T \in \mathrm{R}^n$ be given. The standard bilinear form on R^n is

$$K(x, y) = \sum_{i=1}^{n} x_i y_i \tag{18}$$

Notice that $|x|^2 = K(x, x)$. If $x \neq 0$, we let $x^\dagger = |x|^{-2} x$. Let $c \in \mathrm{R}^n$ and a nonzero $y \in \mathrm{R}^n$ be given. The hyperplane in R^n containing c and perpendicular to y is $H_n(c, y) = \{x \in \mathrm{R}^n : K(x, y) = 0\} + c = \{x \in \mathrm{R}^n : K(x - c, y) = 0\}$.

The reflection τ_y about $H_n(c, y)$ is the homeomorphism of $\hat{\mathrm{R}}^n = \mathrm{R}^n \cup \{\infty\}$ satisfying $\tau_y(x) = x - 2K(x - c, y)y^\dagger, x \in \mathrm{R}^n$, and $\tau_y(\infty) = \infty$.

Notice that $\tau_y(y + c) = -y + c$ and that $\tau_y(x) = x$ every $x \in H_n(c, y)$. The sphere in R^n centered at c and with radius $r > 0$ is given by $S_n(c, r) = \{x \in \mathrm{R}^n : |x - c| = r\}$.

The inversion σ in $S_n(c, r)$ is the homeomorphism of $\hat{\mathrm{R}}^n$ satisfying $\sigma(x) = c + r^2(x - c)^\dagger$, if $x \neq c$, and $\sigma(c) = \infty$.

Let $S^{n-1} = S_n(0,1)$, and let $\Pi : S^{n-1} \to \hat{\mathrm{R}}^{n-1}$ be the stereographic projection defined by $\Pi(x) = (1 - x_n)^{-1}(x_1, \ldots, x_{n-1})$, $x_n \neq 1$, and $\Pi(e_n) = \infty$.

The inversion ϕ_0 in $S_n(e_n, \sqrt{2})$ is special because the restriction of ϕ_0 to $\hat{\mathrm{R}}^{n-1}$ is the inverse of the stereographic projection Π, that is,

$$\Pi^{-1}(\tilde{x}) = \phi_0(\tilde{x}, 0), \tilde{x} \in \hat{\mathrm{R}}^{n-1} \tag{19}$$

(Beardon 1983). Recall, $GM(\hat{\mathrm{R}}^n)$ is the group of Möbius transformations of $\hat{\mathrm{R}}^n$, and $GM(S^{n-1})$ is the subgroup of Möbius transformations of $\hat{\mathrm{R}}^n$ that leave the unit sphere S^{n-1} invariant. The group homomorphism

$$\Pi^* : GM(S^{n-1}) \to GM(\hat{\mathrm{R}}^{n-1}) \tag{20}$$

given by $\Pi^*(f) = \Pi \circ f \circ \Pi^{-1}$, where $f \in GM(S^{n-1})$, is well-defined because of equation (19) and because the finite product of Möbius transformations is a Möbius transformation. We denote any identity mapping by I.

Lemma 6. Let ϕ_1 be the inversion in S^{n-1}. Then, the kernel of Π^* is $\{I, \phi_1\}$. Moreover, Π^* is surjective.

Proof. If $\Pi^*(f) = I$ on $\hat{\mathrm{R}}^{n-1}$, then $f = I$ is the identity mapping on S^{n-1}. The only two Möbius transformations in $GM(\hat{\mathrm{R}}^n)$ that restrict to the identity mapping on S^{n-1} are the inversion ϕ_1 in S^{n-1} and the identity mapping on $\hat{\mathrm{R}}^n$. Thus, the kernel of Π^* is $\{I, \phi_1\}$.

We show that Π^* is surjective. Let $\phi \in GM(\hat{\mathrm{R}}^{n-1})$ be given. We apply the Poincaré extension $\phi^{ext} \in GM(\hat{\mathrm{R}}^n)$ of ϕ satisfying $\phi^{ext}(\tilde{x},0) = (\phi(\tilde{x}),0)$ for every $\tilde{x} \in \hat{\mathrm{R}}^{n-1}$ (Beardon 1983). We recall the inversion ϕ_1 in $S_n(e_n, \sqrt{2})$. The restriction of ϕ_0 to S^{n-1} is the stereographic projection Π that maps S^{n-1} onto $\hat{\mathrm{R}}^{n-1}$. Since ϕ^{ext} leaves $\hat{\mathrm{R}}^{n-1}$ invariant, we find $\phi_0\phi^{ext}\phi_0 \in GM(S^{n-1})$. Applying equation (19), we obtain $\Pi^*(\phi_0\phi^{ext}\phi_0)(\tilde{x}) = \Pi\phi_0\phi^{ext}\phi_0\Pi^{-1}(\tilde{x}) = \Pi\phi_0\phi^{ext}\phi_0\phi_0(\tilde{x},0) = \Pi\phi_0\phi^{ext}(\tilde{x},0)$ since $\phi_0^2 = I$. Moreover, $\Pi\phi_0\phi^{ext}(\tilde{x},0) = \Pi\phi_0(\phi(\tilde{x}),0) = \phi(\tilde{x})$ by applying equation (19) again. Hence, $\Pi^*(\phi_0\phi^{ext}\phi_0) = \phi$ and Π^* is surjective.

