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Gaussian twin Neighborhood Prime Labeling on Fan Digraphs

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Abstract

Gaussian integers are complex numbers of the form $\gamma = x + iy$ where x and y are integers and $i^2 = -1$. The set of Gaussian integers is usually denoted by $\mathbb{Z}[i]$. A Gaussian integer $\gamma = a + ib \in \mathbb{Z}[i]$ is prime if and only if either $\gamma = \pm(1 \pm i)$, $N(\gamma) = a^2 + b^2$ is a prime integer congruent to 1 (mod 4), or $\gamma = p + 0i$ or $= 0 + pi$ where p is a prime integer and $|p| \equiv 3(\text{mod } 4)$. Let $D = (V, A)$ be a digraph with $|V| = n$. An injective function $f: V(D) \to [\gamma_n]$ is said to be a Gaussian twin neighborhood prime labeling of D, if it is both Gaussian in and out neighborhood prime labeling. A digraph which admits a Gaussian twin neighborhood prime labeling is called a Gaussian twin neighborhood prime digraph. In this paper, we introduce some definitions of fan digraphs. Further, we establish the Gaussian twin neighborhood prime labeling in fan digraphs using Gaussian integers.

Keywords: Gaussian integers, neighborhood prime, labeling, digraphs.

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1. Introduction

Graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. The concept of graph labeling was introduced by Rosa in [1]. An useful survey on graph labeling by J.A. Gallian can be found in [2]. Spiral ordering of the Gaussian integer was first introduced by Hunter Lehmann and Andrew Park in [4]. T. J. Rajesh Kumar and T. K. Mathew Varkey [7] introduced the concept of Gaussian neighborhood prime labeling of a graph. K. Palani [6] et.al, introduced the concept of Gaussian twin neighborhood prime labeling in digraphs.

Let $D = (V, A)$ be a digraph of order$n$. Then V is the set of vertices of D with$|V| = n$, and A is the set of arcs of D consisting of ordered pairs of distinct vertices. The in-degree $d^-(v)$ of a vertex v in a digraph D is the number of arcs having v as its terminal vertex. The out-degree $d^+(v)$ of v is the number of arcs having v as its initial vertex [2]. Throughout this article we use only digraphs.

In this article, we introduce the definition of fan and double fan digraphs by orienting fan and double fan graphs. Also, we investigate the existence of Gaussian twin neighborhood prime labeling in fan digraphs.

2. Preliminaries

The following basic definitions and properties are from [4]

Definition 2.1. Gaussian integers are complex numbers of the form $\gamma = x + iy$ where x and y are integers and$i^2 = -1$. The set of Gaussian integers is usually denoted by$\mathbb{Z}[i]$.

A Gaussian integer is even if $1 + i$ dividesγ. Otherwise it is an odd Gaussian integer.

Definition 2.2. A Gaussian integer $\gamma = a + ib \in \mathbb{Z}[i]$ is prime if and only if either
(i) $\gamma = \pm(1 \pm i)$
(ii) $N(\gamma) = a^2 + b^2$ is a prime integer congruent to 1 (mod 4), or
(iii) $\gamma = p + 0i$ Or $= 0 + pi$ where p is a prime integer and$|p| \equiv 3(\text{mod } 4)$.

Definition 2.3. The spiral ordering of the Gaussian integers is recursively defined ordering of the Gaussian integers. We denote the nth Gaussian integer in the spiral ordering by γ_n. The ordering is defined beginning with $\gamma_1 = 1$ and continuing as:

$$\gamma_{n+1} = \begin{cases} \gamma_n + i & \text{if } \mathrm{Re}(\gamma_n) \equiv 1 \ (\text{mod } 2), \mathrm{Re}(\gamma_n) > \mathrm{Im}(\gamma_n) + 1 \\ \gamma_n - 1 & \text{if } \mathrm{Im}(\gamma_n) \equiv 0 \ (\text{mod } 2), \mathrm{Re}(\gamma_n) \le \mathrm{Im}(\gamma_n) + 1, \mathrm{Re}(\gamma_n) > 1 \\ \gamma_n + 1 & \text{if } \mathrm{Im}(\gamma_n) \equiv 1 \ (\text{mod } 2), \mathrm{Re}(\gamma_n) < \mathrm{Im}(\gamma_n) + 1 \\ \gamma_n + i & \text{if } \mathrm{Im}(\gamma_n) \equiv 0 \ (\text{mod } 2), \mathrm{Re}(\gamma_n) = 1 \\ \gamma_n - i & \text{if } \mathrm{Re}(\gamma_n) \equiv 0 \ (\text{mod } 2), \mathrm{Re}(\gamma_n) \ge \mathrm{Im}(\gamma_n) + 1, \mathrm{Im}(\gamma_n) > 0 \\ \gamma_n + 1 & \text{if } \mathrm{Re}(\gamma_n) \equiv 0 \ (\text{mod } 2), \mathrm{Im}(\gamma_n) = 0 \end{cases}$$

and $[\gamma_n]$ denotes the set of first n Gaussian integers in the spiral ordering.

Properties 2.4. 1. A Gaussian integer $\gamma = x + iy$ is called a prime Gaussian integer if its only divisors are $\pm 1, \pm i, \pm\gamma$ or $\pm\gamma i$.

2. Two Gaussian integers x and y are relatively prime if their only common divisors are the units in $\mathbb{Z}[i]$.
3. Let γ be a Gaussian integer and let u be a unit. Then γ and $\gamma + u$ are relatively prime.
4. In the spiral ordering, consecutive Gaussian integers are relatively prime.
5. In the spiral ordering, consecutive odd Gaussian integers are relatively prime.
6. Let α be a prime Gaussian integer and γ be a Gaussian integer. Then γ and $\gamma + \alpha$ are relatively prime if and only if $\alpha \nmid \gamma$.
7. Let γ be an odd Gaussian integer, let t be a positive integer and u be a unit. Then γ and $\gamma + u(1+i)^t$ are relatively prime.

The following definitions are taken from [6]

Definition 2.5. Let $D = (V, A)$ be a digraph with $|V| = n$. An injective function $f: V(D) \to [\gamma_n]$ is called **Gaussian in-neighborhood prime labeling** of D, if for every vertex $v \in V(D)$ with $d^-(v) > 1$, the Gaussian integers in the set $\{f(u): u \in N^-(v)\}$ are relatively prime where $N^-(v) = \{u \in V(D) : \overrightarrow{uv} \in A(D)\}$.

Definition 2.6. Let $D = (V, A)$ be a digraph with $|V| = n$. An injective function $f: V(D) \to [\gamma_n]$ is called **Gaussian out-neighbourhood prime labeling** of D, if for every vertex $v \in V(D)$ with $d^+(v) > 1$, the Gaussian integers in the set $\{f(u): u \in N^+(v)\}$ are relatively prime where $N^+(v) = \{u \in V(D) : \overrightarrow{vu} \in A(D)\}$.

Definition 2.7. Let $D = (V, A)$ be a digraph with $|V| = n$. A function $f: V(D) \to [\gamma_n]$ is said to be a **Gaussian twin neighbourhood prime labeling** of D, if it is both Gaussian in and out neighborhood prime labeling. A digraph which admits Gaussian twin neighborhood prime labeling is called a Gaussian twin neighborhood prime digraph.

Observation 2.8.
1. If D is a digraph such that $N^+(v)$ or $N^-(v)$ are either φ or singleton set, then D admits a Gaussian twin neighborhood prime labeling.
2. A neighborhood prime digraph D in which every vertex is such that either its in-degree or out-degree at most 1 is Gaussian twin neighborhood prime.

The following definitions are referred from [8].

Definition 2.9. Fan graph is defined as the graph $P_n + K_1, n \geq 2$ where K_1 is the empty graph on one vertex and P_n, a path graph on n vertices.

Definition 2.10. A double fan DF_n consists of two fan graphs with a common path. In other words $DF_n = P_n + \overline{K_2}, n \geq 2$.

3. Fan Digraphs

In this section, some new digraphs are introduced by orienting fan graphs in different possible ways and named accordingly. Also we investigate the existence of the Gaussian twin neighborhood prime labeling of those digraphs.

Definition 3.1. In a fan$F_n = P_n + K_1$, orient the edges of the path P_n clockwise or anticlockwise and the spoke edges towards the central vertex. Call the resulting digraph as **in-fan** and denote it by$i\overrightarrow{F_n}$.

Definition 3.2. In a fan$F_n = P_n + K_1$, orient the edges of the path P_n clockwise or anticlockwise and the spoke edges away from the central vertex. Call the resulting digraph as **out-fan** and denote it by$o\overrightarrow{F_n}$.

Definition 3.3. A fan$F_n = P_n + K_1$, is said to be an **alternating fan ($A\overrightarrow{F_n}$)** if the edges of the path P_n are oriented clockwise or anticlockwise and the spoke edges alternately.

Definition 3.4. In a fan$F_n = P_n + K_1$, orient the path edges alternately and the spoke edges towards the central vertex. Call the resulting digraph as **alternating in-fan** and denote it by$Ai\overrightarrow{F_n}$.

Definition 3.5. In a fan$F_n = P_n + K_1$, orient the edges of the path P_n alternately and the spoke edges away from the central vertex. Call the resulting digraph as **alternating out-fan** and denote it by$Ao\overrightarrow{F_n}$.

Definition 3.6. In a fan$F_n = P_n + K_1$, orient the edges of the path P_n alternately and the spoke edges such that either $d^+(v) > 0$ or$d^-(v) > 0$ $\forall\, v \in V(P_n)$. Call the resulting digraph as **sole double alternating fan** and denote it by$SDA\overrightarrow{F_n}$.

