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Topology via Graph Ideals

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Abstract

The study of ideal topological space has started since 1933 and till date it is being developed by several mathematicians. Various classes of open sets, different types of operators and exploration of elementary topological results in ideal topological spaces have been discussed in various research papers. Methods of generating topologies using various relations have been explored by many researchers. Many researchers explored the methods of inducing topologies via graphs. This paper, introduces the notions of graph ideals, graph local function and characterizes some of their properties. It also describes a method of generating a new graph topology on the vertex set of a graph from the graph adjacency topology using Kuratowski closure operator and depicts the nature of open sets with respect to the new topology. Further, it explores the condition for compatibility of the graph adjacency topology with graph ideal.

Keywords: graph ideal, open neighbourhood function, graph local function, graph adjacency topological space

2020 AMS subject classifications: 54H99, 57M15, 54A10, 54A05 [1]

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1 Introduction

Topology is concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending. For a very long time, it was believed that abstract topological structures have limited applications in the generalization of real line and complex plane or some connections to Algebra and other branches of Mathematics. Further, it seems that there is a large gap between these structures and real life applications. Generating topologies by relations and the representation of topological concepts through binary relations will narrow the gap between Topology and its applications. A relation on graph represents a key for bridging Graph Theory and topological structures. The relation induces new types of topological structures to the graph. In 1967, J.W.Evans et al.[5] showed that there is a one-to-one correspondence between the labelled transitive directed graph with n points and the labelled topologies on n points. In 1967, S.S.Anderson and G.Chartrand[1] investigated the lattice-graph of the topologies of transitive directed graphs presented by J.W.Evans et al.[5]. In 2010, C.Marijuan[13] studied the relation between directed graphs and finite topologies. In 2013, S.M.Amiri et al.[2] induced a topology on vertex set of undirectd graph. In 2018, A.Kilicman and K.Abdulkalex[8] associated an incidence topology with vertex set of simple graphs without isolated vertices. In 2018, Shokry Nada et al.[15] generated topologies using the post class relations on the vertex set of graphs and discussed some of its applications in the biomedical field. K.Lalithambigai and P.Gnanachandra in [12], described a method of generating topology using adjacency, incidence relations on vertex set of graphs and studied the properties of closure and interior of vertex set of subgraphs in the graph adjacency topological space. Further, K.Lalithambigai and P.Gnanachandra in [11], introduced graph grills and explored the properties of topologies induced by graph grills on vertex set of graphs. The topic of ideals in general topological spaces is treated in the classic text by Kuratowski[9],[10] and also by R.Vaidyanathaswamy in [16],[17]. The properties which a topological space possess locally and the conditions through which those properties become a global one was studied by O.Njastad in 1966 [14]. Generation of new topologies from the old one via ideals in general topological space was studied by D.Jankovic and T.R.Hamlett[7]. Ideal on a topological space can be used to study the similarity of two structures. Till now, scholars have mostly contributed to the study of ideals in general topological space. It can be noticed that most of the real life problems can be modelled as a graph and can be solved using the graph theory concepts. The aim of this paper is to bridge the gap between abstract and concrete concepts in ideal topological spaces, by defining graph ideals and graph local functions in graph adjacency topological space. This paper explores the basic facts of graph local function and describes the method of generating a new topology from the older via graph ideals. Further, this paper characterizes the nature of the open sets of the new topology in terms of the closure operator and investigates the compatibility of graph adjacency topology with the graph ideal.

2 Preliminaries

Fundamental definitions and preliminaries of graph theory and topology may be found in the sources [3],[4],[6].

A graph G consists of a pair (V(G),X(G)), where V(G) is a nonempty finite set whose elements are called vertices and X(G) is a set of unordered pairs of distinct elements of V(G). The elements of X(G) are called edges of the graph G. An edge joining a vertex to itself is called a loop. Edges joining the same vertices are called multiple edges. A graph without loops and multiple edges is called a simple graph. A graph G is called a bipartite graph if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 to a vertex of V_2. (V_1,V_2) is called a bipartition of G. If further G contains every edge joining the vertex of V_1 to the vertex of V_2, then G is called a complete bipartite graph. If V_1 contains m vertices and V_2 contains n vertices then the complete bipartite graph G is denoted by $K_{m,n}$. It should be noticed that $K_{1,m}$ is called a star for m$\geqslant$1. The degree of a vertex v in a graph G is the number of edges incident with v and it is denoted by deg(v). A graph in which degree of every vertex are the same is called a regular graph. A vertex of degree 0 is called an isolated vertex.

The concept of ideals in topological spaces has been studied by Kuratowski[9] and Vaidyanathaswamy[14].

An ideal on a topological space (X, τ) is a non empty collection I of subsets of X satisfying the following two conditions: (i) If $A \in I$ and $B \subset A$, then $B \in I$

(ii) If $A \in I$ and $B \in I$, then $A \cup B \in I$

An ideal topological space is a topological space (X, τ) with an ideal I on X, and is denoted by (X, τ, I)

Throughout the paper, the graphs under discussion are the simple undirected graphs which are not star graphs.

3 Graph Ideal and Graph Local Function

In this section, graph ideal and graph local function in a graph adjacency topological space are defined with illustrations.

Definition 3.1. *Let G = (V(G),X(G)) be a graph. For v ∈ V(G), the neighbourhood set N_v of v is defined as* $N_v = \{u \in V(G) : uv \in X(G)\}$.

Definition 3.2. *Let G = (V(G),X(G)) be a graph without isolated vertices. Define S_N as the family of N_v for all v ∈ V(G), i.e., $S_N = \{N_v : v \in V(G)\}$. Then S_N forms a subbase for a topology $\mathcal{T}_A$ on V(G) and the pair (V(G),$\mathcal{T}_A$) is called graph adjacency topological space. If W is a vertex induced subgraph of G, then the closure of V(W) is defined by cl(V(W)) = V(W) $\cup\{v \in V(G) : N_v \cap V(W) \neq \phi\}$ and the interior of V(W) is defined by int(V(W)) = $\{v \in V(G) : N_v \subseteq V(G)\}$.*

Definition 3.3. *Let G = (V(G),X(G)) be a graph for which $P(V)$ and $P(X)$ are the power sets of V(G) and X(G) respectively. The set $\mathcal{I} = \{\mathcal{G}' : \mathcal{G}' = (V', X')$, where $V' \subseteq V, X' \subseteq X\}$ is said to be a graph ideal on a graph adjacency topological space (V(G),$\mathcal{T}_A$) if it satisfies the following two conditions*
i) if $\mathcal{G}'$ and $\mathcal{G}'' \in \mathcal{I}$, then $\mathcal{G}' \cup \mathcal{G}'' \in \mathcal{I}$
ii) if $\mathcal{G}'$ is a subgraph of $\mathcal{G}''$ and $\mathcal{G}'' \in \mathcal{I}$, then $\mathcal{G}' \in \mathcal{I}$.