VAHLEN MATRICES

The linear fractional transformations on the extended complex plane are well-defined because any two complex numbers may be multiplied. Similarly, we may define pseudo linear fractional transformations on R^n, since there is an intrinsic multiplication operation in R^n. In fact, there is a natural embedding of R^n into a real Clifford algebra, where R^n inherits the multiplication of the real Clifford algebra. We identify R^n as in (21). Also, there is a natural (n+1)-dimensional Euclidean space V^{n+1} sitting inside the real Clifford algebra as in (23) that contains R^n. We return to our study of the Möbius transformations ϕ of $\hat{\mathrm{R}}^n$. We may extend ϕ to a Möbius transformation of $\hat{\mathrm{V}}^{n+1} = \mathrm{V}^{n+1} \cup \{\infty\}$ by applying the Poincaré extension of ϕ. Then, we apply the results of Ahlfors (1985) and Waterman (1993) to show that ϕ is the restriction to $\hat{\mathrm{R}}^n$ of some pseudo linear fractional transformation that is defined on V^{n+1} (see Lemma 8). The pseudo linear fractional transformations are defined by Vahlen matrices.

We review general facts about Vahlen matrices (Ahlfors 1985). Let $Cl(n)$ be the real Clifford algebra generated by $\{e_1,\dots,e_n\}$ subject to $e_i^2=-1$ and $e_ie_j=-e_je_i$ if $i\neq j$. For $1\le k\le n$, consider $J=\{j_1,\dots,j_k\}\subseteq\{1,\dots,n\}$, where $j_1<\cdots<j_k$. Consider the product $e_J=e_{j_1}\cdots e_{j_k}$, and if $J=\varnothing$, set $e_0=e_\varnothing=1$. The collection $\{e_J : J\subseteq\{1,\dots,n\}\}$ is a basis for the real vector space $Cl(n)$. We identify R^n with the following subspace of $Cl(n)$:

$$\mathrm{R}^n=\left\{\sum_{j=1}^{n}a_je_j : a_j\in\mathrm{R}\right\}\subset Cl(n) \qquad (21).$$

The product of $x,y\in\mathrm{R}^n$ satisfies

$$xy+yx=-2K(x,y) \qquad (22).$$

Let $e_J=e_{j_1}\cdots e_{j_k}$ be a basis element of $Cl(n)$. There are three main involutions in $Cl(n)$. The grade involution is an algebra automorphism of $Cl(n)$ that is denoted and satisfies $\hat{e}_J=(-1)^k e_{j_1}\cdots e_{j_k}$ The conjugation is an algebra antiautomorphism of $Cl(n)$ that is denoted and satisfies $\bar{e}_J=(-1)^k e_{j_k}\cdots e_{j_1}$. The reversion is an algebra antiautomorphism of $Cl(n)$ satisfying $e_J^*=e_{j_k}\cdots e_{j_1}$. One checks that $(e_J^*)^\wedge=(\hat{e}_J)^*=\bar{e}_J$ for every $J=\{j_1,\dots,j_k\}\subseteq\{1,\dots,n\}$.

For an integer $k\ge1$, define the sum $s(k)=1+\cdots+k$, and set $s(0)=0$. If $k\ge0$, let $\beta(k)=(-1)^{s(k)}$ and set $\beta(-1)=1$. We list some basic properties of e_J without proof. The proof of statement 5 follows immediately from Theorem 2 of Waterman (1993).

Lemma 7. Let $J\subseteq\{1,\dots,n\}$ be given. Let $k=\#J$ be the number of elements in J. Let $e=e_1\cdots e_n$ be the product of all the e_i for $1\le i\le n$. If $x\in\mathrm{R}^n$, then

(i) $e_J^*=\beta(k-1)e_J$, $\bar{e}_J=\beta(k)e_J$,

(ii) $e_J^2=\beta(k)$, $e_J\bar{e}_J=1$,

(iii) $exe=-\beta(n-1)x$,

(iv) $ex = (-1)^{n+1} xe$, and

(v) if $a \in Cl(n)$ and $ax = x\hat{a}$ for all $x \in \mathrm{R}^n$, then $a \in \mathrm{R}$.

Let

$$V^{n+1} = \left\{\sum_{j=0}^{n} a_j e_j : a_j \in R\right\} \subset Cl(n) \quad (23).$$

If $v = \sum_{j=0}^{n} a_j e_j \in V^{n+1}$ and $v \neq 0$, the multiplicative inverse of v is $v^{-1} = \bar{v}/\left|v\right|^2$. Let Γ_n be the multiplicative group generated by the nonzero vectors in V^{n+1}. We say that Γ_n is the Clifford group of $Cl(n)$. A Vahlen matrix is a 2-by-2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (24),$$

where $a,b,c,d \in \Gamma_n \cup \{0\}$ for which $ab^*, cd^*, c^* a, d^*b \in \mathrm{V}^{n+1}$, and such that the pseudo-determinant Det_{v} satisfies $Det_v(A) \equiv ad^* - bc^* \in \mathrm{R}-\{0\}$. Let $GL_2(\Gamma_n)$ be the multiplicative group of all Vahlen matrices. If $A, B \in GL_2(\Gamma_n)$, then $Det_v(AB) = Det_v(A)Det_v(B)$, where the matrix multiplication is the usual one (equation (32)). Let $SL_2(\Gamma_n)=\{A\in GL_2(\Gamma_n):Det_v(A)=1\}$. Then, $SL_2(\Gamma_n)$ is a subgroup of $GL_2(\Gamma_n)$.