Definition 3.7. In a fan$F_n = P_n + K_1$, orient the edges of the path P_n alternately and the spoke edges such that neither $d^+(v) > 0$ nor$d^-(v) > 0$ $\forall\, v \in V(P_n)$. Call the resulting digraph as **di-double alternating fan** and denote it by$DDA\overrightarrow{F_n}$.

Theorem 3.8. In-fan ($i\overrightarrow{F_n}$) admits Gaussian twin neighborhood prime labeling for $n \geq 2$.

Proof: Let $n \geq 2$ and let $v_1, v_2, \ldots, v_n$ be the vertices of the directed path $\overrightarrow{P_n}$ and u be the apex vertex.

Then $A\left(i\overrightarrow{F_n}\right) = \{\overrightarrow{v_i v_{i+1}} | 1 \leq i \leq n-1\} \cup \{\overrightarrow{v_i u} | 1 \leq i \leq n\}$ is the arc set.

This digraph has $n+1$ vertices and $2n-1$ arcs.

Define an injective function $f: V\left(i\overrightarrow{F_n}\right) \to [\gamma_{n+1}]$ by $f(u) = \gamma_1$ and $f(v_i) = \gamma_{i+1}$ for$1 \leq i \leq n$.

Here,$d^-(u) > 1$. Further, the labels of the in-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

$N^-(v_1) = \phi$ And$N^-(v_i) = \{v_{i-1}\}$ for $2 \leq i \leq n$.

Therefore, f is a Gaussian in-neighborhood prime labeling.
Next to prove f is also a Gaussian out-nighborhood prime labeling.

Now $d^+(v_i) > 1$ for $1 \leq i \leq n-1$. Here, the out-neighborhood vertices of $v_i(1 \leq i \leq n-1)$ contains the Gaussian integer $\gamma_1 = 1$ and γ_1 is relatively prime to all the Gaussian integers.

Further, $N^+(v_n) = \{u\}$ and$N^+(u) = \phi$.

Therefore, f is a Gaussian out-neighborhood prime labeling.
Which implies f is a Gaussian twin neighborhood prime labeling.
Hence, in-fan $(i\overrightarrow{F_n})$ admits Gaussian twin neighborhood prime labeling for$n \geq 2$.

Theorem 3.9. Out-fan $(o\overrightarrow{F_n})$ admits Gaussian twin neighborhood prime labeling for$n \geq 2$.

Proof: Let $n \geq 2$ and let $v_1, v_2, \ldots, v_n$ be the vertices of the directed path $\overrightarrow{P_n}$ and u be the apex vertex.

Then $A\left(o\overrightarrow{F_n}\right) = \{\overrightarrow{v_i v_{i+1}} | 1 \leq i \leq n-1\} \cup \{\overrightarrow{uv_i} | 1 \leq i \leq n\}$ is the arc set.

This digraph has $n+1$ vertices and $2n-1$ arcs.

Define an injective function $f: V\left(o\overrightarrow{F_n}\right) \to [\gamma_{n+1}]$ by $f(u) = \gamma_1$ and $f(v_i) = \gamma_{i+1}$ for$1 \leq i \leq n$.

Now $d^-(v_i) > 1$ for $2 \leq i \leq n$. In the above labeling, the in-neighborhood vertices of v_i contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all Gaussian integers.

Further, $N^-(v_1) = \{u\}$ and$N^-(u) = \phi$.

Therefore, f is a Gaussian in-neighborhood prime labeling.
Next to prove f is also a Gaussian out-nighborhood prime labeling.

Now $d^+(u) > 1$ and the labels of the out-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Also, $N^+(v_i) = \{v_{i+1}\}$ for $1 \leq i \leq n-1$ and $N^-(v_n) = \phi$

Therefore, f is a Gaussian out-neighborhood prime labeling.
Which implies f is a Gaussian twin neighborhood prime labeling.
Hence, out-fan $(o\overrightarrow{F_n})$ admits Gaussian twin neighborhood prime labeling for $n \geq 2$.

Theorem 3.10. Alternating fan $(A\overrightarrow{F_n})$ admits Gaussian twin neighborhood prime labeling for $n \geq 2$.

Proof: Let $n \geq 2$ and let $v_1, v_2, \ldots, v_n$ be the vertices of the directed path $\overrightarrow{P_n}$ and u be the apex vertex.

This digraph has $n+1$ vertices and $2n-1$ arcs.

Case (i): n is odd

$A(\overrightarrow{AF_n}) = \left\{\overrightarrow{v_{2i-1}v_{2i}} \middle| 1 \le i \le \frac{n-1}{2}\right\} \cup \left\{\overrightarrow{v_{2i}v_{2i+1}} \middle| 1 \le i \le \frac{n-1}{2}\right\} \cup \left\{\overrightarrow{v_{2i-1}u} \middle| 1 \le i \le \frac{n+1}{2}\right\} \cup \left\{\overrightarrow{uv_{2i}} \middle| 1 \le i \le \frac{n-1}{2}\right\}$ is the corresponding arc set.

Define $f: V(\overrightarrow{AF_n}) \to [\gamma_{n+1}]$ by $f(u) = \gamma_1$; $f(v_{2i-1}) = \gamma_{i+1}$ for $1 \le i \le \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{\frac{(n+1)}{2}+i+1}$ for $1 \le i \le \frac{n-1}{2}$.

Clearly $d^-(u) > 1$ and $d^-(v_{2i}) > 1$ for $1 \le i \le \frac{n-1}{2}$.

Now $N^-(u) = \{v_1, v_3, \dots, v_{2i-1}\}$ for $1 \le i \le \frac{n+1}{2}$ and the labels of the in-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and hence are relatively prime.

Further, $N^-(v_{2i}) = \{u, v_{2i-1}\}$ for $1 \le i \le \frac{n-1}{2}$ and the label set of the in-neighbors of v_{2i} contains $\gamma_1 = 1$.

Also, $N^-(v_1) = \phi$ and $N^-(v_{2i-1}) = \{v_{2i-2}\}$ for $2 \le i \le \frac{n+1}{2}$

Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Now $d^+(u) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \le i \le \frac{n+1}{2}$.

$N^+(u) = \{v_2, v_4, \dots, v_{2i}\}$ for $1 \le i \le \frac{n-1}{2}$ and the out-neighborhood vertices of u are labeled with the consecutive Gaussian integers in the spiral ordering and so by the result 1.4(4), the labels are relatively prime.

Now $N^+(v_{2i-1}) = \{u, v_{2i}\}$ for $1 \le i \le \frac{n-1}{2}$ and $N^+(v_n) = \{u\}$.

Also, the out-neighborhood vertices of $v_{2i-1} (1 \le i \le \frac{n-1}{2})$ contains the Gaussian integer $\gamma_1 = 1$.

Further, $N^+(v_{2i}) = \{v_{2i+1}\}$ for $1 \le i \le \frac{n-1}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

f is a Gaussian twin neighborhood prime labeling when n is odd.

Case (ii): n is even

$A(\overrightarrow{AF_n}) = \left\{\overrightarrow{v_{2i-1}v_{2i}} \middle| 1 \le i \le \frac{n}{2}\right\} \cup \left\{\overrightarrow{v_{2i}v_{2i+1}} \middle| 1 \le i \le \frac{n-2}{2}\right\} \cup \left\{\overrightarrow{v_{2i-1}u} \middle| 1 \le i \le \frac{n}{2}\right\} \cup \left\{\overrightarrow{uv_{2i}} \middle| 1 \le i \le \frac{n}{2}\right\}$ is the corresponding arc set.

Define $f: V(\overrightarrow{AF_n}) \to [\gamma_{n+1}]$ by $f(u) = \gamma_1$; $f(v_{2i-1}) = \gamma_{i+1}$ for $1 \le i \le \frac{n}{2}$ and $f(v_{2i}) = \gamma_{\frac{n}{2}+i+1}$ for $1 \le i \le \frac{n}{2}$.

Clearly $d^-(u) > 1$ and $d^-(v_{2i}) > 1$ for $1 \le i \le \frac{n}{2}$.

Now $N^-(u) = \{v_1, v_3, \dots, v_{2i-1}\}$ for $1 \le i \le \frac{n}{2}$ and the labels of the in-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and so are relatively prime.

Further, $N^-(v_{2i}) = \{u, v_{2i-1}\}$ for $1 \leq i \leq \frac{n}{2}$ and the label set of in-neighbors of v_{2i} contains the Gaussian integer$\gamma_1 = 1$.

Also, $N^-(v_1) = \phi$ and $N^-(v_{2i-1}) = \{v_{2i-2}\}$ for $2 \leq i \leq \frac{n}{2}$

Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.
Next to prove f is also a Gaussian out-nighborhood prime labeling.

Now $d^+(u) > 1$ and $d^+(v_{2i-1}) > 1$ for$1 \leq i \leq \frac{n}{2}$.
$N^+(u) = \{v_2, v_4, \dots, v_{2i}\}$ For$1 \leq i \leq \frac{n}{2}$ and the out-neighborhood vertices of u are labeled with the consecutive Gaussian integers in the spiral ordering and so they are relatively prime.
Further, $N^+(v_{2i-1}) = \{u, v_{2i}\}$ for $1 \leq i \leq \frac{n}{2}$ and the out-neighborhood vertices of $v_{2i-1}(1 \leq i \leq \frac{n}{2})$ contains the Gaussian integer$\gamma_1 = 1$.
Also, $N^+(v_{2i}) = \{v_{2i+1}\}$ for $1 \leq i \leq \frac{n-2}{2}$ and$N^+(v_n) = \phi$.
Therefore, f is a Gaussian out-neighborhood prime labeling when n is even.
f is a Gaussian twin neighborhood prime labeling when n is even.
Cases (i) and (ii) imply f is a Gaussian twin neighborhood prime labeling.
Hence, alternating fan $(A\overrightarrow{F_n})$ admits Gaussian twin neighborhood prime labeling for$n \geq 2$.