Example 3.1. *Consider the following graph*

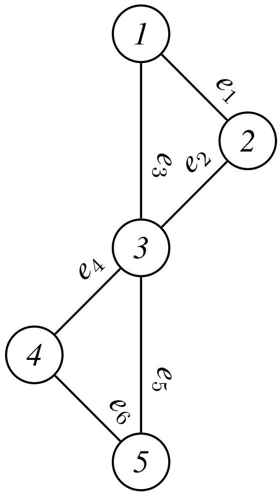


Figure. 3.1

For the graph in Figure 3.1,
$\mathcal{I} = \{(\{1,2\},\{e_1\}),(\{1\},\phi),(\{2\},\phi),(\{1,2\},\phi)\}$ *is a graph ideal.*
$\mathcal{I} = \{(\{1,2,3\},\{e_2\}),(\{4,5\},\{e_6\}),(\{1,2,3,4,5\},\{e_2,e_6\}),(\{1,2\},\{e_1\})$ *is not a graph ideal, since,* $(\{1\},\phi) \notin \mathcal{I},(\{2\},\phi) \notin \mathcal{I},(\{3\},\phi) \notin \mathcal{I},(\{4\},\phi) \notin \mathcal{I},(\{5\},\phi) \notin \mathcal{I}$.

Definition 3.4. *Let $(V(G),\mathcal{T}_A)$ be a graph adjacency topological space such that v ∈ V(G). The open neighbourhood system at v denoted by N(v) is defined as N(v) = $\{U \in \mathcal{T}_A : v \in U\}$.*

Definition 3.5. *Let $(V(G),\mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. Let W be a subgraph of G. Then $(V(W))^*(\mathcal{I},\mathcal{T}_A) = \{v \in V(G) :$ for every $U \in N(v), V(W) \cap U \neq V(\mathcal{G}')$ for any $\mathcal{G}' \in \mathcal{I}\}$ is called the graph local function of V(W) with respect to $\mathcal{I}$ and $\mathcal{T}_A$.*

Example 3.2. *Consider the following graph*

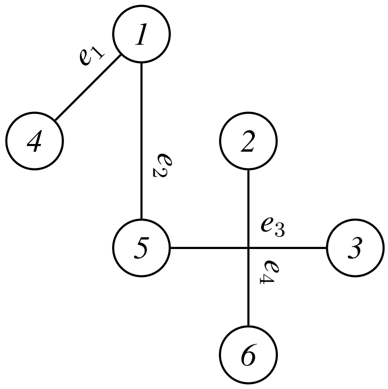


Figure 3.2

Graph local function of a subgraph of the graph in Figure 3.2 is illustrated below:
$S_N = \{\{4,5\},\{6\},\{5\},\{1\},\{1,3\},\{2\}\}$.
B = $\{\phi,\{4,5\},\{6\},\{5\},\{1\},\{1,3\},\{2\}\}$.
$\mathcal{T}_A = \{\phi,\{4,5\},\{6\},\{5\},\{1\},\{1,3\},\{2\},\{4,5,6\},\{4,5,1\},\{1,3,4,5\},\{2,4,5\},\{5,6\},\{1,6\},$
$\{1,3,6\},\{2,6\},\{1,5\},\{1,3,5\},\{2,5\},\{1,2\},\{1,5,6\},\{1,4,5\},\{1,4,5,6\},\{1,5,6\},\{1,3,5,6\},$
$\{1,3,4,5,6\},\{2,4,5,6\},\{2,4,5,1\},\{1,2,3,4,5\},\{2,5,6\},\{1,2,6\},\{1,2,3,6\},\{1,2,5\},$
$\{1,2,3,5\},\{1,2,5,6\},\{1,2,4,5\},\{1,2,4,5,6\},\{1,2,5,6\},\{1,2,3,5,6\},\{1,2,3,4,5,6\}\}$.
Let $\mathcal{I} = \{(\{1\},\phi),(\{5\},\phi),(\{1,5\},\phi),(\{1,5\},e_2)\}$ *be a graph ideal.*
Let $W = (\{1,2,3\},\phi)$ *be a subgraph of the given graph. Then* $(V(\mathcal{W}))^*(\mathcal{I},\mathcal{T}_A) = \{2,3\}$.

Note 3.6.

1. *Let G be a graph with n vertices. For each i = 1,2, ...n,* $\{(\{v_i\},\phi)\}$ *is a graph ideal. In this case, for any subgraph* W, $(V(\mathcal{W}))^*(\mathcal{I},\mathcal{T}_A) = cl(V(\mathcal{W}))$.
2. *Let* $P(G)$ *be the collection of all subgraphs of a graph G. Clearly* $P(G)$ *is a graph ideal. In this case, for any subgraph* W, $(V(W))^*(\mathcal{I},\mathcal{T}_A) = \phi$.

4 Facts Concerning Graph Local Function

This section describes some basic facts concerning the graph location function which are useful in the generation of new topology from the old one.

Theorem 4.1. *Let* $(V(G),\mathcal{T}_A)$ *be a graph adjacency topological space with graph ideals* $\mathcal{I}$ *and* $\mathcal{J}$. *Let* H *and* W *be subgraphs of* G. *Then*

1. $V(H) \subseteq V(W) \Rightarrow (V(H))^* \subseteq (V(W))^*$
2. $\mathcal{I} \subseteq \mathcal{J} \Rightarrow (V(W))^*(\mathcal{J}) \subseteq (V(W))^*(\mathcal{I})$

3. $(V(W))^* \subseteq cl(V(W))$
4. $((V(W))^*)^* \subseteq (V(W))^*$
5. $(V(H \cup W))^* = (V(H))^* \cup (V(W))^*$
6. $(V(H))^* - (V(W))^* \subseteq (V(H) - V(W))^*$
7. $U \in \mathcal{T}_A \Rightarrow U \cap (V(W))^* = U \cap (U \cap V(W))^* \subseteq (U \cap V(W))^*$
8. $(V(W))^*(\mathcal{I}) \cup (V(W))^*(\mathcal{J}) = (V(W))^*(\mathcal{I} \cap \mathcal{J})$.
9. $(V(H) \cap V(W))^* \subseteq (V(H))^* \cap (V(W))^*$.
10. $I = V(G^{'})$ *for some* $G^{'} \in \mathcal{I} \Rightarrow (V(H) \cup I)^* = (V(H))^* = (V(H) - I)^*$.
11. $(V(H))^*$ - $((V(H))^*)^* \subseteq (V(H) - (V(H))^*)^*$.