Let $\hat{\mathrm{V}}^{n+1} = \mathrm{V}^{n+1} \cup \{\infty\}$. A Vahlen matrix $A \in GL_2(\Gamma_n)$ induces a bijection F_A of $\hat{\mathrm{V}}^{n+1}$ into itself defined by

$$F_A(v) = (av+b)(cv+d)^{-1}, v \in \hat{\mathrm{V}}^{n+1} \quad (25).$$

We say that F_A is a pseudo-linear fractional transformation of $\hat{\mathrm{V}}^{n+1}$. One checks that $F_A F_b = F_{AB}$ for every $A, B \in GL_2(\Gamma_n)$. Furthermore, $F_A = F_{AB}$, if and only if $A = \alpha B$ for some nonzero $\alpha \in$ R. In such a case, we write $A \sim B$. Notice that ~ is an equivalence relation on $GL_2(\Gamma_n)$. For $A \in GL_2(\Gamma_n)$ let $[A]$ denote the equivalence class of A under ~. Let $PGL_2(\Gamma_n) = \{[A] : A \in GL_2(\Gamma_n)\}$ be a projective group of Vahlen matrices, and let $PSL_2(\Gamma_n) = \{[A] : A \in SL_2(\Gamma_n)\}$.

Let $GM(\hat{\mathrm{V}}^{n+1})$ denote the group of Möbius transformations of $\hat{\mathrm{V}}^{n+1}$, that is, the group generated by inversions about n-dimensional spheres and reflections about hyperplanes in $\hat{\mathrm{V}}^{n+1}$. Let $M(\hat{\mathrm{V}}^{n+1})$ denote the subgroup of orientation preserving

Möbius transformations of $\hat{V}^{n+1}$. For $n \ge 1$, the mapping $[A] \mapsto F_A$ defines the group isomorphism (Theorem B of Ahlfors (1985); Theorem 5 of Waterman (1993)) in

$$PSL_2(\Gamma_n) \to M(\hat{V}^{n+1}). \tag{26}$$

Notice that for even $n \ge 2$, the mapping $x \mapsto -x$ is an orientation reversing orthogonal transformation of $\hat{V}^{n+1}$ and the mapping $[A] \mapsto F_A$ defines a group isomorphism (Ahlfors 1985; Waterman 1993) in

$$PGL_2(\Gamma_n) \to GM(\hat{V}^{n+1}) \tag{27}.$$

We continue to identify R^n as the subspace of $\hat{V}^{n+1}$ according to equation (21). We prove that every Möbius transformation of $\hat{R}^n$ can be expressed as the restriction to $\hat{R}^n$ of a pseudo-linear fractional transformation F_A of $\hat{V}^{n+1}$ as in (25).

Lemma 8. Let ϕ be a Möbius transformation of $\hat{R}^n$. Then, there exists a Vahlen matrix $A \in GL_2(\Gamma_n)$, such that the restriction of the corresponding pseudo linear fractional transformation F_A satisfies $F_A\big|_{\hat{R}^n} = \phi$.

Proof. We regard the Poincaré extension ϕ^{ext} of ϕ as a Möbius transformation of $\hat{V}^{n+1}$ satisfying $\phi^{ext}\big|_{\hat{R}^n} = \phi$. For even $n \ge 2$, there exists $A \in GL_2(\Gamma_n)$, such that $F_A = \phi^{ext}$ because of the isomorphism (27). For odd $n \ge 1$, if ϕ^{ext} is orientation preserving, then $F_A = \phi^{ext}$ for some $A \in SL_2(\Gamma_n)$ by the isomorphism (26); and if ϕ^{ext} is orientation reversing, let $\phi^{ext2} = \phi_2 \circ \phi^{ext}$, where ϕ_2 is the reflection about $H(0, e_0)$ in $\hat{V}^{n+1}$, and notice that ϕ^{ext2} is an orientation preserving Möbius transformation of $\hat{V}^{n+1}$, so that $\phi^{ext2} = F_A$ for some $A \in SL_2(\Gamma_n)$ and $F_A\big|_{\hat{R}^n} = \phi$.

Since $R^n \subset V^{n+1}$, we may extend an orthogonal transformation ϕ of R^n to an orthogonal transformation ϕ^{ext} of $\hat{V}^{n+1}$, such that $\phi^{ext}\big|_{R^n} = \phi$. There are exactly two orthogonal extensions ϕ^{ext} of ϕ : with one extension orientation preserving and the other orientation reversing. Moreover, either $\phi^{ext}(I) = 1$ or $\phi^{ext}(I) = -1$.

If $y \in R^n$ with $|y| = 1$ and if ϕ is the reflection about the hyperplane $H(0,y)$, then $\phi(x) = x - 2K(x, y)y = yxy = F_V\big|_{R^n}(x)$, where $x \in R^n$ and $V = \begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix} \in SL_2(\Gamma_n)$. Notice that F_V is an orientation preserving orthogonal transformation of V^{n+1} by isomorphism (26), but $F_V\big|_{R^n}$ is orientation reversing. Moreover, by the Cartan-Diedonne theorem, any orthogonal transformation ψ of R^n (being a product of reflections) may be expressed as $\psi = F_W\big|_{R^n}$ for some $W = \begin{pmatrix} \gamma & 0 \\ 0 & \hat{\gamma} \end{pmatrix} \in SL_2(\Gamma_n)$. We describe another specific case of Lemma 8.

Lemma 9. Let $\phi \in GM(\hat{\mathrm{R}}^n)$ be the indicated Möbius transformation. There exists $A \in GL_2(\Gamma_n)$ that induces a pseudo linear fractional transformation F_A satisfying $\phi = F_A\big|_{\hat{\mathrm{R}}^n}$.

(i) If ϕ is the inversion in $S_n(e_n, \sqrt{2})$, then $A = \begin{pmatrix} e_n & -1 \\ 1 & -e_n \end{pmatrix}$.

(ii) If ϕ is the reflection about $H_n(0, v)$, $v \in R^n$, and $|v| = 1$, then $A = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}$.