Theorem 3.11. Alternating in-fan $(Ai\overrightarrow{F_n})$ is a Gaussian twin neighborhood prime digraph for$n \geq 2$.

Proof. Let $n \geq 2$ and let $v_1, v_2, \dots, v_n$ be the vertices of the directed path $\overrightarrow{P_n}$ and u be the apex vertex.

Case (i): n is odd

$A\left(Ai\overrightarrow{F_n}\right) = \left\{\overrightarrow{v_{2i-1}v_{2i}} \middle| 1 \leq i \leq \frac{n-1}{2}\right\} \cup \left\{\overrightarrow{v_{2i+1}v_{2i}} \middle| 1 \leq i \leq \frac{n-1}{2}\right\} \cup \left\{\overrightarrow{v_{2i-1}u} \middle| 1 \leq i \leq \frac{n+1}{2}\right\} \cup \left\{\overrightarrow{v_{2i}u} \middle| 1 \leq i \leq \frac{n-1}{2}\right\}$ is the corresponding arc set.

Define an injective function $f: V\left(Ai\overrightarrow{F_n}\right) \to [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$f(v_{2i-1}) = \gamma_{i+1}$ for $1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{\frac{(n+1)}{2}+i+1}$ for $1 \leq i \leq \frac{n-1}{2}$.
Here $d^-(u) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$.
Clearly, the label set of in-neighborhood vertices of u contains consecutive Gaussian integers in the spiral ordering and so they are relatively prime.
Now $N^-(v_{2i}) = \{v_{2i-1}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the labels of the in-neighborhood vertices of v_{2i} are consecutive Gaussian integers in the spiral ordering
Further, $N^-(v_{2i-1}) = \phi$ for $1 \leq i \leq \frac{n+1}{2}$
Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.
Next to prove f is also a Gaussian out-neighborhood prime labeling.

Now $d^{+}(v_{2i-1}) > 1$ for $1 \le i \le \frac{n+1}{2}$ and the labels of vertices in $N^{+}(v_{2i-1})$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.
Further, $N^{+}(v_{2i}) = \{u\}$ for $1 \le i \le \frac{n-1}{2}$ and $N^{+}(u) = \phi$.
Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.
By (1) and (2), f is a Gaussian twin neighborhood prime labeling when n is odd.

Case (ii): n is even

$A(Ai\overrightarrow{F_n}) = \left\{\overrightarrow{v_{2i-1}v_{2i}} \middle| 1 \le i \le \frac{n}{2}\right\} \cup \left\{\overrightarrow{v_{2i+1}v_{2i}} \middle| 1 \le i \le \frac{n-2}{2}\right\} \cup \left\{\overrightarrow{v_{2i-1}u} \middle| 1 \le i \le \frac{n}{2}\right\} \cup \left\{\overrightarrow{v_{2i}u} \middle| 1 \le i \le \frac{n}{2}\right\}$ is the arc set.

Define an injective function $f: V(Ai\overrightarrow{F_n}) \to [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$f(v_{2i-1}) = \gamma_{i+1}$ for $1 \le i \le \frac{n}{2}$ and $f(v_{2i}) = \gamma_{\frac{n}{2}+i+1}$ for $1 \le i \le \frac{n}{2}$.

Here $d^{-}(u) > 1$ and $d^{-}(v_{2i}) > 1$ for $1 \le i \le \frac{n-2}{2}$.
Clearly, the label set of in-neighborhood vertices of u contains consecutive Gaussian integers in the spiral ordering and so they are relatively prime.
Now $N^{-}(v_{2i}) = \{v_{2i-1}, v_{2i+1}\}$ for $1 \le i \le \frac{n-2}{2}$ and the label set of in-neighborhood vertices of v_{2i} are consecutive Gaussian integers in the spiral ordering.
Also, $N^{-}(v_n) = \{v_{n-1}\}$ and $N^{-}(v_{2i-1}) = \phi$ for $1 \le i \le \frac{n+1}{2}$.
Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.
Now $d^{+}(v_{2i-1}) > 1$ for $1 \le i \le \frac{n}{2}$ and the out-neighborhood vertices of v_{2i-1} contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.
Further, $N^{+}(v_{2i}) = \{u\}$ for $1 \le i \le \frac{n}{2}$.
Therefore, f is a Gaussian out-neighborhood prime labeling when n is even.
By (3) and (4), f is a Gaussian twin neighborhood prime labeling when n is even.
Cases (i) and (ii) imply f is a Gaussian twin neighborhood prime labeling.
Thus, an alternating in-fan $(Ai\overrightarrow{F_n})$ is a Gaussian twin neighborhood prime digraph for $n \ge 2$.

Theorem 3.12. Alternating out-fan $(Ao\overrightarrow{F_n})$ is a Gaussian twin neighborhood prime digraph for $n \ge 2$.

Proof: Let $n \ge 2$.

Let $v_1, v_2, \dots, v_n$ be the vertices of the directed path $\overrightarrow{P_n}$ and u be the apex vertex.

This digraph has $n + 1$ vertices and $2n - 1$ arcs.

Case (i): n is odd

$A(Ao\overrightarrow{F_n}) = \left\{\overrightarrow{v_{2i-1}v_{2i}} \middle| 1 \le i \le \frac{n-1}{2}\right\} \cup \left\{\overrightarrow{v_{2i+1}v_{2i}} \middle| 1 \le i \le \frac{n-1}{2}\right\} \cup \left\{\overrightarrow{uv_{2i-1}} \middle| 1 \le i \le \frac{n+1}{2}\right\} \cup \left\{\overrightarrow{uv_{2i}} \middle| 1 \le i \le \frac{n-1}{2}\right\}$ is the corresponding arc set.

Define an injective function $f: V(Ao\overrightarrow{F_n}) \to [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$f(v_{2i-1}) = \gamma_{i+1}$ for$1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{\frac{(n+1)}{2}+i+1}$ for $1 \leq i \leq \frac{n-1}{2}$.

Now $d^{-}(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$ and the label set of in-neighbors of v_{2i} are the consecutive Gaussian integers in the spiral ordering.

Further, $N^{-}(v_{2i-1}) = \{u\}$ for $1 \leq i \leq \frac{n+1}{2}$ and$N^{-}(u) = \phi$.

Therefore, f is a Gaussian in- neighborhood prime labeling when n is odd.

Next to prove f is a Gaussian out-neighborhood prime labeling.

Now $d^{+}(u) > 1$ and $d^{+}(v_{2i-1}) > 1$ for$2 \leq i \leq \frac{n-1}{2}$.

Clearly, the label set of out-neighborhood vertices of u contains all the vertices of the path $\overrightarrow{P_n}$ which are labeled with the consecutive Gaussian integers in the spiral ordering.

Further, $N^{+}(v_{2i-1}) = \{v_{2i-2}, v_{2i}\}$ for $2 \leq i \leq \frac{n-1}{2}$ and the labels of the out-neighborhood vertices of v_{2i-1} are consecutive Gaussian integers in the spiral ordering.

Also, $N^{+}(v_1) = \{v_2\}$, $N^{+}(v_n) = \{v_{n-1}\}$ and $N^{+}(v_{2i}) = \phi$ for $1 \leq i \leq \frac{n-1}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

(1) and (2) imply f is a Gaussian twin neighborhood prime labeling if n is odd.

Case (ii): n is even

$A\left(Ao\overrightarrow{F_n}\right) = \left\{\overrightarrow{v_{2i-1}v_{2i}}\middle|1 \leq i \leq \frac{n}{2}\right\} \cup \left\{\overrightarrow{v_{2i+1}v_{2i}}\middle|1 \leq i \leq \frac{n-2}{2}\right\} \cup \left\{\overrightarrow{uv_{2i-1}}\middle|1 \leq i \leq \frac{n}{2}\right\} \cup \left\{\overrightarrow{uv_{2i}}\middle|1 \leq i \leq \frac{n}{2}\right\}$ is the corresponding arc set.

Define an injective function $f: V\left(Ao\overrightarrow{F_n}\right) \to [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$f(v_{2i-1}) = \gamma_{i+1}$ for $1 \leq i \leq \frac{n}{2}$ and $f(v_{2i}) = \gamma_{\frac{n}{2}+i+1}$ for $1 \leq i \leq \frac{n}{2}$.

Now $d^{-}(v_{2i}) > 1$ for $1 \leq i \leq \frac{n}{2}$ and the label set of the in-neighborhood vertices of v_{2i} are consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Further, $N^{-}(u) = \phi$ and $N^{-}(v_{2i-1}) = \{u\}$ for $1 \leq i \leq \frac{n}{2}$.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Here $d^{+}(u) > 1$ and $d^{+}(v_{2i-1}) > 1$ for $2 \leq i \leq \frac{n}{2}$.

Clearly, the labels out-neighborhood vertices of u contains consecutive Gaussian integers in the spiral ordering and so are relatively prime.

Further, $N^{+}(v_{2i-1}) = \{v_{2i-2}, v_{2i}\}$ for $2 \leq i \leq \frac{n}{2}$ and the labels of out-neighborhood vertices of v_{2i-1} are consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Also, $N^{+}(v_1) = \{v_2\}$ and $N^{+}(v_{2i}) = \phi$ for $1 \leq i \leq \frac{n}{2}$.