Proof. 1. Let $v \notin (V(W))^*$.
Hence there exists $U \in N(v)$ such that $V(W) \cap U = V(\mathcal{G}^{'})$, for some $\mathcal{G}^{'} \in \mathcal{I}$.
Since $V(H) \subseteq V(W), V(H) \cap U \subseteq V(W) \cap U$.
Hence there exists $U \in N(v)$ such that $V(H) \cap U = V(\mathcal{G}^{'})$, for some $\mathcal{G}^{'} \in \mathcal{I}$.
So $v \notin (V(H))^*$. Hence $(V(H))^* \subseteq (V(W))^*$.
2. Let $v \in (V(W))^*(\mathcal{J})$.
Hence, for every $U \in N(v), V(W) \cap U \neq V(\mathcal{G}^{'})$, for any $\mathcal{G}^{'} \in \mathcal{J}$.
Since $\mathcal{I} \subseteq \mathcal{J}, V(W) \cap U \neq V(\mathcal{G}^{'})$, for any $\mathcal{G}^{'} \in \mathcal{I}$.
Hence for every $U \in N(v), V(W) \cap U \neq V(\mathcal{G}^{'})$, for any $\mathcal{G}^{'} \in \mathcal{I}$.
Hence $(V(W))^*(\mathcal{J}) \subseteq (V(W))^*(\mathcal{I})$
3. Let $v \in (V(W))^*$.
Hence for every $U \in N(v), V(W) \cap U \neq V(\mathcal{G}^{'})$, for any $\mathcal{G}^{'} \in \mathcal{I}$.
Thus $V(W) \cap N_v \neq \phi$ and $v \in cl(V(W))$.
Hence $(V(W))^* \subseteq cl(V(W))$.
4. Let $v \in ((V(W))^*)^*$.
Hence for every $U \in N(v), (V(W))^* \cap U \neq V(\mathcal{G}^{'})$, for any $\mathcal{G}^{'} \in \mathcal{I}$.
For every $U \in N(v), V(W) \cap U \neq V(\mathcal{G}^{'})$, for any $\mathcal{G}^{'} \in \mathcal{I}$.
So $v \in (V(W))^*$ and $((V(W))^*)^* \subseteq (V(W))^*$.
5. Let $v \notin (V(H))^* \cup (V(W))^*$.
Then $v \notin (V(H))^*$ and $v \notin (V(W))^*$.
So there exists $U_1 \in N(v)$ such that $V(H) \cap U_1 = V(\mathcal{G}^{'})$, for some $\mathcal{G}^{'} \in \mathcal{I}$ and $U_2 \in N(v)$ such that $V(W) \cap U_2 = V(\mathcal{G}^{''})$, for some $\mathcal{G}^{''} \in \mathcal{I}$.
Hence $(V(H) \cup V(W)) \cap U = V(\mathcal{G}^{'''})$, where $U = U_1 \cup U_2$ and $V(\mathcal{G}^{'''}) = V(\mathcal{G}^{'}) \cup V(\mathcal{G}^{''})$.
So there exists $U \in N(v)$ such that $(V(H) \cup V(W)) \cap U = V(\mathcal{G}^{'''})$, for some $\mathcal{G}^{'''} \in \mathcal{I}$.

Hence $v \notin (V(H \cup W))^*$ and $(V(H \cup W))^* \subseteq (V(H))^* \cup (V(W))^*$.
Thus $V(H) \subseteq V(H \cup W)$ and $V(W) \subseteq V(H \cup W)$.
By (1), $(V(H))^* \subseteq (V(H \cup W))^*$ and $(V(W))^* \subseteq (V(H \cup W))^*$.
Therefore $(V(H))^* \cup (V(W))^* \subseteq (V(H \cup W))^*$.
Hence $(V(H))^* \cup (V(W))^* = (V(H \cup W))^*$.
6. Let $v \notin (V(H) - V(W))^*$.
Hence there exists $U \in N(v)$ such that $(V(H) - V(W)) \cap U = V(\mathcal{G}')$, for some $\mathcal{G}' \in \mathcal{I}$.
So $(V(H) \cap U) - (V(W) \cap U) = V(\mathcal{G}')$, for some $\mathcal{G}' \in \mathcal{I}$.
Thus $(V(H) \cap U) = V(\mathcal{G}')$ and $V(W) \cap U \neq V(\mathcal{G}')$, for some $\mathcal{G}' \in \mathcal{I}$.
So $v \notin (V(H))^*$ and $v \in (V(W))^*$.
Thus $v \notin (V(H))^* - (V(W))^*$.
Hence $(V(H))^* - (V(W))^* \subseteq (V(H) - V(W))^*$.
7. Let $U \in \mathcal{T}_A$ and $v \in U \cap (V(W))^*$.
$v \in U$ and $v \in (V(W))^*$.
$v \in U$ and $U \in \mathcal{T}_A \Rightarrow U \in N(v)$.
$v \in (V(W))^* \Rightarrow$ for every $U_1 \in N(v), V(W) \cap U_1 \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$.
$U \cap (V(W) \cap U_1) \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$.
$(U \cap V(W)) \cap U_1 \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$.
Hence $v \in (U \cap V(W))^*$.
So $v \in U \cap (U \cap V(W))^*$.
$U \cap (V(W))^* \subseteq U \cap (U \cap V(W))^*$.
By reserving the above steps, it follows that $U \cap (U \cap V(W))^* \subseteq U \cap (V(W))^*$.
So $U \cap (V(W))^* = U \cap (U \cap V(W))^*$.
Moreover, $U \cap (U \cap V(W))^* \subseteq (U \cap V(W))^*$.
8. Since $\mathcal{I} \cap \mathcal{J} \subseteq \mathcal{I}$ and $\mathcal{I} \cap \mathcal{J} \subseteq \mathcal{J}, (V(W))^*(\mathcal{I}) \subseteq (V(W))^*(\mathcal{I} \cap \mathcal{J})$ and $(V(W))^*(\mathcal{J}) \subseteq (V(W))^*(\mathcal{I} \cap \mathcal{J})$.
Hence $(V(W))^*(\mathcal{I}) \cup (V(W))^*(\mathcal{J}) \subseteq (V(W))^*(\mathcal{I} \cap \mathcal{J})$.
Let $v \in (V(W))^*(\mathcal{I} \cap \mathcal{J})$. So, for every $U \in N(v), V(W) \cap U \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I} \cap \mathcal{J}$.
For every $U \in N(v), V(W) \cap U \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$ or $V(W) \cap U \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{J}$.
So $v \in (V(W))^*(\mathcal{I})$ or $v \in (V(W))^*(\mathcal{J})$ and so $v \in (V(W))^*(\mathcal{I}) \cup (V(W))^*(\mathcal{J})$.
$(V(W))^*(\mathcal{I} \cap \mathcal{J}) \subseteq (V(W))^*(\mathcal{I}) \cup (V(W))^*(\mathcal{J})$.
So $(V(W))^*(\mathcal{I} \cap \mathcal{J}) = (V(W))^*(\mathcal{I}) \cup (V(W))^*(\mathcal{J})$.
9. Since $V(H) \cap V(W) \subseteq V(H)$ and $V(H) \cap V(W) \subseteq V(W)$, by (1),
$(V(H) \cap V(W))^* \subseteq (V(H))^*$ and $(V(H) \cap V(W))^* \subseteq (V(W))^*$.
Hence $(V(H) \cap V(W))^* \subseteq (V(H))^* \cap (V(W))^*$.
10. Since $V(H) \subseteq V(H) \cup I$ and $V(H) - I \subseteq V(H)$, by (1),
$(V(H))^* \subseteq (V(H) \cup I)^*$ and $(V(H) - I)^* \subseteq (V(H))^*$.
Now, $v \notin (V(H) - I)^*$