Proof. The proof of the second statement is in the discussion prior to Lemma 9. To prove the first statement, let $\psi(x) = \sqrt{2}x + e_n$, and let ϕ be the inversion in S^{n-1}. The inversion ϕ_0 in $S_n(e_n, \sqrt{2})$ satisfies $\phi_0 = \psi \circ \phi_1 \circ \psi^{-1}$ (Beardon 1983). Thus, the product of the Vahlen matrices for ψ, ϕ_1, and ψ^{-1} is the Vahlen matrix $\begin{pmatrix} \sqrt{2} & e_n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -e_n \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} e_n & -1 \\ 1 & -e_n \end{pmatrix}$, which is the matrix in the first statement.

EMBEDDING THE ORTHOGONAL GROUP $O(n)$

Let ϕ_0 be the inversion in $S_n(e_n, \sqrt{2})$, and let

$$V_0 = \begin{pmatrix} e_n & -1 \\ 1 & -e_n \end{pmatrix}, \tag{28}$$

be as in Statement 1 of Lemma 9. Then, $\phi_0 = F_{V_0}\big|_{\hat{R}^n}$. Since $\phi_0^{-1} = \phi_0$, the inverse of the Vahlen matrix V_0 satisfies $F_{V_0^{-1}} = F_{V_0}$ and $\phi_0 = F_{V_0^{-1}}\big|_{\hat{R}^n}$.

Given $f \in GM(S^{n-1})$, there exists a Vahlen matrix $A \in GL_2(\Gamma_n)$, such that $f(x) = F_A(x)$ for every $x \in \hat{R}^n$ (see Lemma 8). Applying the group homomorphism (20) induced by stereographic projection, and equation (19), we obtain $\Pi^*(f) = \Pi \circ f \circ \Pi^{-1} = \phi_0\big|_{S^{n-1}} \circ f \circ \phi_0\big|_{\hat{R}^{n-1}} = F_{V_0 A V_0^{-1}}\big|_{\hat{R}^{n-1}}$.

This leads us to inner-conjugate the Vahlen matrices in Statement 2 of Lemma 9 by V_0. We present the results without proof. The calculations are straightforward since $V_0^{-1} = \frac{1}{2}\begin{pmatrix} -e_n & 1 \\ -1 & e_n \end{pmatrix}$.

Lemma 10. If $i = 1, \cdots, n-1$, then the following Vahlen matrices lie in $SL_2(\Gamma_{n-1})$:

(i) $V_0\begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix}V_0^{-1} = \begin{pmatrix} -e_i & 0 \\ 0 & e_i \end{pmatrix}$

(ii) $V_0\begin{pmatrix} e_n & 0 \\ 0 & -e_n \end{pmatrix}V_0^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

For each $v = \sum_{i=1}^{n} a_i e_i \in S^{n-1}$, let $f^v \in O(n)$ denote the reflection about the hyperplane $H_n(0,v)$. We may express $\Pi^*(f^v)$ as a pseudo-linear fractional transformation. Applying Lemmata 9 and 10, we obtain $\Pi^*(f^v) = F_V\big|_{\hat{R}^{n-1}}$, where $V = V_0\begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}V_0^{-1}$ $\in SL_2(\Gamma_{n-1})$.

Now, define the mapping $m_1 : S^{n-1} \to SL_2(\Gamma_{n-1})$, where $m_1(v) = V$. Notice that $v^2 = -1$ and that $V^2 = -I_2$. Let A^* denote the conjugate transpose of a Vahlen matrix $A = \{a_{i,j}\}$. Here, the (i, j)-entry of A^* is $\bar{a}_{j,i}$. One checks that $V_0V_0^* = V_0^*V_0 = 2I$, so that $VV^* = V^*V = I$.

For $i \in \{1, \dots, k\}$, let $v_i \in S^{n-1}$ be a unit vector and let $m_1(v_i) = V_i$. If the product of reflections satisfy $f^{v_1} \cdots f^{v_k} = I_n$ and the identity mapping on R^n, then $\Pi^*(f^{v_1} \cdots f^{v_k}) = I_{n-1}$. However, we have $\Pi^*(f^{v_1} \cdots f^{v_k}) = F_W\big|_{\hat{R}^{n-1}}$, where $W = V_1 \cdots V_k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\Gamma_{n-1})$. Then, $(ax+b)(cx+d)^{-1} = x$ for every $x \in \hat{R}^{n-1}$. Taking $x = 0$, we obtain $b = 0$. Looking at the (1,1) entries of the equality $WW^* = I$, we have $a\bar{a} = 1$; looking at the (2,1) entries, we have $c\bar{a} = 0$, so that $c = 0$; and looking at the (2,2) entries, we get $d\bar{d} = 1$. Now, $ad^* = Det_v(W) = 1$ and we have $d^{-1} = a^* = \hat{a}^{-1}$. Notice that $(ax+b)(cx+d)^{-1} = x$ becomes $(ax)(d)^{-1} = x$, that is, $ax = xd$. Hence, we have $ax = x\hat{a}$ for each $x \in \mathrm{R}^{n-1}$. Since $a \in \Gamma_{n-1} \subset Cl(n-1)$, Statement 5 of Lemma 7 guarantees that $a \in \mathrm{R}$. Consequently, $a = d = \pm 1$ and $W = \pm I_2$.

Theorem 11. If $A \in O(n)$ and if $A = f^{v_1} \cdots f^{v_k}$ is a product of reflections, then the mapping

$$\xi_{e_n} : O(n) \to PSL_2(\Gamma_{n-1}) \tag{29}$$

given by $\xi_{e_n}(A) = [m_1(v_1) \cdots m_1(v_k)]$ is an injective group homomorphism.

