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Reach Energy of Digraphs

V. Mahalakshmi*
B. Vijaya Praba†
K. Palani‡

Abstract

A Digraph D consists of two finite sets $(V,\mathcal{A})$, where V denotes the vertex set and $\mathcal{A}$ denotes the arc set. For vertices $u,v \in V$, if there exists a directed path from u to v then v is said to be reachable from u and vice versa. The Reachability matrix of D is the $n \times n$ matrix $R(D) = [r_{ij}]$, where $r_{ij} = 1$, if v_j is reachable from v_i and $r_{ij} = 0$ otherwise. The eigen values corresponding to the reachability matrix are called reach eigen values. The reach energy of a digraph is defined by $E_R(D) = \sum_{i=1}^{n} |\lambda_i|$where λ_i is the eigen value of the reachability matrix. In this paper we introduce the reach spectrum of a digraph and study its properties and bounds. Moreover, we compute reach spectrum for some digraphs.

Keywords: Reachable, Reachability matrix, reach eigen values, reach spectrum, Reach energy.

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*Assistant Professor (PG & Research Department of Mathematics, A.P.C. Mahalaxmi College for Women Thoothukudi, India); Email - mahalakshmi@apcmcollege.ac.in
†Research Scholar Reg.No: 21212012092002 (PG & Research Department of Mathematics, A.P.C. Mahalaxmi College for Women Thoothukudi, India); Email - vijayapraba.b@gmail.com
‡ Third Author's Affiliation (PG & Research Department of Mathematics, A.P.C. Mahalaxmi College for Women Thoothukudi, India); Email - palani@apcmcollege.ac.in
Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamilnadu, India;
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1. Introduction

In this paper we considered simple and connected graph. A directed graph or digraph D consists of two finite sets $(V, \mathcal{A})$ where V denotes the vertex set and $\mathcal{A}$ denotes the arc set. For two vertices u and v, an arc from u to v is denoted by uv. Two vertices u and v is said to be adjacent if either $uv \in \mathcal{A}$ or $vu \in \mathcal{A}$. In 1978 Gutman [4] defined the energy of a simple graph as the sum of the absolute values of its eigen values and it is denoted by $E(G)$. i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$. The concept of graph energy was extended to digraph by Pena and Rada [8] and Adiga et al. [1]. Khan et al. [5] defined a new notion of energy of digraph called iota energy. In this paper, we investigate the properties and some bounds on reach energy.

Definition 1.1. A path is said to be directed path in which all the edges are directed either in clockwise or in anticlockwise direction and it is denoted as $\overrightarrow{P_n}$. Let $\{v_1, v_2, \dots, v_n\}$ be the vertex set of a directed path. Then the set $\{v_i v_{i+1} |\ i = 1,2,\dots, n-1\}$ is the arc set of $\overrightarrow{P_n}$.

Definition 1.2. A path is said to be alternate path in which the edges are given alternate direction and it is denoted as $\overrightarrow{AP_n}$

Definition 1.3. A star graph $K_{1,n}$ in which all the edges are directed towards the root vertex is called an instar and is denoted as $\overrightarrow{\iota K_{1,n}}$

Definition 1.4. A star graph $K_{1,n}$ in which all the edges are directed away from the root vertex is called an outstar and is denoted as $\overrightarrow{oK_{1,n}}$

2. Reach Energy

Definition 2.1. Let $D = (V, A)$ be a directed graph with n vertices. The reachability matrix [2] $R(D) = [r_{ij}]$ is the $n \times n$ matrix with $r_{ij} = 1$, if v_j is reachable from v_i and $r_{ij} = 0$ otherwise. We assume that each vertex is reachable from itself. The characteristic polynomial of $R(D) = [r_{ij}]$ is denoted by $f(D, \lambda) = det(R(D) - I\lambda)$. Let $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be the reach eigen values of D. The reach eigen values of the graph D are the eigen values of $R(D)$ and is called as reach spectrum of D. The spectrum of D is denoted by

$$spec\ D = \begin{Bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ m_1 & m_2 & \dots & m_n \end{Bmatrix}$$

where m_i is the algebraic multiplicity of the eigen values λ_i, for $1 \leq i \leq n$

Then the reach energy of D is defined as the sum of absolute values of reach spectrum of D.

i.e., $E_R(D) = \sum_{i=1}^{n} |\lambda_i|$

Example 2.2.