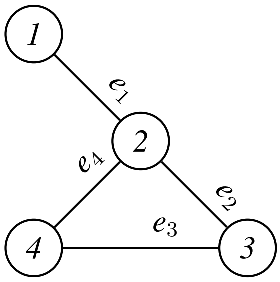
$\Rightarrow$ there exists $U \in N(v)$ such that $U \cap (V(H) - I) = V(\mathcal{G}')$, for some $\mathcal{G}' \in \mathcal{I}$.
$\Rightarrow$ there exists $U \in N(v)$ such that $U \cap (V(H) - V(\mathcal{G}'')) = V(\mathcal{G}')$, for some $\mathcal{G}', \mathcal{G}'' \in \mathcal{I}$.
So there exists $U \in N(v)$ such that $(U \cap V(H)) - (U \cap V(\mathcal{G}'')) = V(\mathcal{G}')$, for some $\mathcal{G}', \mathcal{G}'' \in \mathcal{I}$.
This implies that there exist $U \in N(v)$ such that $U \cap V(H) = V(\mathcal{G}')$, for some $\mathcal{G}' \in \mathcal{I}$.
$\Rightarrow v \notin (V(H))^*$.
So $(V(H))^* \subseteq (V(H) - I)^*$.
Let $v \in (V(H) \cup I)^*$.
For every $U \in N(v), U \cap (V(H) \cup I) \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$.
For every $U \in N(v), (U \cap V(H)) \cup (U \cap I) \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$.
But $I = V(\mathcal{G}'')$, for some $\mathcal{G}'' \in \mathcal{I}$.
So $U \cap V(H) \neq V(\mathcal{G}''')$, for any $\mathcal{G}''' \in \mathcal{I}$.
So $v \in (V(H))^*$.
Hence $(V(H) \cup I)^* \subseteq (V(H))^*$.
So $(V(H) \cup I)^* = (V(H))^* = (V(H) - I)^*$.
11. $v \in (V(H))^* - ((V(H))^*)^*$.
So $v \in (V(H))^*$ and $v \notin ((V(H))^*)^*$.
For every $U \in N(v), U \cap V(H) \neq V(\mathcal{G}')$, for any $\mathcal{G}' \in \mathcal{I}$ and there exists $U_1 \in N(v)$ such that $U_1 \cap (V(H))^* = V(\mathcal{G}'')$, for some $\mathcal{G}'' \in \mathcal{I}$.
For every $U \in N(v), (U \cap V(H)) - (U \cap (V(H))^*) \neq V(\mathcal{G}''')$, for any $\mathcal{G}''' \in \mathcal{I}$.
For every $U \in N(v), U \cap (V(H) - (V(H))^*) \neq V(\mathcal{G}''')$, for any $\mathcal{G}''' \in \mathcal{I}$.
$v \in (V(H) - (V(H))^*)^*$.
So $(V(H)^* - ((V(H))^*)^* \subseteq (V(H) - (V(H))^*)^*$. $\square$

5 Topology via Graph Ideals

This section describes a method of generating a new topology from the old one using Kuratowski closure operator cl^*.

Definition 5.1. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. Let W be a subgraph of G. Define $cl^*(V(W)) = V(W) \cup (V(W))^*$ and $\mathcal{T}_A^*(\mathcal{I}) = \{V(H) \subseteq V(G) : cl^*(V(G) - V(H)) = V(G) - V(H)\}$. $\mathcal{T}_A^*(\mathcal{I})$ is called the graph adjacency topology generated by cl^*. When there is no ambiguity, it is denoted as $\mathcal{T}_A^*$.*

Example 5.1. *Consider the following graph*



$S_N = \{\{2\}, \{1,3,4\}, \{2,4\}, \{2,3\}\}$
$\mathsf{B} = \{\phi, \{2\}, \{1,3,4\}, \{2,4\}, \{2,3\}, \{4\}, \{3\}\}$
$\mathcal{T}_A = \{\phi, \{2\}, \{1,3,4\}, \{2,4\}, \{2,3\}, \{4\}, \{3\}, \{3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$
$\mathcal{I} = \{(\{1,2\}, \{e_1\}), (\{1\}, \phi), (\{2\}, \phi), (\{1,2\}, \phi)\}$
$N(1) = \{\{1,3,4\}, \{1,2,3,4\}\}, N(2) = \{\{2\}, \{2,4\}, \{2,3\}, \{2,3,4\}, \{1,2,3,4\}\},$
$N(3) = \{\{1,3,4\}, \{2,3\}, \{3\}, \{3,4\}, \{2,3,4\}, \{1,2,3,4\}\},$
$N(4) = \{\{1,3,4\}, \{2,4\}, \{4\}, \{3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$
$\mathcal{T}_A^* = \{\phi, \{3\}, \{4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$.