Proof. The discussion before Theorem 11 shows that the group homomorphism (29) is well-defined. It suffices to show that the group homomorphism (29) is injective. Let $A = f^{v_1} \cdots f^{v_k}$ be a product of reflections, and let $B = f^{w_1} \cdots f^{w_l}$ with $w_j \in S^{n-1}$ for each $j = 1, \dots, l$. Suppose $\xi_{e_n}(A) = \xi_{e_n}(B)$. Let each $W_j = m(w_j) \in PSL_2(\Gamma_{n-1})$. Since $\xi_{e_n}(B) = [W_1 \cdots W_l]$, we obtain $V_1 \cdots V_k = \pm W_1 \cdots W_l$, so that $F_{V_1 \cdots V_k} = F_{W_1 \cdots W_l}$ and $\Pi^*(A) = \Pi^*(B)$.

Recall that the kernel of the group homomorphism Π^* is $\{I_n, \phi 1\}$, where ϕ_1 is the inversion in S^{n-1} (see Lemma 6). Since AB^{-1} is a real linear mapping in the kernel of Π^*, we conclude that $A = B$. Hence, ξ_{e_n} is injective.

The homomorphism (29) is a higher dimensional analogue but not a generalization of the homomorphism ξ in (8). In fact, when $n = 4$, ξ_{e_n} does not reduce to ξ as in (8), since the quaternionic matrices in (5) are not necessarily Vahlen matrices. Moreover, the pseudo determinant of $m(y)$ in (5) is not necessarily a real number for $Det_v(m(y))=p^2+q^2-r^2+y_2{}^2+2rqi-2prj$.

We compare Theorem 11 with well-known facts studied by many math students. First, if z_0 is a complex number with modulus $|z_0| = 1$, then multiplication by z_0 defines a rotation $f_0(z) = z_0 z$ in the complex plane that leaves the group (isomorphic to S^1) of complex numbers with modulus 1 invariant. Let $z_1 \in S^1$ be a square root of z_0. Then, the diagonal complex matrix $A_0 = diag(z_1, z_1^{-1})$ is a Vahlen matrix in $SL_2(\Gamma_1)$ that defines a linear fractional transformation F_{A_0} satisfying $F_{A_0} = f_0$. Next, we may identify the sets S^2 (which is not a group) and S^3 with the set of pure unit quaternions and the group of unit quaternions, respectively. If $w_0 \in S^3$ is a unit quaternion, then the mapping $g_0(w) = w_0 w \overline{w}_0, w \in S^2$ is an orientation-preserving orthogonal transformation that leaves S^2 invariant. Conversely, any orthogonal transformation of R^3 (identified with the set of pure quaternions) that leaves S^2 invariant is of the form g_0 (Theorem 3.3 of Coxeter (1946)).

In Theorem 11, the homomorphism ξ_{e_n} depends on the stereographic projection Π. Suppose that we change Π to another stereographic projection. We claim that there is an injective group homomorphism from *O(n)* into a subgroup of $PSL_2(\Gamma_n)$ that is inner-conjugate to $PSL_2(\Gamma_{n-1})$. Notice that, by an application of the Poincaré extension, each Möbius transformation in $GM(\hat{R}^{n-1})$ is the restriction to $\hat{R}^{n-1}$ of some Möbius transformation in $GM(\hat{R}^n)$. Moreover, for any hyperplane $\mathrm{H} \subset R^n$, there is a Möbius transformation p in $GM(\hat{R}^n)$ that maps $\hat{R}^{n-1}$ onto $\hat{\mathrm{H}} = \mathrm{H} \cup \{\infty\}$. Let $GM(\hat{\mathrm{H}})$ be the group of Möbius transformations of $\hat{\mathrm{H}}$ generated by inversions in spheres in $\hat{\mathrm{H}}$ of dimension $n - 2$ and reflections in hyperplanes of $\hat{\mathrm{H}}$. Then, we may associate a Möbius transformation φ of $\hat{\mathrm{H}}$ to a Möbius transformation $p^{-1}\varphi p$ of $\hat{R}^{n-1}$ by inner-conjugation. A bijection of $\hat{\mathrm{R}}^n$ is a Möbius transformation if and only if the bijection preserves cross ratios (cf. Theorem 3.2.7 of Beardon (1983)). Thus, each Möbius transformation in $GM(\hat{\mathrm{H}})$ is the restriction to $\hat{\mathrm{H}}$ of some Möbius transformation in $GM(\hat{\mathrm{R}}^n)$.

Let $s \in S^{n-1}$, and let ψ be the inversion in the sphere $S(s, \sqrt{2}) \subset R^n$. Then, ψ, maps the unit sphere S^{n-1} onto the hyperplane $\hat{\mathrm{H}}$, perpendicular to s, that is, if $x \in S^{n-1}$ and $x \neq s$, then $K(\psi_s(x), s) = 0$. The restriction of ψ_s to the unit sphere S^{n-1} is a stereographic projection, and ψ_s induces a group homomorphism $\psi_S^* : GM(S^{n-1}) \to GM(\hat{\mathrm{H}}_s)$, such that if $f \in GM(S^{n-1})$, we define $\psi_s^*(f) = \psi_s \circ f \circ \psi_s\big|_{\hat{\mathrm{H}}_s}$. By Lemma 7, we have $f = F_A\big|_{\hat{\mathrm{R}}^n}$ and $\psi_s = F_D\big|_{\hat{\mathrm{R}}^n}$, where A, D are Vahlen matrices with entries in Γ_n, and F_A is the pseudo linear fractional transformation of $\hat{\mathrm{V}}^{n+1}$ defined by A. Using a proof similar to Statement 1 of Lemma 9, we can show that $D = \begin{pmatrix} s & -1 \\ 1 & -s \end{pmatrix}$. Hence, $\psi_s^*(F_A) = F_{DAD^{-1}}\big|_{\hat{\mathrm{H}}_s}$.