Therefore, f is a Gaussian out- neighborhood prime labeling when n is even.

(3) and (4) imply f is a Gaussian twin neighborhood prime labeling if n is even.

From the cases (i) and (ii), f is a Gaussian twin neighborhood prime labeling.

Thus, an alternating outfan $(Ao\overrightarrow{F_n})$ is a Gaussian twin neighborhood prime digraph for $n \geq 2$.

Theorem 3.13. Sole double alternating fan ($SDA\overrightarrow{F_n}$) is a Gaussian twin neighborhood prime digraph for $n \geq 2$.

Proof: Let $V\left(SDA\overrightarrow{F_n}\right) = \{u, v_i | 1 \leq i \leq n\}$ be the vertex set where v_i represent the ith vertex of the directed path $\overrightarrow{P_n}$ and u is the apex vertex.

This digraph has $n+1$ vertices and $2n-1$ arcs.

Case (i): n is odd

$A\left(SDA\overrightarrow{F_n}\right) = \left\{\overrightarrow{v_{2\iota-1}v_{2\iota}} \middle| 1 \leq i \leq \frac{n-1}{2}\right\} \cup \left\{\overrightarrow{v_{2\iota+1}v_{2\iota}} \middle| 1 \leq i \leq \frac{n-1}{2}\right\} \cup \left\{\overrightarrow{v_{2\iota-1}u} \middle| 1 \leq i \leq \frac{n+1}{2}\right\} \cup \left\{\overrightarrow{uv_{2\iota}} \middle| 1 \leq i \leq \frac{n-1}{2}\right\}$ is the corresponding arc set.

Define an injective function $f: V\left(SDA\overrightarrow{F_n}\right) \to [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$f(v_{2i-1}) = \gamma_{i+1}$ for$1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{\frac{(n+1)}{2}+i+1}$ for $1 \leq i \leq \frac{n-1}{2}$.

Now $d^-(u) > 1$ and $d^-(v_{2i}) > 1$ for$1 \leq i \leq \frac{n-1}{2}$.

The label set of in-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and so are relatively prime.
Also, the in-neighborhood vertices of v_{2i} contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.
Further, $N^-(v_{2i-1}) = \{u\}$ for$1 \leq i \leq \frac{n+1}{2}$.
Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.
Next to prove f is also a Gaussian out-nighborhood prime labeling.

Now $d^+(u) > 1$ and $d^+(v_{2i-1}) > 1$ for$1 \leq i \leq \frac{n+1}{2}$.

The label set of out-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and so those are relatively prime.
Also, the label set of out-neighborhood vertices of v_{2i-1} contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.
$N^+(v_{2i}) = \phi$ for$1 \leq i \leq \frac{n-1}{2}$.
Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

Case (ii): n is even

$A\left(SDA\overrightarrow{F_n}\right) = \left\{\overrightarrow{v_{2\iota-1}v_{2\iota}} \middle| 1 \leq i \leq \frac{n}{2}\right\} \cup \left\{\overrightarrow{v_{2\iota+1}v_{2\iota}} \middle| 1 \leq i \leq \frac{n-2}{2}\right\} \cup \left\{\overrightarrow{v_{2\iota-1}u} \middle| 1 \leq i \leq \frac{n}{2}\right\} \cup \left\{\overrightarrow{uv_{2\iota}} \middle| 1 \leq i \leq \frac{n}{2}\right\}$ is the arc set.

Define an injective function $f: V\left(SDA\overrightarrow{F_n}\right) \to [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$f(v_{2i-1}) = \gamma_{i+1}$ for $1 \leq i \leq \frac{n}{2}$ and $f(v_{2i}) = \gamma_{\frac{n}{2}+i+1}$ for $1 \leq i \leq \frac{n}{2}$.

Now $d^-(u) > 1$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n}{2}$.

The label set of of in-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and so those are relatively prime.

Further, the label set of in-neighborhood vertices of v_{2i} contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.
Also, $N^-(v_{2i-1}) = \phi$ for $1 \leq i \leq \frac{n}{2}$.
Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.
Next to prove f is also a Gaussian out-nighborhood prime labeling.

Now $d^+(u) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n}{2}$.

The label set of out-neighborhood vertices of u are consecutive Gaussian integers and so they are relatively prime. Then the out-neighborhood vertices of v_{2i-1} contains the Gaussian integer$\gamma_1 = 1$.
Also, $N^+(v_{2i}) = \phi$ for$1 \leq i \leq \frac{n}{2}$.
Therefore, f is a Gaussian out-neighborhood prime labeling when n is even.
Case (i) and (ii) imply f is a Gaussian twin neighborhood prime labeling.
Hence, the Sole-double alternating fan ($SDA\overrightarrow{F_n}$) is a Gaussian twin neighborhood prime digraph for$n \geq 2$.

Theorem 3.14. Di-double alternating fan $DDA\overrightarrow{F_n}$ is a Gaussian twin neighborhood prime digraph for$n \geq 2$.

Proof: Let $n \geq 2$ and let $V\left(DDA\overrightarrow{F_n}\right) = \{u, v_i | 1 \leq i \leq n\}$ be the vertex set where v_i represent the ith vertex of the directed path $\overrightarrow{P_n}$ and u is the apex vertex.

This digraph has $n + 1$ vertices and $2n - 1$ arcs.

Case (i): n is odd

$A\left(DDA\overrightarrow{F_n}\right) = \left\{\overrightarrow{v_{2i-1}v_{2i}} \middle| 1 \leq i \leq \frac{n-1}{2}\right\} \cup \left\{\overrightarrow{v_{2i+1}v_{2i}} \middle| 1 \leq i \leq \frac{n-1}{2}\right\} \cup \left\{\overrightarrow{v_{2i}u} \middle| 1 \leq i \leq \frac{n-1}{2}\right\} \cup \left\{\overrightarrow{uv_{2i-1}} \middle| 1 \leq i \leq \frac{n+1}{2}\right\}$ is the arc set.

Define an injective function $f: V\left(DDA\overrightarrow{F_n}\right) \to [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$f(v_{2i-1}) = \gamma_{i+1}$ for$1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{\frac{(n+1)}{2}+i+1}$ for $1 \leq i \leq \frac{n-1}{2}$.

Now$d^-(u) > 1$. The label set of in-neighborhood vertices of u are consecutive Gaussian integers in the in the spiral ordering and so they are relatively prime.

Also, $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$ and the in-neighborhood vertices of v_{2i} are labeled with the consecutive Gaussian integers.

Further, $N^-(v_{2i-1}) = \{u\}$ for $1 \leq i \leq \frac{n+1}{2}$.
Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.
Now $d^+(u) > 1$ and the out-neighborhood vertices of u are consecutive Gaussian integers in the labeling and so they are relatively prime.

Also, $d^+(v_{2i-1}) > 1$ for $2 \leq i \leq \frac{n-1}{2}$ and the label set of the out-neighborhood vertices of v_{2i-1} contains the consecutive Gaussian integers in the spiral ordering.

Further, $N^+(v_1) = \{v_2\}$ and $N^+(v_n) = \{v_{n-1}\}$.

$N^+(v_{2i}) = \{u\}$ for $1 \leq i \leq \frac{n-1}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

Case (ii): n is even

$A(DDA\overrightarrow{F_n}) = \left\{\overrightarrow{v_{2i-1}v_{2i}} \middle| 1 \leq i \leq \frac{n}{2}\right\} \cup \left\{\overrightarrow{v_{2i+1}v_{2i}} \middle| 1 \leq i \leq \frac{n-2}{2}\right\} \cup \left\{\overrightarrow{v_{2i}u} \middle| 1 \leq i \leq \frac{n}{2}\right\} \cup \left\{\overrightarrow{uv_{2i-1}} \middle| 1 \leq i \leq \frac{n}{2}\right\}$ is the arc set.

Define an injective function $f: V(DDA\overrightarrow{F_n}) \to [\gamma_{n+1}]$ by $f(u) = \gamma_1$;

$f(v_{2i-1}) = \gamma_{i+1}$ for $1 \leq i \leq \frac{n}{2}$ and $f(v_{2i}) = \gamma_{\frac{n}{2}+i+1}$ for $1 \leq i \leq \frac{n}{2}$.

Now $d^-(u) > 1$ and labeling of the in-neighborhood vertices of u are consecutive Gaussian integers in the spiral ordering and they are relatively prime.

Also, $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-2}{2}$ and the in-neighborhood vertices of v_{2i} are labeled with the consecutive Gaussian integers and so are relatively prime.

Further, $N^-(v_n) = \{v_{n-1}\}$ and $N^-(v_{2i-1}) = \{u\}$ for $1 \leq i \leq \frac{n}{2}$.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.

Next to prove f is also a Gaussian out-nighborhood prime labeling.

Now $d^+(u) > 1$ and the labeling of the out-neighborhood vertices of u are consecutive Gaussian integers.

Also, $d^+(v_{2i-1}) > 1$ for $2 \leq i \leq \frac{n}{2}$ and the label set of the out-neighborhood vertices of $v_{2i-1} (2 \leq i \leq \frac{n}{2})$ are consecutive Gaussian integers in the spiral ordering and so are relatively prime.

Also, $N^+(v_1) = \{v_2\}$ and $N^+(v_{2i}) = \{u\}$ for $1 \leq i \leq \frac{n}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is even.

Cases (i) and (ii) imply f is a Gaussian twin neighborhood prime labeling.

Thus the di-double alternating fan ($DDA\overrightarrow{F_n}$) is a Gaussian twin neighborhood prime digraph for$n \geq 2$.