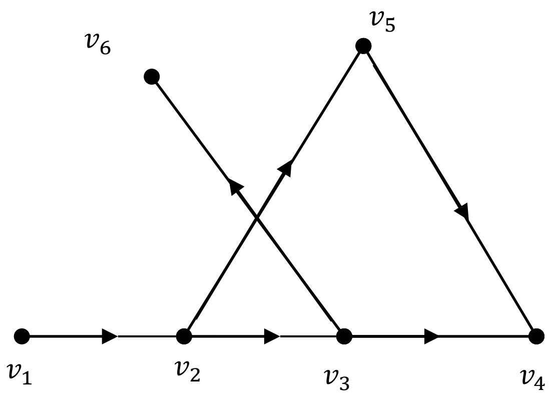

Figure 1.

Reachability matrix is

$$R(D) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Characteristic polynomial of $R(D)$ is given by
$f(D,\lambda) = \lambda^6 - 6\lambda^5 + 15\lambda^4 - 20\lambda^3 + 15\lambda^2 - 6\lambda + 1$.
Hence, the reach spectrum is $\begin{Bmatrix} 1 \\ 6 \end{Bmatrix}$.
Therefore, the reach energy of D is $E_R(D) = 6$.

3. Reach Energy of Some Graphs

Theorem 3.1. Directed path and alternate path attains same Reach Energy.
Proof: Let $P_n(D)$ be the directed path with vertex set $V = \{v_1, v_2, \dots, v_n\}$
Let $AP_n(D)$ be the alternate path with vertex set $V = \{v_1', v_2', \dots, v_n'\}$.
The reachability matrix of $P_n(D)$ is in the upper triangular matrix form with the entries 1.
The reachability matrix of $AP_n(D)$ is of the form

$$R(AP_n(D)) = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

Characteristic polynomial of P_n is $f(P_n(D),\lambda) = \det(R(P_n(D)) - \lambda\, I_n)$

$$f(P_n(D),\lambda) = \begin{vmatrix} 1-\lambda & 1 & \cdots & 1 \\ 0 & 1-\lambda & 1 & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & \cdots & 0 & 1-\lambda \end{vmatrix}$$

$= (-1)^n\,(\lambda-1)^n$.

Characteristic polynomial of AP_n is $f(AP_n(D),\lambda) = \det(R(AP_n(D)) - \lambda\, I_n)$

$$f(AP_n(D),\lambda) = \begin{vmatrix} 1-\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1-\lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 1 & 0 \\ \vdots & \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1-\lambda \end{vmatrix}$$

$= (-1)^n\,(\lambda-1)^n$.
Clearly, $f(P_n(D),\lambda) = f(AP_n(D),\lambda)$.

Since the characteristic polynomial of $P_n(D)$ and $AP_n(D)$ are same, Spectrum of $R(P_n)$ and $R(AP_n)$ are same.
Hence, the Reach Spectrum of $R(P_n(D))$ and $R(AP_n(D))$ are $\begin{Bmatrix} 1 \\ n \end{Bmatrix}$ and its Reach Energy is

$$E_R(P_n(D)) = E_R(AP_n(D)) = \sum_{1}^{n} 1 = n$$

Therefore, the directed path and alternate path attains same Reach Energy.

Theorem 3.2. Reach energy of directed star is independent of its orientation.
Proof: Let $K_{1,n-1}(D)$ be the directed instar with vertex set $v_1, v_2, \ldots, v_{n-1}$
Let $K'_{1,n-1}$(D) be the directed outstar with vertex set $v_1', v_2', \ldots, v_{n-1}'$
The reachability matrix of $K_{1,n-1}(D)$ is of the form

$$R\left(K_{1,n-1}(D)\right) = I_n + \begin{pmatrix} 0 & 0_{1\times n-1} \\ J_{n-1\times 1} & 0_{n-1\times n-1} \end{pmatrix}$$

The reachability matrix of $K'_{1,n-1}(D)$ is of the form

$$R\left(K'_{1,n-1}(D)\right) = I_n + \begin{pmatrix} 0 & J_{1\times n-1} \\ 0_{n-1\times 1} & 0_{n-1\times n-1} \end{pmatrix}$$

Characteristic polynomial of $K_{1,n-1}(D)$ is

$f\left(K_{1,n-1}(D),\lambda\right) = \det(R\left(K_{1,n-1}(D)\right) - \lambda\, I_n)$

$$f\left(K_{1,n-1}(D),\lambda\right) = \begin{vmatrix} 1-\lambda & 0 & \cdots & 0 \\ 1 & 1-\lambda & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 1 & 0 & 0 & 1-\lambda \end{vmatrix}$$

$= (-1)^n\,(\lambda - 1)^n$

Characteristic polynomial of $K'_{1,n-1}(D)$ is

$f\left(K'_{1,n-1}(D),\lambda\right) = \det(R\left(K'_{1,n-1}(D)\right) - \lambda\, I_n)$

$$f\left(K'_{1,n-1}(D),\lambda\right) = \begin{vmatrix} 1-\lambda & 1 & \cdots & 1 \\ 0 & 1-\lambda & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1-\lambda \end{vmatrix}$$

$= (-1)^n\,(\lambda - 1)^n$.