Let $T = \psi_s \circ \phi_0$ where ϕ_0 is the inversion in the sphere $S(e_n, \sqrt{2})$. Then, $B = DV_0$ is a Vahlen matrix for the pseudo linear fractional transformation that defines T. For each $v \in S^{n-1}$, we obtain $\left[D\begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} D^{-1} \right] = \left[BV_0^{-1}\begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}(BV_0^{-1})^{-1} \right] \in PSL_2(\Gamma_n)$. Since the homomorphism ξ_{e_n} in equation (29) is a function, the relation $\xi_s : O(n) \to PSL_2(\Gamma_n)$ satisfying $\xi_s(f^v) = \left[D\begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} D^{-1} \right]$ extends to a well-defined group homomorphism.

Thus, by changing the stereographic projection $\Pi = \phi_0\big|_{S^{n-1}} = \psi_{e_n}\big|_{S^{n-1}}$ to another stereographic projection $\psi_s\big|_{S^{n-1}}$, the images of ξ_{e_n} and ξ_s are subgroups of $PSL_2(\Gamma_n)$ that are inner-conjugate to each other.

LINEAR MAPS ON CLIFFORD ALGEBRAS

We study 2-by-2 matrices with entries in a real Clifford algebra $Cl(n)$. We discuss an analog of real orthogonal matrices and call these Cl(n)-orthogonal matrices. Moreover, we discuss the eigenvalues of such 2-by-2 matrices with entries in $Cl(n)$.

The bilinear form (18) in R^n extends to the Clifford algebra $Cl(n)$ as follows: $K\left(\sum_J a_J e_J, \sum_J b_J e_J\right) = \sum_J a_J b_J$, where $a_J, b_J \in \mathrm{R}$, and the sum is indexed over all subsets $J \subseteq \{1, \dots, n\}$. The norm satisfies $\left|\sum_J a_J e_J\right|^2 = \sum_J a_J^2 = K\left(\sum_J a_J e_J, \sum_J a_J e_J\right)$. Let $Cl(n)^{(2)}$ denote the set of column vectors $(v_1, v_2)^T$ where $v_1, v_2 \in Cl(n)$. Let $K^{(2)}$ be a bilinear form on defined by $K^{(2)}\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = K(v_1, w_1) + K(v_2, w_2)$.

Let $gl_2(n)$ denote the set of all 2-by-2 matrices with entries in $Cl(n)$. For each $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in gl_2(n)$, define a right $Cl(n)$-linear map on $Cl(n)^{(2)}$ where

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_{11}v_1 + x_{12}v_2 \\ x_{21}v_1 + x_{22}v_2 \end{pmatrix} \quad (30).$$

The composite of two linear maps and is given by their matrix product

$$XY = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} x_{11}y_{11} + x_{12}y_{21} & x_{11}y_{12} + x_{12}y_{22} \\ x_{21}y_{11} + x_{22}y_{21} & x_{21}y_{12} + x_{22}y_{22} \end{pmatrix} \quad (31).$$

Then, $gl_2(n)$ is a group under matrix multiplication. The collection $\{e_J : J \subseteq \{1,\ldots,n\}\}$ is an orthonormal basis for $Cl(n)$ and satisfies

$$K(e_J a, e_J b) = K(a,b) = K(ae_J, be_J) \quad (32)$$

for all $a, b \in Cl(n)$.

Let X^* denote the conjugate transpose of the matrix $X = \{x_{i,j}\}$. Here, the (i,j)-entry of X^* is $\bar{x}_{j,i}$. Applying equation (32) and component-wise matrix multiplication, we find that X^* is the adjoint of X, that is $K^{(2)}(Xv, w) = K^{(2)}(v, X^* w)$, for all $v, w \in Cl(n)^{(2)}$. We say that X is Cl(n)-orthogonal if $K^{(2)}(Xv, w) = K^{(2)}(v, X^* w)$ for all $v, w \in Cl(n)^{(2)}$. Notice that X is Cl(n)-orthogonal, if and only if (the matrix products) $XX^* = X^* X = I$.

Let $v \in \mathrm{R}^n$, $|v| = 1$, and let $e = e_1 \cdots e_n$ be the product of the standard basis elements of R^n. Then the following three matrices are Cl(n)-orthogonal matrices:

$$A = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}, B = \begin{pmatrix} 0 & (-1)^n e \\ e & 0 \end{pmatrix}, C = \frac{1}{\sqrt{2}}\begin{pmatrix} (-1)^{n+1} ee_n & (-1)^n e \\ e & -ee_n \end{pmatrix}.$$

In addition to the Vahlen matrices in Lemma 9, we claim that B and C induce pseudo linear fractional transformations whose restrictions to $\hat{\mathrm{R}}^n$ are the inversions ϕ_1 and ϕ_0 in the spheres $S_n(0,1)$ and $S_n(e_n, \sqrt{2})$, respectively. Let $x \in \mathrm{R}^n$ be nonzero. Then $F_B(x) = (-1)^n e(ex)^{-1}$. By Lemma 7, we obtain $(ex)^{-1} = \beta(n)x^{-1}e$, and $F_B(x) = (-1)^{n+1}$ $\beta(n)exe/|x|^2 = x/|x|^2 = \phi_1(x)$.