Observation 5.2.

1. *If $\mathcal{I} = \{(\{v_i\}, \phi)\}$ is a graph ideal, then $(V(W))^*(\mathcal{I}, \mathcal{T}_A) = cl(V(W))$ for every $V(W) \subseteq V(G)$.*
   Hence $cl^*(V(W)) = cl(V(W))$.
   $\mathcal{T}_A^* = \{V(H) \subseteq V(G) : cl^*(V(G) - V(H)) = V(G) - V(H)\}$
   $= \{V(H) \subseteq V(G) : cl(V(G) - V(H)) = V(G) - V(H)\}$
   $= \mathcal{T}_A$.

2. *If $P(G)$ is the collection of all subgraphs of a graph G. Then $P(G)$ is a graph ideal. Hence for any subgraph W, $(V(W))^*(\mathcal{I}, \mathcal{T}_A) = \phi$ and $cl^*(V(W)) = V(W)$.*
   $\mathcal{T}_A^* = \{V(H) \subseteq V(G) : cl^*(V(G) - V(H)) = V(G) - V(H)\}$
   $= \{V(H) \subseteq V(G) : V(G) - V(H) = V(G) - V(H)\}$
   $= P(G)$
   Hence $\mathcal{T}_A^*$ is a discrete topology.

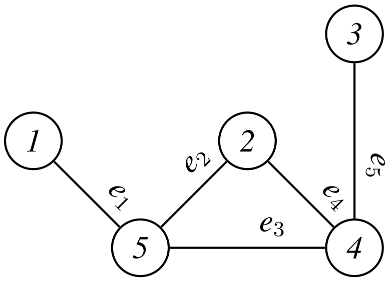
3. *For every graph ideal $\mathcal{I}, \{(\{v_i\}, \phi)\} \subseteq \mathcal{I} \subseteq P(G)$. Hence $\mathcal{T}_A \subseteq \mathcal{T}_A^* \subseteq$ discrete topology.*

Proposition 5.1. *If $\mathcal{I}$ and $\mathcal{J}$ are graph ideals such that $\mathcal{I} \subseteq \mathcal{J}$, then $\mathcal{T}_A^*(\mathcal{I}) \subseteq \mathcal{T}_A^*(\mathcal{J})$.*

Proof. $\mathcal{I} \subseteq \mathcal{J} \Rightarrow (V(H))^*(\mathcal{J}) \subseteq (V(H))^*(\mathcal{I})$. Thus $\{V(H) \cup (V(H))^*(\mathcal{J})\} \subseteq \{V(H) \cup (V(H))^*(\mathcal{I})\}$. Therefore $cl^*(V(H))(\mathcal{J}) \subseteq cl^*(V(H))(\mathcal{I})$. This means $cl^*(V(G) - V(H))(\mathcal{I}) \subseteq cl^*(V(G) - V(H))(\mathcal{J})$. Hence $\mathcal{T}_A^*(\mathcal{I}) \subseteq \mathcal{T}_A^*(\mathcal{J})$. $\square$

Definition 5.3. *The derived set of V(H) in V(G), $\mathcal{T}_A$) denoted by $(V(H))^d$ is defined by v $\in (V(H))^d$ if and only if $(U - \{v\}) \cap V(H) \neq \phi$ for every U $\in$ N(v). The derived set of V(H) in V(G), $\mathcal{T}_A^*$) is denoted by $(V(H))^{d*}$.*

Example 5.2. *Consider the following graph*



$S_N = \{\{1\}, \{5,4\}, \{4\}, \{3,2,5\}, \{1,2,4\}\}$
$\mathsf{B} = \{\phi, \{1\}, \{5,4\}, \{4\}, \{3,2,5\}, \{1,2,4\}, \{2\}\}$
$\mathcal{T}_A = \{\phi, \{1\}, \{5,4\}, \{4\}, \{3,2,5\}, \{1,2,4\}, \{2\}, \{1,2,4,5\}, \{2,5\}, \{2,3,4,5\}, \{2,4,5\}, \{2,4\}, \{1,2,3,4,5\}\}$
Let $V(H) = \{3,5\}$. *Then* $(V(H))^d = \{3\}$.

Note 5.4.

1. $v \in (V(H))^{d*} \Leftrightarrow v \in cl^*(V(H) - \{v\})$
   $\Leftrightarrow v \in (V(H) - \{v\}) \cup (V(H) - \{v\})^*$
   $\Leftrightarrow v \in (V(H) - \{v\})^*$
   $\Leftrightarrow$ *for every* $U \in N(v), (V(H) - \{v\}) \cap U \neq V(G')$, *for any* $G' \in \mathcal{I}$
   Hence $v \in (V(H))^{d*} \Leftrightarrow$ *for every* $U \in N(v), (V(H) - \{v\}) \cap U \neq V(G')$, *for any* $G' \in \mathcal{I}$.
2. $(V(H))^{d*} \subseteq (V(H))^*$

Definition 5.5. *A graph which has either an infinite number of vertices or edges or both is called an infinite graph.*

Definition 5.6. *Let G be an infinite graph. Let $\mathcal{I}_f$ be the graph ideal of finite subgraphs of G and H be a subgraph of G. A vertex v is said to be an ω-accumulation point of V(H) if and only if $U \cap V(H)$ is infinite for every $U \in N(v)$. The set of all ω-accumulation points of V(H) is denoted by $(V(H))^\omega$.*

Observation 5.7. $(V(H))^\omega = (V(H))^*(\mathcal{I}_f)$.