Clearly, $f\left(K_{1,n-1}(D)\,,\lambda\right) = f\left(K'_{1,n-1}(D)\,,\lambda\right)$.

Since the characteristic polynomial of $K_{1,n-1}(D)$ and $K'_{1,n-1}(D)$ are same, Spectrum of $R(K_{1,n-1}(D)\,)$ and $R(K'_{1,n-1}(D)\,)$ are same.

Hence, the Reach Spectrum of $R(K_{1,n-1}(D)\,)$ and $R(K'_{1,n-1}(D)\,)$ are $\begin{cases}1\\ n\end{cases}$ and its Reach Energy

$$E_R\left(K_{1,n-1}(D)\right) = E_R\left(K'_{1,n-1}(D)\right) = \sum_{1}^{n} 1 = n$$

Therefore, the Reach Energy of directed star is independent of its orientation.

4. Properties of Reach Eigen values

Theorem 4.1: Let D be any digraph. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the reach eigen values of $R(D)$, then the following condition holds.

i. $\sum_{i=1}^{n} \lambda_i = n$

ii. $\sum_{i=1}^{n} {\lambda_i}^2 = n + \alpha + \beta$;

where $\alpha = \sum_{i>j} r_{ij} r_{ji}$ and $\beta = \sum_{i<j} r_{ij} r_{ji}$

Proof:

i. Sum of eigen values of $R(D)$ is same as the trace of $R(D)$.

i.e., $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} r_{ii}$;

Since each vertex is reachable from itself, all the diagonal entries must be 1.

$= 1 + 1 + 1 + \cdots + 1$ (n times)

Therefore,

$$\sum_{i=1}^{n} \lambda_i = n$$

ii. Since, the sum of squares of the eigen values of R is the trace of $[R(D)]^2$

$$\sum_{i=1}^{n} {\lambda_i}^2 = \sum_{i=1}^{n}\sum_{i=1}^{n} r_{ij}\, r_{ji}$$

$$= \sum_{i=j=1}^{n} r_{ii} r_{ii} + \sum_{i \neq j=1}^{n} r_{ij} r_{ji}$$

$$= \sum_{i=j} (r_{ii})^2 + \sum_{i>j} r_{ij} r_{ji} + \sum_{i<j} r_{ij} r_{ji}$$

$= n + \alpha + \beta$; where $\alpha = \sum_{i>j} r_{ij} r_{ji}$ and $\beta = \sum_{i<j} r_{ij} r_{ji}$

Therefore,

$$\sum_{i=1}^{n} {\lambda_i}^2 = n + \alpha + \beta$$

5. Bounds for Reach Energy

Theorem 5.1: Let D be a directed graph. Let Z be the absolute value of determinant of the reachability matrix R of D i.e., $Z = |\det R(D)|$

Then

$$n\sqrt{n + \alpha + \beta} \leq E_R(D) \leq \sqrt{(n + \alpha + \beta) + n(n-1)Z^{2/n}}$$

Proof:

We know that Cauchy Schwarz inequality is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i\right)^2 \left(\sum_{i=1}^{n} b_i\right)^2$$

Put $a_i = 1$, $b_i = |\lambda_i|$

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)^2 \left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

$$[E_R(D)]^2 \leq n^2(n + \alpha + \beta)$$

$$E_R(D) \leq n\sqrt{n + \alpha + \beta} \qquad (1)$$

Since arithmetic mean is not smaller than geometric mean, we have

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

$$= \left(\prod_{i=1}^{n}|\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$

$$= \prod_{i=1}^{n}|\lambda_i|^{\frac{2}{n}}$$

$$= \left|\prod_{i=1}^{n}\lambda_i\right|^{\frac{2}{n}}$$

$$= |\det R(D)|^{\frac{2}{n}} = Z^{\frac{2}{n}}$$

Therefore,

$$\sum_{i\neq j}|\lambda_i||\lambda_j| \geq n(n-1)\, Z^{\frac{2}{n}} \qquad (2)$$

Now consider,

$$[E_R(D)]^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)^2$$

$$= \sum_{i=1}^{n}|\lambda_i|^2 + \sum_{i\neq j}|\lambda_i||\lambda_i|$$

$$\geq (n+\alpha+\beta) + n(n-1)\, Z^{\frac{2}{n}});\ by\ (2)$$

Hence, $E_R(D) \geq \sqrt{(n+\alpha+\beta\,) + n(n-1)\, Z^{\frac{2}{n}}}$

From (1) and (2),

$$n\sqrt{n+\alpha+\beta} \leq E_R(D) \leq \sqrt{(n+\alpha+\beta) + n(n-1)Z^{2/n}}$$

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