The proof that C induces ϕ_0 is similar to the proof of Statement 1 of Lemma 9. Let $\psi(x) = \sqrt{2}x + e_n$. Then, $\psi\phi_1\psi^{-1}$ is the inversion in $S_n(e_n, \sqrt{2})$ (Beardon 1983). Consequently, the product of the Vahlen matrices for ψ, ϕ_1, ψ^{-1} is a Vahlen for the inversion in $S_n(e_n, \sqrt{2})$: $\begin{pmatrix} \sqrt{2} & e_n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & (-1)^n e \\ e & 0 \end{pmatrix}\begin{pmatrix} 1 & -e_n \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} (-1)^{n+1} ee_n & (-1)^n e \\ e & -ee_n \end{pmatrix}$. We multiply the factor $1/\sqrt{2}$ to the Vahlen matrix in the right side to guarantee that $CC^* = I$.

We say X is positive semidefinite if $K^{(2)}(Xv, v) \geq 0$ for every $v \in Cl(n)^{(2)}$. We write $X \geq 0$ if X is positive semidefinite. For every $X \in gl_2(n)$, we have $XX^* \geq 0$ and $X^*X \geq 0$. For $r > 0$, the matrix for the inversion in $S_n(0,r) \geq 0$ can be written as $\begin{pmatrix} 0 & (-1)^n r^2 e \\ e & 0 \end{pmatrix}$ = $\begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & (-1)^n e \\ e & 0 \end{pmatrix}$ and resembles the polar decomposition of real matrices, that is, a product of a positive semidefinite matrix and an orthogonal matrix in $gl_2(n)$.

In discussing eigenvalues, we use the quadratic norm $\Delta(a) = \bar{a}a$, $a \in Cl(n)$. Notice that it is not necessarily true that $a\bar{a} = |a|^2$; if $a = e_1e_2 + e_3$, then $a\bar{a} = 2(1 - e_1e_2e_3)$. However, if $v \in V^{n+1}$, then $\Delta(v) = v\bar{v} = |v|^2$. We restrict and choose the nonzero eigenvalues and the nonzero components of eigenvectors from the set $N^* = \{a \in Cl(n)$: $0 \neq \Delta(a) \in R\}$.

Because Γ_n is the multiplicative group generated by the nonzero vectors in V^{n+1}, we have $\Gamma_n \subseteq N^*$. It is known that $\Delta(ab) = \Delta(a)\Delta(b)$ for $a, b \in N^*$(Proposition 5.7.2 of Garling (2011)). Moreover, if $a \in N^*$, then $\hat{a}, a^*, \bar{a} \in N^*$ and $|a|^2 = \Delta(a) = \Delta(\hat{a}) = \Delta(a^*) = \Delta(\bar{a})$.

Let $X \in gl_2(n)$ be given. Suppose that $v_1, v_2 \in N = N^* \cup \{0\}$ are not both zero. Let $\lambda \in n$ be given. We say that $v = (v_1, v_2)^T$ is an eigenvector of X with right eigenvalue λ if $X_v = v\lambda$; v is an eigenvector of X with left eigenvalue λ if $X_v = \lambda v$. If v is an eigenvector of A with right eigenvalue λ, and if $y \in N^*$, then notice that vy^{-1} is also an eigenvector of A with right eigenvalue $y\lambda y^{-1}$ because

$$A(vy^{-1}) = v\lambda y^{-1} = vy^{-1}(y\lambda y^{-1}) \qquad (33).$$

If A is the Vahlen matrix in (24), we claim that $cv + d \in \Gamma_n \cup \{0\}$ for any $v \in V^{n+1}$. In fact, if $c \neq 0$, then $cv + d = c(v + c^{-1}d) \in \Gamma_n \cup \{0\}$; while if $c = 0$, clearly $cv + d \in \Gamma_n$. If the corresponding pseudo linear fractional transformation satisfies $F_A(v)=v$, then a standard calculation shows that $(v,1)^T$ is an eigenvector of A with right eigenvalue

$cv + d$. In particular, Vahlen matrices for inversions and reflections have eigenvectors since they have fixed points. A good discussion of the eigenvalues of quaternionic matrices is discussed by Zhang (2007).

Lemma 12. Let $\lambda \in N$, $r > 0$, and $v \in \mathbb{R}^n$ with $v \neq 0$.

(i) If λ is a left or right eigenvalue of $A = \begin{pmatrix} 0 & (-1)^n r^2 e \\ e & 0 \end{pmatrix}$, then $|\lambda| = r$.

(ii) Then, λ is a left eigenvalue of $B = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}$, if and only if $\lambda = \pm v$. Moreover, λ is a right eigenvalue of B, if and only if $\lambda = \pm bvb^{-1}$ for some nonzero $b \in N$.

Proof. Let $v = (v_1, v_2)^T$ be an eigenvector of A with right eigenvalue λ. Then, $\left((-1)^n r^2 e v_2, e v_1\right)^T = \left(v_1\lambda, v_2\lambda\right)^T$. Notice that λ, v_1 and v_2 are each nonzero, otherwise $\lambda = v_1 = v_2 = 0$, and v is not an eigenvector. We recall the function β from Lemma 7. Since $e^{-1} = \beta(n)e$, we find $v_2(-1)^n r^2 = (-1)^n r^2 v_2 = \beta(n)ev_1\lambda = \beta(n)v_2\lambda^2 = v_2\beta(n)\lambda^2$. Then, $(-1)^n r^2 = \beta(n)\lambda^2$. Hence, $|\lambda|^4 = \Delta(\lambda)^2 = \Delta(\lambda^2) = \Delta(r^2) = r^4$, so that $|\lambda| = r$. A similar calculation shows that, if λ is a left eigenvalue, then $|\lambda| = r$.