Definition 5.8. *Let G be an infinite graph. Let $\mathcal{I}_c$ be the graph ideal of countable subgraphs of G and H be a subgraph of G. A vertex v is said to be a condensation point of V(H) if U $\cap$ V(H) is uncountable for every U $\in$ N(v). The set of all condensation points of V(H) is denoted by $(V(H))^*(\mathcal{I}_c)$.*

Definition 5.9. *A graph adjacency topological space $(V(G), \mathcal{T}_A)$ is said to be a T_1 space if for each pair of distinct vertices v_i and v_j, v_i belong to every $U \in N(v_i)$ for which v_i does not belong to any $U \in N(v_j)$ and v_j belongs to every $U \in N(v_j)$ and v_j does not belong to any $U \in N(v_i)$.*

Remark 5.1.

1. *Let $\mathcal{I}_f$ be a graph ideal of finite subgraphs of G. Since $(\{v_i\}, \phi) \in \mathcal{I}_f$ for each $v_i \in V(G), (V(H))^* = (V(H))^{d*}$.*
   Also $(V(H))^* = (V(H))^{\omega}$. Hence ω-accumulation points of V(H) in $(V(G), \mathcal{T}_A)$ are precisely the limit points of V(H) in $(V(G), \mathcal{T}_A^*(\mathcal{I}_f))$. Since $(\{v_i\}, \phi) \in \mathcal{I}_f, (\{v_i\}, \phi)^* = \phi$. So $cl^*(\{v_i\}) = \{v_i\}$.Hence $(V(G), \mathcal{T}_A^*)$ is T_1.
   In T_1 spaces, ω-accumulation point and limit point coincides. Hence the set of ω-accumulation points of V(H) in $(V(G), \mathcal{T}_A)$ and $(V(G), \mathcal{T}_A^*)$ coincide.
   $(V(H))^*(\mathcal{I}_f) = (V(H))^d$ iff $\mathcal{T}_A = \mathcal{T}_A^*(\mathcal{I}_f))$ iff $(V(G), \mathcal{T}_A)$ is T_1.

2. *Let $\mathcal{I}_c$ be a graph ideal of countable subgraphs of G. Since $(V(H))^*$ is the set of condensation points of V(H) and $(V(H))^* = (V(H))^{d*}$, the condensation points of V(H) in (V(G), $\mathcal{T}_A$) are precisely the limit points of V(H) in $(V(G), \mathcal{T}_A^*(\mathcal{I}_c))$.*

Definition 5.10. *Let H be a subgraph of a graph G. We say that V(H) is closed and discrete in $(V(G), \mathcal{T}_A)$ if and only if $(V(H))^d = \phi$.*

Lemma 5.1. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. If $I \in \mathcal{I}$, then I is closed and discrete in $(V(G), \mathcal{T}_A^*)$.*

Proof. I $\in \mathcal{I} \Rightarrow I^* = \phi \Rightarrow I^{d*} = \phi$. □

Remark 5.2. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space. Let $\mathcal{I}_{cd} = \{H : H$ is a subgraph of G and $(V(H))^d = \phi\}$.*
Let us prove $\mathcal{I}_{cd}$ is a graph ideal.
$H_1, H_2 \in \mathcal{I}_{cd} \Rightarrow (V(H_1))^d = \phi$ and $(V(H_2))^d = \phi$.
Hence for every $v \in V(G), (U - \{v\}) \cap V(H_1) = \phi$ and $(U - \{v\}) \cap V(H_2) = \phi$.
So for every $v \in V(G), (U - \{v\}) \cap V(H_1 \cup H_2) = \phi$.
So $H_1 \cup H_2 \in \mathcal{I}_{cd}$.
Let $H_1 \in \mathcal{I}_{cd}$ and H_2 be a subgraph of H_1.
$H_1 \in \mathcal{I}_{cd} \Rightarrow (V(H_1))^d = \phi$.
Hence for every $v \in V(G), (U - \{v\}) \cap V(H_1) = \phi$ and so for every $v \in V(G), (U - \{v\}) \cap V(H_2) = \phi$.
So $(V(H_2))^d = \phi$ which implies $H_2 \in \mathcal{I}_{cd}$.
$\mathcal{I}_{cd}$ is a graph ideal. Also $(V(H))^d \subseteq (V(H))^*$. Hence $\mathcal{T}_A^* = \mathcal{T}_A$.
Lemma **??** *implies that $\mathcal{I}_{cd}$ is the largest graph ideal with the property that $\mathcal{T}_A^* = \mathcal{T}_A$.*
Finally, $(V(H))^* = (V(H))^d$ iff $(V(G), \mathcal{T}_A)$ is T_1.

Example 5.3. *Let* $(V(G), \mathcal{T}_A)$ *be a graph adjacency topological space. Let* $\mathcal{I} = \{G' : G'$ *is a subgraph of G and* $int(cl(V(G'))) = \phi\}$*, i.e.,* $\mathcal{I}$ *is the collection of nowhere dense subgraphs of* $(V(G), \mathcal{T}_A)$*.*
Let $G', G'' \in \mathcal{I}$*. Hence* $int(cl(V(G'))) = \phi$ *and* $int(cl(V(G''))) = \phi$*.*
Now, $int(cl(V(G' \cup G''))) = int(cl(V(G') \cup cl(V(G''))) \supseteq int(cl(V(G')) \cup int(cl(V(G'')) = \phi$*.*
Hence $int(cl(V(G' \cup G''))) \supseteq \phi$*.*
So $\mathcal{I}$ *is not a graph ideal.*

6 The Open sets of $\mathcal{T}_A^*$

Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. Let H be a subgraph of G. Then V(H) is said to be $\mathcal{T}_A^*$-closed if and only if $((V(H))^* \subseteq V(H)$. Hence V(H) is $\mathcal{T}_A^*$-open if and only if V(G) - V(H) is $\mathcal{T}_A^*$-closed, i.e., if and only if $(V(G) - V(H))^* \subseteq V(G) - V(H)$, i.e., if and only if $V(H) \subseteq V(G) - (V(G) - V(H))^*$.
Hence V(H) is $\mathcal{T}_A^*$-open if and only if $v \in V(H) \Rightarrow v \notin (V(G) - V(H))^*$. So V(H) is $\mathcal{T}_A^*$-open if and only if $v \in V(H) \Rightarrow$ there exists $U \in N(v)$ such that $U \cap (V(G) - V(H)) = V(G')$, for some $G' \in \mathcal{I}$.
Let $W = U \cap (V(G) - V(H))$. V(H) is $\mathcal{T}_A^*$-open if and only if, for $v \in V(H)$ there exists $U \in N(v)$ such that $W = V(G')$, for some $G' \in \mathcal{I}$.
If $v \in V(H)$ then $v \notin V(G) - V(H)$. So $v \notin W$. Also $v \in U$. Hence $v \in U - W$. V(H) is $\mathcal{T}_A^*$-open if and only if, for $v \in V(H)$, there exists $U \in N(v)$ such that $v \in U - W$ and $W = V(G')$ for some $G' \in \mathcal{I}$.
Let $\beta(\mathcal{I}, \mathcal{T}_A^*) = \{U - W : U \in \mathcal{T}_A, W = V(G')$ for some $G' \in \mathcal{I}\}$. $\beta(\mathcal{I}, \mathcal{T}_A^*)$ is a basis for $\mathcal{T}_A^*$.