4. Double fan digraphs

In this section, some new digraphs are introduced by orienting double fan graphs in different possible ways and named accordingly. Also, the Gaussian twin neighborhood prime labeling is proved for those digraphs.

Definition 4.1. In a double fan$DF_n = P_n + \overline{K_2}$, orient the edges of the common path P_n clockwise or anticlockwise and the spoke edges are towards the central vertex. Call the resulting digraph as **double in-fan** and denote it by$Di\overrightarrow{F_n}$.

Definition 4.2. In a double fan$DF_n = P_n + \overline{K_2}$, orient the edges of the common path P_n clockwise or anticlockwise and the spoke edges away from the central vertex. Call the resulting digraph as **double out-fan** and denote it by$Do\overrightarrow{F_n}$.

Definition 4.3. A double fan$DF_n = P_n + \overline{K_2}$, is said to be a **double alternating fan** ($DA\overrightarrow{F_n}$) if the edges of the common path P_n are oriented in clockwise or anticlockwise and the spoke edges alternately.

Definition 4.4. In a double fanDF_n, orient the edges of the common path P_n alternately and the spoke edges are towards the central vertices. Call the resulting digraph as **double alternating in-fan** and denote it by$DAo\overrightarrow{F_n}$.

Definition 4.5. In a double fanDF_n, orient the edges of the common path P_n alternately and the spoke edges are away the central vertices. Call the resulting digraph is called a **double alternating out-fan** and denote it by$DAi\overrightarrow{F_n}$.

Definition 4.6. In a double fan DF_n, orient the edges of the common path P_n alternately and the spoke edges such that either $d^+(v) = 0$ or $d^-(v) = 0\ \forall\ v \in V(P_n)$. The resulting digraph is called a **double sole-double alternating fan** and denoted it as $DSDA\overrightarrow{F_n}$.

Definition 4.7. In a double fan DF_n, orient the edges of the common path P_n alternately and the spoke edges such that neither $d^+(v) = 0$ nor $d^-(v) = 0\ \forall\ v \in V(P_n)$. The resulting digraph is called a **double di-double alternating fan** and denote it by $DDDA\overrightarrow{F_n}$.

Theorem 4.8. Double in-fan ($Di\overrightarrow{F_n}$)is a Gaussian twin neighborhood prime digraph.

Proof: Let $V\left(Di\overrightarrow{F_n}\right) = \{u, w, v_i |\ 1 \leq i \leq n\}$ where u and w are the apex vertices and v_i represent the ith vertex of the directed path $\overrightarrow{P_n}$.

Then $A\left(Di\overrightarrow{F_n}\right) = \{\overrightarrow{v_i v_{i+1}}|\ 1 \leq i \leq n-1\} \cup \{\overrightarrow{v_i u}| 1 \leq i \leq n\} \cup \{\overrightarrow{v_i w}|\ 1 \leq i \leq n\}$ is the arc set.

This digraph graph has $n + 2$ vertices and $3n - 1$ edges.

Define an injective function $f: V\left(Di\overrightarrow{F_n}\right) \to [\gamma_{n+2}]$ by$(u) = \gamma_1$, $f(w) = \gamma_2$ and $f(v_i) = \gamma_{i+2}$ for $1 \leq i \leq n$.

Here $d^-(u) > 1$, $d^-(w) > 1$.

By the definition of f, the in-neighborhood vertices of u and w are labeled by consecutive Gaussian integers γ_1 and γ_2 in the spiral ordering and so are relatively prime.

Now, $N^-(v_1) = \phi$ and $N^-(v_i) = \{v_{i-1}\}$ for $2 \leq i \leq n$.

Therefore, f is a Gaussian in-neighborhood prime labeling.

Next to prove f is also Gaussian out-neighborhood prime labeling.
Now, $d^+(v_i) > 1$ for $1 \leq i \leq n$.
Further, $N^+(v_i) = \{u, v_{i+1}, w\}$ for $1 \leq i \leq n-1$ and the labels of vertices in $N^+(v_i)$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.
Also, $N^+(v_n) = \{u, w\}$ and labels of u and w are consecutive Gaussian integers.
Further, $N^+(u) = N^+(w) = \phi$.
Therefore, f is a Gaussian out- neighborhood prime labeling.
f is a Gaussian twin neighborhood prime labeling.
Thus, double fan($Di\overrightarrow{F_n}$) is a Gaussian twin neighborhood prime digraph.

Theorem 4.9. Double out-fan($Do\overrightarrow{F_n}$) admits Gaussian twin neighborhood prime labeling.
Proof: Let $V\left(Do\overrightarrow{F_n}\right) = \{u, w, v_i |\ 1 \leq i \leq n\}$ where u and w are the apex vertices and v_i represent the ith vertex of the directed path $\overrightarrow{P_n}$.
Then $A\left(Do\overrightarrow{F_n}\right) = \{\overrightarrow{v_\imath v_{\imath+1}} |\ 1 \leq i \leq n-1\} \cup \{\overrightarrow{uv_\imath} | 1 \leq i \leq n\} \cup \{\overrightarrow{wv_\imath} |\ 1 \leq i \leq n\}$ is the arc set.
This digraph has $n+2$ vertices and $3n-1$ edges.
Define an injective function $f: V\left(Do\overrightarrow{F_n}\right) \to [\gamma_{n+2}]$ by
$f(u) = \gamma_1$, $f(w) = \gamma_2$ and $f(v_i) = \gamma_{i+2}$ for $1 \leq i \leq n$.
Now, $d^-(v_i) > 1$ for $1 \leq i \leq n$.
$N^-(v_1) = \{u, w\}$ and $N^-(v_i) = \{u, v_{i-1}, w\}$ for $2 \leq i \leq n$.
Clearly, label set of vertices in $N^-(v_i)$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.
Also, $N^-(u) = N^-(w) = \phi$.
Therefore, f is a Gaussian in-neighborhood prime labeling.
Next to prove f is also Gaussian out-neighborhood prime labeling.
Here $d^+(u) > 1$, $d^+(w) > 1$.
By the definition of f, the set of out-neighborhood vertices of u and w are labeled by the consecutive Gaussian integers in the spiral ordering and which are relatively prime.
Further, $N^+(v_i) = \{v_{i+1}\}$ for $1 \leq i \leq n-1$ and $N^+(v_n) = \phi$.
Therefore, f is a Gaussian out-neighborhood prime labeling.
f is a Gaussian twin neighborhood prime labeling.
Hence, double out-fan ($Do\overrightarrow{F_n}$) admits Gaussian twin neighborhood prime labeling.

Theorem 4.10. Double alternating fan ($DA\overrightarrow{F_n}$) is a Gaussian twin neighborhood prime digraph.
Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of the path $\overrightarrow{P_n}$ and u, w be the apex vertices.
Let $V\left(DA\overrightarrow{F_n}\right) = \{u, w, v_i | 1 \leq i \leq n\}$ be the vertex set.
This digraph has $n+2$ vertices and $3n-1$ arcs.
Case (i): n is even
$A\left(DA\overrightarrow{F_n}\right) = \{\overrightarrow{v_{2\imath-1}v_{2\imath}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2\imath}v_{2\imath+1}} | 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{v_{2\imath-1}u} | 1 \leq i \leq \frac{n}{2}\} \cup$
$\{\overrightarrow{v_{2\imath-1}w} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{uv_{2\imath}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{wv_{2\imath}} | 1 \leq i \leq \frac{n}{2}\}$ is the arc set.

Define an injective function $f: V(\overrightarrow{DAF_n}) \to [\gamma_{n+2}]$ by $f(u) = \gamma_1; f(w) = \gamma_2$ and $f(v_{2i-1}) = \gamma_{i+2}$ and $f(v_{2i}) = \gamma_{\frac{n}{2}+i+2}$ for $1 \le i \le \frac{n}{2}$.
Here, $d^-(u) > 1$, $d^-(w) > 1$ and $d^-(v_{2i}) > 1$ for $1 \le i \le \frac{n}{2}$.
Clearly, the label set of in-neighborhood vertices of u and w are consecutive Gaussian integers in the spiral ordering.
Further, $N^-(v_{2i}) = \{v_{2i-1}, u, w\}$ for $1 \le i \le \frac{n}{2}$. By the definition of f, the vertex u is labeled as $\gamma_1 = 1$, which is relatively prime to all the Gaussian integers.
Also, $N^-(v_1) = \phi$ and $N^-(v_{2i-1}) = \{v_{2i-2}\}$ for $2 \le i \le \frac{n}{2}$.
Therefore, f is Gaussian in-neighborhood prime labeling when n is even.
Next to prove f is also Gaussian out-neighborhood prime labeling.
Now $d^+(u) > 1$, $d^+(w) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \le i \le \frac{n}{2}$.
Clearly, the label set of out-neighborhood vertices of u and w are labeled by consecutive Gaussian integers in the spiral ordering.
Further, $N^+(v_{2i-1}) = \{v_{2i}, u, w\}$ for $1 \le i \le \frac{n}{2}$ and by the definition of f, the label of u is $\gamma_1 = 1$, which is relatively prime to all the Gaussian integers.
Also, $N^+(v_{2i}) = \{v_{2i+1}\}$ for $1 \le i \le \frac{n-2}{2}$ and $N^+(v_n) = \phi$.
Therefore, f is a Gaussian out neighborhood prime labeling when n is even.
f is a Gaussian twin neighborhood prime labeling when n is even.
Case (ii): n is odd.
Then, $A(\overrightarrow{DAF_n}) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i}v_{2i+1}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i-1}u} | 1 \le i \le \frac{n+1}{2}\} \cup \{\overrightarrow{v_{2i-1}w} | 1 \le i \le \frac{n+1}{2}\} \cup \{\overrightarrow{uv_{2i}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{wv_{2i}} | 1 \le i \le \frac{n-1}{2}\}$ is the arc set.
Define an injective function $f: V(\overrightarrow{DAF_n}) \to [\gamma_{n+2}]$ by $f(u) = \gamma_1; f(w) = \gamma_2$ and $f(v_{2i-1}) = \gamma_{i+2}$ and $f(v_{2i}) = \gamma_{\left(\frac{n+1}{2}\right)+i+2}$ for $1 \le i \le \frac{n-1}{2}$.
Here, $d^-(u) > 1$, $d^-(w) > 1$ and $d^-(v_{2i}) > 1$ for $1 \le i \le \frac{n-1}{2}$.
Clearly, the label set of in-neighborhood vertices of u and w are consecutive Gaussian integers in the spiral ordering.
Further, $N^-(v_{2i}) = \{v_{2i-1}, u, w\}$ for $1 \le i \le \frac{n-1}{2}$ and by the definition of f, the vertex u has the label $\gamma_1 = 1$, which is relatively prime to all the Gaussian integers.
Also, $N^-(v_1) = \phi$ and $N^-(v_{2i-1}) = \{v_{2i-2}\}$ for $2 \le i \le \frac{n+1}{2}$.
Therefore, f is a Gaussian in- neighborhood prime labeling when n is odd.
Next to prove f is also a Gaussian out-nighborhood prime labeling.