Notice that $(1,0)^T$ and $(0,1)^T$ are eigenvectors of B with left eigenvalues v and $-v$, respectively. Moreover, $\lambda = 0$ is neither a left nor a right eigenvalue of B. Let $\lambda \in N^*$ be a left eigenvalue of B with eigenvector $(x,y)^T$. Then, $(vx, -vy)^T = B(x,y)^T = \lambda(x,y)^T$. If $x \neq 0$, then $v = \lambda$ and $y = 0$; if $y \neq 0$, then $v = -\lambda$ and $x = 0$.

Let λ' be a right eigenvalue of B corresponding to the eigenvector $w = (w_1, w_2)^T$, that is, suppose $Bw = w\lambda'$. Then, $(vw_1, -vw_2)^T = (w_1\lambda', w_2\lambda')$. If $w_1 \neq 0$, then $\lambda' = -w_1^{-1}vw_1$. Either $w_2 = 0$ or $\lambda' = -w_2^{-1}vw_2$. If $w_1 = 0$, then $w_2 \neq 0$ and $\lambda' = -w_2^{-1}vw_2$. Conversely, if a nonzero $b \in N$ is given, then $\lambda = -bvb^{-1}$ is a right eigenvalue of B corresponding to the eigenvector $w = (b^{-1}, 0)^T$ while $\lambda = -bvb^{-1}$ is a right eigenvalue of B corresponding to the eigenvector $w = (0, b^{-1})^T$.

For the matrix A in Lemma 12, suppose n is odd, $x \in \mathbb{R}^n$, and $|x| = r$. Applying Statement 4 of Lemma 7, we find that ex is a left and a right eigenvalue of A with eigenvector $(x, 1)^T$. Thus, if n is odd, A has infinitely many left eigenvalues, which is a partial answer to an analogous question raised for quaternionic matrices in (Question 5.1 of Zhang (2007)). In Statement 2 of Lemma 12, the set of right eigenvalues of B consists of either one or two distinct orbits in N under inner-conjugation. We compare this with Corollary 5.2 of Zhang (2007), where a general n-by- n quaternionic matrix has exactly n distinct such orbits.

FINAL REMARKS

We proved that the mapping ξ in (8) is a differentiable function between Lie groups. We determined a matrix representation of the differential $d\xi$ of ξ. In Theorem 11, we described a higher dimensional analog ξ_{e_n} of ξ. Our methods involved Möbius transformations, real Clifford algebras, Vahlen matrices, and pseudo linear fractional transformations. We also discussed linear maps defined by 2-by-2 matrices with entries in a real Clifford algebra, and evaluated some eigenvalues of such matrices.

We conjecture that the homomorphsim $\xi_{e_n} : O(n) \to PSL_2(\Gamma_{n-1})$ in Theorem 11 is continuous. Moreover, it would be interesting to determine the eigenvalues of an arbitrary 2-by-2 matrix with entries in a Clifford algebra.

ACKNOWLEDGEMENTS

The authors would like to thank the referees for correcting a mistake in one of the matrices in Section 4, and for their suggestions in making this paper more understandable.

REFERENCES

Ahlfors LV. 1985. Möbius transformations and Clifford numbers. In: Chavel I, Farkas HM, editors. Differential Geometry and Complex Anaysis. Berlin: Springer-Verlag. p. 65-73.

Beardon AF. 1983. The Geometry of Discrete Groups. Graduate Texts in Mathematics, Vol. 91.

Becker H, Kechris AS. 1996. The Descriptive Set Theory of Polish Group Actions. Cambridge University Press. London Mathematical Society Lecture Note Series 232.

Brocker T, Dieck T. 1985. Representations of Compact Lie Groups. Graduate Texts in Mathematics, Vol. 98.

Canlubo C, Reyes E. 2012. Reflections on and quaternionic Möbius transformations. J. Lie Theory. 22:839–844.

Conway J, Smith D. 2003. On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry. Massachusetts: A.K. Peters, Ltd.

Coxeter HSM. 1946. Quaternions and reflections. American Mathematical Monthly. 53:136-146.

Garling DJH. 2011. Clifford Algebras: An Introduction. (London Mathematical Society Student Texts)The London Mathematical Society Student Texts). Vol. 78. London: Cambridge University Press.

Helgason SH. 1978, Differential Geometry, Lie Groups, and Symmetric Spaces. New York: Academic Press.

Koecher M, Remmert R. Hamilton's Quaternions. In: Ewing JH, editor. Numbers (Graduate Texts in Mathematics). Vol. 123. New York (NY); Springer; c1991.

Lawson J. 2010. Clifford algebras, Möbius transformations, Vahlen matrices, and B-loops. Commentations in Math Universitas Carolinae. 51:319–331.

Waterman PL. 1993. Möbius transformations in several dimensions. Advances in Mathematics. 101:87-113.

Zhang F. 2007. Geršgorin type theorems for quaternionic matrices. Linear Algebra and Its Applications. 424:139-153.

Dennis I. Merino <dmerino@selu.edu>, **Edgar N. Reyes**, and **Gary A. Walls** are professors in the Department of Mathematics at Southeastern Louisiana University (SLU) in Hammond, Louisiana, USA. They conduct weekly Mathematics seminar during the fall and spring semesters on various topics. **Bimal Kunwor** attended such a seminar when he was a Mathematics undergraduate student at SLU. He is currently a Mathematics graduate student at Louisiana State University in Baton Rouge, Louisiana, USA.