Note 6.1. *If* $\mathcal{I}$ *and* $\mathcal{J}$ *are graph ideals then* $\mathcal{I} \vee \mathcal{J} = \{I \cup J : I \in \mathcal{I}, J \in \mathcal{J}\}$ *is also a graph ideal.*

Theorem 6.1. *Let* $(V(G), \mathcal{T}_A)$ *be a graph adjacency topological space with a graph ideals* $\mathcal{I}$ *and* $\mathcal{J}$*. Let H be a subgraph of G. Then* $(V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A) = (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J})) \cap (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I}))$*.*

Proof. Let $v \notin (V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A)$.
Hence there exists $U \in N(v)$ such that $U \cap V(H) = V(G')$, for some $G' \in \mathcal{I} \vee \mathcal{J}$.
Let $G'' \in \mathcal{I}$ and $G''' \in \mathcal{J}$ such that $U \cap V(H) = V(G'''')$ where $G'''' = G'' \cup G'''$.
Assume that $V(G'') \cap V(G''') = \phi$.
Hence $(U \cap V(H)) - V(G'') = V(G''')$ and $(U \cap V(H)) - V(G''') = V(G'')$.
So $(U - V(G'')) \cap V(H) = V(G''')$, where $G''' \in \mathcal{J}$ and $(U - V(G''')) \cap V(H) = V(G'')$, where $G'' \in \mathcal{I}$.

$\Rightarrow v \notin (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I}))$ or $v \notin (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J}))$.
$\Rightarrow v \notin (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I})) \cap (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J}))$.
Hence $(V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J})) \cap (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I})) \subseteq (V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A)$.
$v \notin (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J}))$ implies that there exists $U \in N(v)$ such that $(U - V(G''))\cap V(H) = V(G')$ for some $G' \in \mathcal{I}, G'' \in \mathcal{J}$.
Assume that $V(G'') \subseteq V(H)$. Hence $U \cap V(H) = V(G') \cup V(G'') = V(G''')$, where $G''' \in \mathcal{I} \vee \mathcal{J}$. So $v \notin (V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A)$.
$(V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A) \subseteq (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J}))$.
Similarly, $(V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A) \subseteq (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I}))$.
$(V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A) \subseteq (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J})) \cap (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I}))$.
Hence $(V(H))^*(\mathcal{I} \vee \mathcal{J}, \mathcal{T}_A) = (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{J})) \cap (V(H))^*(\mathcal{J}, \mathcal{T}_A^*(\mathcal{I}))$. □

Remark 6.1. *Given a graph adjacency topological space $(V(G), \mathcal{T}_A)$ and a graph ideal $\mathcal{I}$, $\mathcal{T}_A^{**} = (\mathcal{T}_A^*(\mathcal{I}))^*(\mathcal{I})$ is a topology on $V(G)$ and $\mathcal{T}_A^{**}$ is finer than $\mathcal{T}_A^*$.*

Corollary 6.1. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. Then $(V(H))^*(\mathcal{I}, \mathcal{T}_A) = (V(H))^*(\mathcal{I}, \mathcal{T}_A^*)$ and hence $\mathcal{T}_A^* = \mathcal{T}_A^{**}$.*

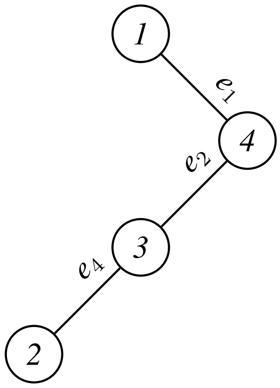
Proof. $(V(H))^*(\mathcal{I}, \mathcal{T}_A) =$
$(V(H))^*(\mathcal{I} \vee \mathcal{I}, \mathcal{T}_A) = (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{I})) \cap (V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{I}))$
= $(V(H))^*(\mathcal{I}, \mathcal{T}_A^*(\mathcal{I}))$
$\mathcal{T}_A^{**} = \mathcal{T}_A^*(\mathcal{I}))^*(\mathcal{I}) = \mathcal{T}_A^*$. □

7 Compatibility of $\mathcal{T}_A$ with $\mathcal{I}$

This section introduces the definition of compatibility of $\mathcal{T}_A$ with the graph ideal $\mathcal{I}$ and explores its significance.

Definition 7.1. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. The graph adjacency topology $\mathcal{T}_A$ is said to be compatible with the graph ideal $\mathcal{I}$, denoted $\mathcal{T}_A \backsim \mathcal{I}$, if the following holds for every vertex induced subgraph H of G :*
if for every $v \in V(H)$, there exists a $U \in N(v)$ such that $U \cap V(H) = V(G')$, for some $G' \in \mathcal{I}$, then $V(H) = V(G'')$, for some $G'' \in \mathcal{I}$.

Example 7.1. *Consider the following graph*



$S_N = \{\{4\}, \{3\}, \{2, 4\}, \{1\}\}$.
$\mathsf{B} = \{\phi, \{4\}, \{3\}, \{2, 4\}, \{1\}\}$.
$\mathcal{T}_A = \{\phi, \{4\}, \{3\}, \{2, 4\}, \{1\}, \{3, 4\}, \{1, 4\}, \{2, 3, 4\}, \{1, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$.
$N(1) = \{\{1\}, \{1, 4\}, \{1, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$.
$N(2) = \{\{2, 4\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$.
$N(3) = \{\{3\}, \{3, 4\}, \{2, 3, 4\}, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$.
$N(4) = \{\{4\}, \{2, 4\}, \{3, 4\}, \{1, 4\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$.
$\mathcal{T}_A$ is compatible with the graph ideal $\mathcal{I} = \{(\{1\}, \phi), (\{2\}, \phi), (\{1, 2\}, \phi)\}$ and not compatible with the graph ideal $\mathcal{I} = \{(\{1\}, \phi), (\{4\}, \phi), (\{1, 4\}, \phi)\}$.