Now $d^+(u) > 1$, $d^+(w) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \le i \le \frac{n+1}{2}$.
By the definition of f, the label set of out-neighborhood vertices of u and w consecutive Gaussian integers in the spiral ordering.
Also, $N^+(v_{2i-1}) = \{v_{2i}, u, w\}$ for $1 \le i \le \frac{n-1}{2}$ and the label of u is $\gamma_1 = 1$ is relatively prime to the Gaussian integers.

$N^+(v_{2i}) = \{v_{2i+1}\}$ for $1 \le i \le \frac{n-1}{2}$.
Further,$N^+(v_n) = \{u, w\}$. The vertices u and v are labeled by γ_1 and γ_2 respectively. Since γ_1 and γ_2 are consecutive Gaussian integers in the spiral ordering and so they are relatively prime.
Therefore, f is a Gaussian out- neighborhood prime labeling when n is odd.
f is a Gaussian twin neighborhood prime labeling.
From both the cases, f is a Gaussian twin neighborhood prime labeling.
Hence, double alternating fan $(DA\overrightarrow{F_n})$ is a Gaussian twin neighborhood prime digraph.

Theorem 4.11. Double alternating in-fan $(DAi\overrightarrow{F_n})$ admits Gaussian twin neighborhood prime labeling.
Proof. Let $V(DAi\overrightarrow{F_n}) = \{u, w, v_i | 1 \le i \le n\}$ where $u, w$the apex vertices are and v_i represent the ith vertex of the common path$\overrightarrow{P_n}$.
This digraph has $n + 2$ vertices and $3n - 1$ arcs.
Case (i): n is odd.
$A(DAi\overrightarrow{F_n}) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i-1}u} | 1 \le i \le \frac{n+1}{2}\} \cup \{\overrightarrow{v_{2i-1}w} | 1 \le i \le \frac{n+1}{2}\} \cup \{\overrightarrow{v_{2i}u} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i}w} | 1 \le i \le \frac{n-1}{2}\}$ is the corresponding arc set.
Define an injective function $f: V(DAi\overrightarrow{F_n}) \to [\gamma_{n+2}]$ by$(u) = \gamma_1$, $f(w) = \gamma_2$,
$f(v_{2i-1}) = \gamma_{i+2}$ for $1 \le i \le \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{\left(\frac{n+1}{2}\right)+i+2}$ for $1 \le i \le \frac{n-1}{2}$.
Here, $d^-(u) > 1$, $d^-(w) > 1$ and $d^-(v_{2i}) > 1$ for $1 \le i \le \frac{n-1}{2}$.
$N^-(u) = N^-(w) = \{v_1, v_2, \dots, v_n\}$. The vertices $v_1, v_2, \dots, v_n$ are labeled with the consecutive Gaussian integers in the spiral ordering and so they are relatively prime.
Further, $N^-(v_{2i}) = \{v_{2i-1}, v_{2i+1}\}$ for $1 \le i \le \frac{n-1}{2}$ and the labels of $N^-(v_{2i})$ are consecutive Gaussian integers in the spiral ordering.
Also, $N^-(v_{2i-1}) = \phi$ for$1 \le i \le \frac{n+1}{2}$.
Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.
Next to prove f is also Gaussian out-neighborhood prime labeling.
Here $d^+(v_{2i-1}) > 1$ for $1 \le i \le \frac{n-1}{2}$ and $d^+(v_{2i}) > 1$ for $1 \le i \le \frac{n-1}{2}$
Further, $N^+(v_1) = \{v_2, u, w\}$ and $N^+(v_{2i-1}) = \{u, w, v_{2i-2}, v_{2i}\}$ for$2 \le i \le \frac{n-1}{2}$.
and the label set of out-neighborhood vertices $v_{2i-1}(1 \le i \le \frac{n-1}{2})$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.
Also, $N^+(v_{2i}) = \{u, w\}$ for $1 \le i \le \frac{n-1}{2}$ and the vertices u and w are labeled by the Gaussian integers $\gamma_1 = 1$ and $\gamma_2 = 1 + i$ respectively.
Now, $N^+(u) = N^+(w) = \phi$.
Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.
f is a Gaussian twin neighborhood prime labeling when n is odd
Case (ii): n is even

$A\left(DAi\overrightarrow{F_n}\right) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \le i \le \frac{n}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \le i \le \frac{n-2}{2}\} \cup \{\overrightarrow{v_{2i-1}u} | 1 \le i \le \frac{n}{2}\} \cup \{\overrightarrow{v_{2i-1}w} | 1 \le i \le \frac{n}{2}\} \cup \{\overrightarrow{v_{2i}u} | 1 \le i \le \frac{n}{2}\} \cup \{\overrightarrow{v_{2i}w} | 1 \le i \le \frac{n}{2}\}$ is the arc set.

Define an injective function $f: V\left(DAi\overrightarrow{F_n}\right) \to [\gamma_{n+2}]$ by $f(u) = \gamma_1; f(w) = \gamma_2$ and $f(v_{2i-1}) = \gamma_{i+2}$ and $f(v_{2i}) = \gamma_{\frac{n}{2}+i+2}$ for $1 \le i \le \frac{n}{2}$.

Here, $d^-(u) > 1$, $d^-(w) > 1$ and $d^-(v_{2i}) > 1$ for $1 \le i \le \frac{n-2}{2}$.

By the definition of f, the label set of the in-neighborhood vertices of u and w are consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Further, $N^-(v_{2i}) = \{v_{2i-1}, v_{2i+1}\}$ for $1 \le i \le \frac{n-2}{2}$ and the vertices in $N^-(v_{2i})$ are labeled by consecutive Gaussian integers in the spiral ordering.

Also, $N^-(v_n) = \{v_{n-1}\}$ and $N^-(v_{2i-1}) = \phi$ for $1 \le i \le \frac{n}{2}$.

Therefore, f admits a Gaussian in-neighborhood prime labeling when n is even.

Now $d^+(v_{2i-1}) > 1$ and $d^+(v_{2i}) > 1$ for $1 \le i \le \frac{n}{2}$.

Here, $N^+(u) = N^+(w) = \phi$.

Also, $N^+(v_1) = \{v_2, u, w\}$ and $N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}, u, w\}$ for $2 \le i \le \frac{n}{2}$ and the labels of $N^+(v_{2i-1})(1 \le i \le \frac{n}{2})$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

$N^+(v_{2i}) = \{u, w\}$ for $1 \le i \le \frac{n}{2}$ and the vertices u and w are labelled by the consecutive Gaussian integers $\gamma_1 = 1$ and $\gamma_2 = 1 + i$ respectively. So the labels of vertices in $N^+(v_{2i})$ are relatively prime.

Therefore, f is a Gaussian out- neighborhood prime labeling when n is even.

f is a Gaussian twin neighborhood prime labeling.

Hence, double alternating in-fan ($DAi\overrightarrow{F_n}$) admits Gaussian twin neighborhood prime labeling.

Theorem 4.12. Double alternating out-fan ($DAo\overrightarrow{F_n}$) is a Gaussian twin neighborhood prime digraph.

Proof. Let $V\left(DAo\overrightarrow{F_n}\right) = \{u, w, v_i | 1 \le i \le n\}$ where u, w are the apex vertices and v_i represent the ith vertex of the common path $\overrightarrow{P_n}$.

This digraph has $n + 2$ vertices and $3n - 1$ arcs.

Case (i): n is odd.

$A\left(DAo\overrightarrow{F_n}\right) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{uv_{2i-1}} | 1 \le i \le \frac{n+1}{2}\} \cup \{\overrightarrow{uv_{2i}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{wv_{2i-1}} | 1 \le i \le \frac{n+1}{2}\} \cup \{\overrightarrow{wv_{2i}} | 1 \le i \le \frac{n-1}{2}\}$ is the arc set.

Define an injective function $f: V\left(DAo\overrightarrow{F_n}\right) \to [\gamma_{n+2}]$ by $(u) = \gamma_1, f(w) = \gamma_2$, $f(v_{2i-1}) = \gamma_{i+2}$ for $1 \le i \le \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{\left(\frac{n+1}{2}\right)+i+2}$ for $1 \le i \le \frac{n-1}{2}$.

Here, $d^-(v_{2i-1}) > 1$ for $1 \le i \le \frac{n+1}{2}$ and $d^-(v_{2i}) > 1$ for $1 \le i \le \frac{n-1}{2}$.

The vertices u and w has no in-neighbors.

That is, $N^-(u) = N^-(w) = \phi$.

Now $N^{-}(v_{2i-1}) = \{u, w\}$ for $1 \le i \le \frac{n+1}{2}$ and the vertices u and w are labelled by the consecutive Gaussian integers $\gamma_1 = 1$ and $\gamma_2 = 1 + i$ respectively. So the labels of vertices in $N^{-}(v_{2i})$ are relatively prime.

Further, $N^{-}(v_{2i}) = \{u, w, v_{2i-1}, v_{2i+1}\}$ for $1 \le i \le \frac{n-1}{2}$ and the label set of in-neighborhood vertices v_{2i} contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.

Next to prove f is also Gaussian out-neighborhood prime labeling.

Now, $d^{+}(u) > 1$, $d^{+}(w) > 1$ and $d^{+}(v_{2i-1}) > 1$ for $2 \le i \le \frac{n-1}{2}$.

The out-neighborhood vertices of u and w are labeled by consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Also, $N^{+}(v_{2i-1}) = \{v_{2i-2}, v_{2i}\}$ for $2 \le i \le \frac{n-1}{2}$ and the vertices v_{2i-2} and v_{2i} are labeled by $\gamma_{\left(\frac{n+1}{2}\right)+i+1}$ and $\gamma_{\left(\frac{n+1}{2}\right)+i+2}$ which are consecutive Gaussian integers in the spiral ordering.

$N^{+}(v_1) = \{v_2\}$ and $N^{+}(v_n) = \{v_{n-1}\}$.

Then, $N^{+}(v_{2i}) = \phi$ for $1 \le i \le \frac{n-1}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

f is a Gaussian twin neighborhood prime labeling when n is odd

Case (ii): n is even

$A\left(DAo\overrightarrow{F_n}\right) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \le i \le \frac{n}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \le i \le \frac{n-2}{2}\} \cup \{\overrightarrow{uv_{2i-1}} | 1 \le i \le \frac{n}{2}\} \cup \{\overrightarrow{uv_{2i}} | 1 \le i \le \frac{n}{2}\} \cup \{\overrightarrow{wv_{2i-1}} | 1 \le i \le \frac{n}{2}\} \cup \{\overrightarrow{wv_{2i}} | 1 \le i \le \frac{n}{2}\}$ is the corresponding arc set.

Define an injective function $f: V\left(DAo\overrightarrow{F_n}\right) \to [\gamma_{n+2}]$ by $(u) = \gamma_1, f(w) = \gamma_2$,
$f(v_{2i-1}) = \gamma_{i+2}$ for $1 \le i \le \frac{n}{2}$ and $f(v_{2i}) = \gamma_{\left(\frac{n}{2}\right)+i+2}$ for $1 \le i \le \frac{n}{2}$.

Here, $d^{-}(v_{2i-1}) > 1$ for $1 \le i \le \frac{n}{2}$ and $d^{-}(v_{2i}) > 1$ for $1 \le i \le \frac{n}{2}$.

Further, $N^{-}(u) = N^{-}(w) = \phi$.

Also, $N^{-}(v_{2i-1}) = \{u, w\}$ for $1 \le i \le \frac{n}{2}$ and the vertices u and w are labeled with the consecutive Gaussian integers $\gamma_1 = 1$ and $\gamma_2 = 1 + i$ respectively.

Here, $N^{-}(v_{2i}) = \{u, w, v_{2i-1}, v_{2i+1}\}$ for $1 \le i \le \frac{n-2}{2}$ and $N^{-}(v_n) = \{u, w, v_{n-1}\}$.

Further, the label of the in-neighborhood vertices of $v_{2i}(1 \le i \le \frac{n}{2})$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.

Next to prove f is also a Gaussian out-neighborhood prime labeling.

Now, $d^{+}(u) > 1$, $d^{+}(w) > 1$ and $d^{+}(v_{2i-1}) > 1$ for $2 \le i \le \frac{n}{2}$.

Clearly, the label of the out-neighborhood vertices of u and w are labeled by consecutive Gaussian integers in the spiral ordering and so are relatively prime.

Also, $N^{+}(v_1) = \{v_2\}$.

$N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}\}$ For$2 \leq i \leq \frac{n}{2}$. The vertices v_{2i-2} and v_{2i} are labeled by $\gamma_{\frac{n}{2}+i+1}$ and $\gamma_{\frac{n}{2}+i+2}$ which are consecutive Gaussian integers in the labeling in spiral ordering.

Also, $N^+(v_{2i}) = \phi$ for$1 \leq i \leq \frac{n}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is even.

Therefore, f is a Gaussian twin neighborhood prime labeling when n is even.

Thus, double alternating out-fan $(DAo\overrightarrow{F_n})$ is a Gaussian twin neighborhood prime digraph.

Theorem 4.13. Double sole double alternating fan $(DSDA\overrightarrow{F_n})$ admits Gaussian twin neighborhood prime labeling.

Proof: Let $V\left(DSDA\overrightarrow{F_n}\right) = \{u, w, v_i | 1 \leq i \leq n\}$ be the vertex set where v_i represent the ith vertex of the common path $\overrightarrow{P_n}$ and u, w be the apex vertices.

This digraph has $n+2$ vertices and $3n-1$ arcs.

Case (i) n is odd.

$A\left(DSDA\overrightarrow{F_n}\right) = \{\overrightarrow{v_{2\iota-1}v_{2\iota}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2\iota+1}v_{2\iota}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2\iota-1}u} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{uv_{2\iota}} | 1 \leq i \leq \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2\iota-1}w} | 1 \leq i \leq \frac{n+1}{2}\} \cup \{\overrightarrow{wv_{2\iota}} | 1 \leq i \leq \frac{n-1}{2}\}$ is the arc set.

Define an injective function $f: V\left(DSDA\overrightarrow{F_n}\right) \to [\gamma_{n+2}]$ by $(u) = \gamma_1, f(w) = \gamma_2$,

$f(v_{2i-1}) = \gamma_{i+2}$ For $1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{\left(\frac{n+1}{2}\right)+i+2}$ for$1 \leq i \leq \frac{n-1}{2}$.

Here, $d^-(u) > 1, d^-(w) > 1$ and $d^-(v_{2i}) > 1$ for$1 \leq i \leq \frac{n+1}{2}$.

Further, $N^-(u) = N^-(w) = \{v_1, v_3, \ldots, v_{2i-1}\}$ for $1 \leq i \leq \frac{n+1}{2}$ and the labels of vertices in $N^-(u)$ and $N^-(w)$ are consecutive Gaussian integers in the spiral ordering and hence are relatively prime.

Also $N^-(v_{2i}) = \{u, w, v_{2i-1}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the label of the vertex u is $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Here, $N^-(v_{2i-1}) = \phi$ for$1 \leq i \leq \frac{n+1}{2}$.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd.

Next to prove f is also Gaussian out-neighborhood prime labeling.

Here, $d^+(u) > 1, d^+(w) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n+1}{2}$.

Now $N^+(u) = N^+(w) = \{v_2, v_4, \ldots, v_{2i}\}$ for$1 \leq i \leq \frac{n-1}{2}$.

Clearly, the label set of out-neighborhood vertices of u and w are labeled by the consecutive Gaussian integers in the spiral ordering and so they are relatively prime.

Also, $N^+(v_1) = \{u, w, v_2\}$, $N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}, u, w\}$ for $2 \leq i \leq \frac{n-3}{2}$ and

$N^+(v_n) = \{u, w, v_{n-1}\}$.

Further, the label of vertices in $N^+(v_{2i-1})$ for $1 \leq i \leq \frac{n-1}{2}$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Also, $N^+(v_{2i}) = \phi$ for $1 \leq i \leq \frac{n-1}{2}$.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.
Therefore, f is a Gaussian twin neighborhood prime labeling when n is odd.
Case (ii): n is even
$A(DSDA\overrightarrow{F_n}) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \le i \le \frac{n}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \le i \le \frac{n-2}{2}\} \cup \{\overrightarrow{v_{2i-1}u} | 1 \le i \le \frac{n}{2}\} \cup \{\overrightarrow{uv_{2i}} | 1 \le i \le \frac{n}{2}\} \cup \{\overrightarrow{v_{2i-1}w} | 1 \le i \le \frac{n}{2}\} \cup \{\overrightarrow{wv_{2i}} | 1 \le i \le \frac{n}{2}\}$ is the arc set.
Define an injective function $f: V(DSDA\overrightarrow{F_n}) \to [\gamma_{n+2}]$ by $f(u) = \gamma_1, f(w) = \gamma_2$,
$f(v_{2i-1}) = \gamma_{i+2}$ for $1 \le i \le \frac{n}{2}$ and $f(v_{2i}) = \gamma_{\left(\frac{n}{2}\right)+i+2}$ for $1 \le i \le \frac{n}{2}$.
Here, $d^-(u) > 1, d^-(w) > 1$ and $d^-(v_{2i}) > 1$ for $1 \le i \le \frac{n}{2}$.
Now $N^-(u) = N^-(w) = \{v_1, v_3, \dots, v_{2i-1}\}$ for $1 \le i \le \frac{n}{2}$ and the label set of vertices in $N^-(u)$ and $N^-(w)$ are consecutive Gaussian integers in the spiral ordering.
Further, $N^-(v_{2i}) = \{u, w, v_{2i-1}, v_{2i+1}\}$ for $1 \le i \le \frac{n-2}{2}$ and $N^-(v_n) = \{u, w, v_{n-1}\}$.
By the definition of f, the label of the vertex u is γ_1 which is relatively prime to all the Gaussian integers.
$N^-(v_{2i-1}) = \phi$ for $1 \le i \le \frac{n}{2}$.
Therefore, f is a Gaussian in-neighborhood prime labeling when n is even.
Next to prove f is also a Gaussian out-nighborhood prime labeling.
Here $d^+(u) > 1$, $d^+(w) > 1$ and $d^+(v_{2i-1}) > 1$ for $1 \le i \le \frac{n}{2}$.
Now $N^+(u) = N^+(w) = \{v_2, v_4, \dots, v_{2i}\}$ for $1 \le i \le \frac{n}{2}$ and the out-neighborhood vertices of u and w are labeled by the consecutive Gaussian integers$\gamma_{\left(\frac{n}{2}\right)+3}$,$\gamma_{\left(\frac{n}{2}\right)+4}$, …, $\gamma_{\left(\frac{n}{2}\right)+i+2}$ and so they are relatively prime.
Further, $N^+(v_1) = \{u, w, v_2\}$ and $N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}, u, w\}$ for $2 \le i \le \frac{n}{2}$.
and the labels of vertices in $N^+(v_1)$ and $N^+(v_{2i-1})$ for $2 \le i \le \frac{n}{2}$ contains the Gaussian integer $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.
Also, $N^+(v_{2i}) = \phi$ for $1 \le i \le \frac{n}{2}$.
Therefore, f is a Gaussian out- neighborhood prime labeling when n even.
From both the cases, f is a Gaussian twin neighborhood prime labeling.
Hence double sole double alternating fan ($DSDA\overrightarrow{F_n}$) admits a Gaussian twin neighborhood prime labeling.