Observation 7.2. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. Then the following are implied by $\mathcal{T}_A \backsim \mathcal{I}$:*

(a) For every vertex induced subgraph H of G, $V(H) \cap (V(H))^* = \phi \Rightarrow (V(H))^* = \phi$.

(b) For every vertex induced subgraph H of G, $(V(H) - (V(H))^*)^* = \phi$.

(c) For every vertex induced subgraph H of G, $(V(H) \cap (V(H))^*)^* = (V(H))^*$.

(d) $\beta(\mathcal{I}, \mathcal{T}_A^*) = \{U - W : U \in \mathcal{T}_A, W = V(G'), \text{for some } G' \in \mathcal{I}\} = \mathcal{T}_A^*$.

Theorem 7.1. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$. Then the following are equivalent:*
(1) $\mathcal{T}_A \backsim \mathcal{I}$
(2) For every vertex induced subgraphs H of G, $V(H) \cap (V(H))^* = \phi \Rightarrow V(H) = V(G')$, for some $G' \in \mathcal{I}$.
(3) For every vertex induced subgraphs H of G, $V(H) - (V(H))^* = \phi$ or $V(H) - (V(H))^* = V(G')$, for some $G' \in \mathcal{I}$.
(4) For every $\mathcal{T}_A^*$ closed subgraph H, $V(H) - (V(H))^* = V(G')$ for some $G' \in \mathcal{I}$.
(5) For every vertex induced subgraphs H of G, if V(H) contains no induced subgraph W with $V(W) \subseteq (V(W))^*$, then $V(H) = V(G')$ for some $G' \in \mathcal{I}$.

Proof. (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (3): For every vertex induced subgraphs H of G, $V(H) - (V(H))^* = \phi$ or $(V(H)-(V(H))^*)\cap(V(H)-(V(H))^*)^* = \phi$. So by (2), $V(H)-(V(H))^* = \phi$ or $V(H) - (V(H))^* = V(G')$, for some $G' \in \mathcal{I}$.
(3) $\Rightarrow$ (4) is straightforward.
(4) $\Rightarrow$ (1): Let H be a vertex induced subgraph of G and assume that for every v $\in$ V(H), there exists U $\in$ N(v) such that $U \cap V(H) = V(G')$ for some $G' \in \mathcal{I}$.
Then $V(H) \cap (V(H))^* = \phi$.
Since $(V(H)\cup(V(H))^*)^* = (V(H))^*\cup((V(H))^*)^* \subseteq (V(H))^*\cup V(H), V(H)\cup (V(H))^*$ is $\mathcal{T}_A^*$ closed.
So by (4), $(V(H) \cup (V(H))^*) - (V(H) \cup (V(H))^*)^* = V(G')$ for some $G' \in \mathcal{I}$.
But $(V(H)\cup(V(H))^*)-(V(H)\cup(V(H))^*)^* = (V(H)\cup(V(H))^*)-((V(H))^*\cup (V(H))^{**}) = (V(H) \cup (V(H))^*) - (V(H))^* = V(H)$.
Hence $V(H) = V(G')$, for some $G' \in \mathcal{I}$.
So $\mathcal{T}_A \backsim \mathcal{I}$
(3) $\Rightarrow$ (5): Let H be a vertex induced subgraph of G and assume that V(H) contains no induced subgraph W with $V(W) \subseteq (V(W))^*$.
Since $V(H)-(V(H))^* = V(G')$, for some $G' \in \mathcal{I}, V(H)\cap(V(H))^* \subseteq (V(H)\cap (V(H))^*)^*, V(H) \cap (V(H))^* = \phi$.
So $V(H) = V(H) - (V(H))^*$ and $V(H) = V(G')$, for some $G' \in \mathcal{I}$.
(5) $\Rightarrow$ (3) : Since $(V(H)-(V(H))^*)\cap(V(H)-(V(H))^*)^* = \phi, V(H)-(V(H))^*$ contains no vertex induced subgraph W such that $V(W) \subseteq (V(W))^*$.
So by (5), $V(H) - (V(H))^* = V(G')$ for some $G' \in \mathcal{I}$.

□

Theorem 7.2. *Let $(V(G), \mathcal{T}_A)$ be a graph adjacency topological space with a graph ideal $\mathcal{I}$ and $\mathcal{T}_A \backsim \mathcal{I}$. Let H be a vertex induced subgraph of G. If V(H) is $\mathcal{T}_A^*$ closed then it is the union of V(W), where $(V(W))^* \subseteq cl(V(W))$, and $V(G')$ for some $G' \in \mathcal{I}$.*

Proof. Let V(H) be $\mathcal{T}_A^*$ closed. Then $(V(H))^* \subseteq V(H)$. So $V(H) = (V(H) - (V(H))^*) \cup (V(H))^*$.
By theorem **??**(3), $V(H) - (V(H))^* = V(G')$ for some $G' \in \mathcal{I}$.
Also $(V(H))^* \subseteq cl(V(H))$.
So V(H)is the union of V(W), where $(V(W))^* \subseteq cl(V(W))$, and $V(G')$ for some $G' \in \mathcal{I}$. □

8 Conclusions

Graph ideal and graph local function in a graph adjacency topological space have been defined and the basic facts concerning graph local function have been proved. The method of generating a new topology from the older one using Kuratowski closure operator via graph ideal has been discussed. Further, the characteristics of the open sets of the new topology in terms of the closure operator have been analyzed. New topologies generated by two different ideals have been compared. The feature of graph location function based on the union of two ideals has been studied. The compatibility of the graph adjacency topology with graph ideal has been defined and the equivalent conditions for compatibility of the graph adjacency topology with graph ideal have been investigated. Further, the results in this paper are useful in the study of some new sets and topologies in graph adjacency topological space with graph ideal. These concepts can be studied further by stating graph local function of induced subgraphs of a graph with respect to graph adjacency topology using graph prime ideal and graph principal ideal. The open sets of topologies generated via prime ideals and principal ideals can be compared and investigated in future studies.

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