Theorem 4.14. Double di-double alternating fan $DDDA\overrightarrow{F_n}$ is a Gaussian twin neighborhood prime digraph.
Proof: Let $V(DDDA\overrightarrow{F_n}) = \{u, w, v_i | 1 \le i \le n\}$ be the vertex set where v_i represent the ith vertex of the common path $\overrightarrow{P_n}$ and u, w be the apex vertices.
This digraph has $n + 2$ vertices and $3n - 1$ arcs.
Case (i): n is odd
$A(DDDA\overrightarrow{F_n}) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{uv_{2i-1}} | 1 \le i \le \frac{n+1}{2}\} \cup \{\overrightarrow{v_{2i}u} | 1 \le i \le \frac{n-1}{2}\} \cup \{\overrightarrow{wv_{2i-1}} | 1 \le i \le \frac{n+1}{2}\} \cup \{\overrightarrow{v_{2i}w} | 1 \le i \le \frac{n-1}{2}\}$ is the arc set.

Define an injective function $f: V\left(DDDA\overrightarrow{F_n}\right) \rightarrow [\gamma_{n+2}]$ by $f(u) = \gamma_1; f(w) = \gamma_2$ and $f(v_{2i-1}) = \gamma_{i+2}$ for $1 \leq i \leq \frac{n+1}{2}$ and $f(v_{2i}) = \gamma_{\left(\frac{n+1}{2}\right)+i+2}$ for $1 \leq i \leq \frac{n-1}{2}$.

Here, $d^-(u) > 1, d^-(w) > 1$, $d^-(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n+1}{2}$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$.

Now $N^-(u) = N^-(w) = \{v_2, v_4, \dots, v_{2i}\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the label set of in-neighborhood vertices of u and w are consecutive Gaussian integers in the spiral ordering and so those are relatively prime.

Also, $N^-(v_{2i-1}) = \{u, w\}$ for $1 \leq i \leq \frac{n+1}{2}$ and the vertices u and w are labeled with consecutive Gaussian integers $\gamma_1 = 1$ and $\gamma_2 = 1 + i$. Since the consecutive Gaussian integers in the spiral ordering are relatively prime.

Further, $N^-(v_{2i}) = \{v_{2i-1}, v_{2i+1}\}$ for $1 \leq i \leq \frac{n-1}{2}$ and the labels of the vertices in $N^-(v_{2i})$ are consecutive Gaussian integers and so are relatively prime.

Therefore, f is a Gaussian in-neighborhood prime labeling when n is odd. Here$d^+(u) > 1$, $d^+(w) > 1$, $d^+(v_{2i-1}) > 1$ for $2 \leq i \leq \frac{n-1}{2}$ and $d^+(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-1}{2}$.

Now $N^+(u) = N^+(w) = \{v_1, v_3, \dots, v_{2i-1}\}$ for $1 \leq i \leq \frac{n+1}{2}$.

By the definition of$f$, the out-neighborhood vertices of u and w are labeled by the consecutive Gaussian integers$\gamma_3, \gamma_4, \dots, \gamma_{i+2}$. Since the consecutive Gaussian integers in the spiral ordering are relatively prime.

$N^+(v_1) = \{v_2\}$ and $N^+(v_n) = \{v_{n-1}\}$.

Now $N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}\}$ for $2 \leq i \leq \frac{n-1}{2}$. The vertices $v_{2i-2}$and v_{2i} are labeled by the consecutive Gaussian integers and which are relatively prime.

Also, $N^+(v_{2i}) = \{u, w\}$ for $1 \leq i \leq \frac{n-1}{2}$. Since the label of the vertex u is $\gamma_1 = 1$ which is relatively prime to all the Gaussian integers.

Therefore, f is a Gaussian out-neighborhood prime labeling when n is odd.

(I) and (II) imply, f is a Gaussian twin neighborhood prime labeling when n is odd.

Case (ii): n is even

$A\left(DDDA\overrightarrow{F_n}\right) = \{\overrightarrow{v_{2i-1}v_{2i}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i+1}v_{2i}} | 1 \leq i \leq \frac{n-2}{2}\} \cup \{\overrightarrow{v_{2i}u} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{uv_{2i-1}} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{v_{2i}w} | 1 \leq i \leq \frac{n}{2}\} \cup \{\overrightarrow{wv_{2i-1}} | 1 \leq i \leq \frac{n}{2}\}$ is the arc set.

Define an injective function $f: V\left(DDDA\overrightarrow{F_n}\right) \rightarrow [\gamma_{n+2}]$ by $(u) = \gamma_1, f(w) = \gamma_2$, $f(v_{2i-1}) = \gamma_{i+2}$ for $1 \leq i \leq \frac{n}{2}$ and $f(v_{2i}) = \gamma_{\left(\frac{n}{2}\right)+i+2}$ for $1 \leq i \leq \frac{n}{2}$.

Here, $d^-(u) > 1, d^-(w) > 1$, $d^-(v_{2i-1}) > 1$ for $1 \leq i \leq \frac{n}{2}$ and $d^-(v_{2i}) > 1$ for $1 \leq i \leq \frac{n-2}{2}$.

Now $N^-(u) = N^-(w) = \{v_2, v_4, \dots, v_{2i}\}$ for $1 \leq i \leq \frac{n}{2}$. Further, the labels of vertices in $N^-(u)$ and $N^-(w)$ are consecutive Gaussian integers in the spiral ordering and so those are relatively prime.

Also, $N^-(v_{2i}) = \{v_{2i-1}, v_{2i+1}\}$ for $1 \le i \le \frac{n-2}{2}$ and the vertices in $N^-(v_{2i})$ are labeled with the consecutive Gaussian integers and so those are relatively prime.
$N^-(v_n) = \{v_{n-1}\}$.
$N^-(v_{2i-1}) = \{u, w\}$ for $1 \le i \le \frac{n}{2}$. Since the vertices u and w are labeled with the consecutive Gaussian integers $\gamma_1 = 1$ and $\gamma_2 = 1 + i$ respectively. Then γ_1 and γ_2 are relatively prime.
Therefore, f is a Gaussian in- neighborhood prime labeling when n is even.
Next to prove f is also Gaussian out-neighborhood prime labeling.
Here $d^+(u) > 1$, $d^+(w) > 1$, $d^+(v_{2i-1}) > 1$ for $2 \le i \le \frac{n}{2}$ and $d^+(v_{2i}) > 1$ for $1 \le i \le \frac{n}{2}$.
Now $N^+(u) = N^+(w) = \{v_1, v_3, \dots, v_{2i-1}\}$ for $1 \le i \le \frac{n}{2}$.
By the definition of f, the label set of the out-neighborhood vertices of u and w are labeled by the consecutive Gaussian integers γ_3, γ_4, …, γ_{i+2} and so those are relatively prime.
Also, $N^+(v_1) = \{v_2\}$.
Now $N^+(v_{2i-1}) = \{v_{2i-2}, v_{2i}\}$ for $2 \le i \le \frac{n}{2}$ and the vertices v_{2i-2} and v_{2i} are labeled by the consecutive Gaussian integers $\gamma_{\frac{n}{2}+i+1}$ and $\gamma_{\frac{n}{2}+i+2}$.
$N^+(v_{2i}) = \{u, w\}$ for $1 \le i \le \frac{n}{2}$. Since the label of the vertex u is $\gamma_1 = 1$ which is relatively prime to all the consecutive Gaussian integers.
Therefore, f is a Gaussian out-neighborhood prime labeling when n is even.
(III) and (IV) imply f is a Gaussian twin neighborhood prime labeling when n is even.
From cases (i) and (ii), f is a Gaussian twin neighborhood prime labeling.
Hence, double di-double alternating fan $\overrightarrow{DDDAF_n}$ is a Gaussian twin neighborhood prime digraph.

5 Conclusions

In this article, we established Gaussian twin neighborhood prime labeling in fan and double fan digraphs. In this way, we extend our thoughts to different types of labeling in digraphs.

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