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Relationship Between Weight Function and 1 – Norm

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Abstract

The δ function on a subset E of $\mathbb{R}$ is the function defined by $\delta(x) = \begin{cases} 0\ if\ x = 0 \\ 1\ if\ x \neq 0 \end{cases}$.

For x= $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we define $\delta(x) = (\delta(x_1), \delta(x_2), \dots, \delta(x_n))$. The Hamming weight $w(x)$ of x is the number of non – zero coordinates of x, where $x \in \mathbb{R}^n$. From this one could see that $w(x) = \|\delta(x)\|_1$, where $\| \quad \|_1$ is the 1 – norm of x given by $\|x\|_1 = \sum_{j=1}^{n} |x_j|$,where $= (x_1, x_2, \dots, x_n)$. This gives a relationship between the weight function and the 1 – norm.In this paper we establish certain properties of the weight function using the properties of norms.

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1. Introduction

Let $X = \mathbb{R}^n$, where $\mathbb{R}$ is the set of real numbers. Then, X is a vector space over $\mathbb{R}$ of dimension n. The Hamming weight function on X is the function $w_H : X \to \mathbb{R}$ given by $w_H(x)$ = number of non –zero co-ordinates of x,
For $x = (x(1), x(2), \dots, x(n)) \in \mathbb{R}^n$.
Thus w_H satisfies the conditions:
$w_H(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $w_H(x) = 0$ if and only if $x = 0$.(1)
$w_H(\alpha x) = w_H(x)$ for all $x \in \mathbb{R}^n$ and $0 \neq \alpha \in \mathbb{R}$. (2)
$w_H(x + y) \leq w_H(x) + w_H(y)$ for all $x, y \in \mathbb{R}^n$.(3)

A norm on $\mathbb{R}^n$ is a function $\| \ \| : X \to \mathbb{R}$ satisfying $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$ and if and only if $x = 0$(4)
$\|\alpha x\| = |\alpha| \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{R}$. (5)
and $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{R}^n$.(6)
We see that w_H satisfies the condition of a norm except the condition (5). Instead, it satisfies (2). We may call such a function a mininorm. Let us formalize the definition.

Definition:1.1 Let X be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$. A mininorm on X is a function $p: X \to \mathbb{R}$ satisfying the following conditions:

$p(x) \geq 0$ for all $x \in X$ and $p(x) = 0$ if and only if $x = 0$(7)

$p(\alpha x) = p(x)$ for all $x \in X$ and $0 \neq \alpha \in K$. (8)

$p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$. (9)

a vector space with a mininorm defined on it is called a mininormed spaces.
It is clear that w_H is a mininorm on $\mathbb{R}^n$.

2.The weight function and the 1- norm

The 1- norm or $\| \ \|_1$ on $\mathbb{R}^n$ is defined by

$\|x\|_1 = \sum_{j=1}^{n} |x(j)|$, where $x = (x(1), x(2), \dots, x(n))$.(10)

We cannot connect the weight function with the 1- norm using the δ - function.

The δ – function on $\mathbb{R}$ is defined by $\delta(x) = \begin{cases} 0 \ if \ x = 0 \\ 1 \ if \ x \neq 0. \end{cases}$(11)

The δ – function can be extended to $\mathbb{R}^n$ in the following way:

$\delta(x) = (\delta(x_1), \delta(x_2), \dots, \delta(x_n))$, (12)

where $x = (x(1), x(2), \dots, x(n))$.

This δ – function satisfies the following [1] :

$\delta(x) \geq 0$ for all $x \epsilon \mathbb{R}^n$ and $\delta(x) = 0$ if and only if $x = 0$(13)

$\delta(\alpha x) = \delta(x)$ for all $x \epsilon \mathbb{R}^n$ and $0 \neq \alpha \epsilon \mathbb{R}$. (14)

and $\delta(x+y) \leq \delta(x) + \delta(y)$ for all $x, y \epsilon \mathbb{R}^n$. (15)

Hence the partial order relation $\leq$ on $\mathbb{R}^n$ is defined as follows:

For $x = (x(1), x(2), \dots, x(n))$ and $y = (y(1), y(2), \dots, y(n))$ in $\mathbb{R}^n$,

$x \leq y$ if and only if $x(j) \leq y(j)$, $j = 1, 2, \dots, n$.(16)

Now let $x = (x(1), x(2), \dots, x(n)) \epsilon \mathbb{R}^n$.

Then, $\delta(x) = (\delta(x_1), \delta(x_2), \dots, \delta(x_n))$

Now, $\delta(x_j) = \begin{cases} 0 \; if \; x_j = 0 \\ 1 \; if x_j \neq 0. \end{cases}$

Hence $\| \delta(x) \|_1 = \sum_{j=1}^{n} |\delta(x_{(j)})|^2$ = number of non- zero components of x.

Thus, $\| \delta(x) \|_1 = w_H(x)$.(17)

This gives the connection between the Hamming weight function and the 1- norm, via the
δ – function.

3.Topological Properties of the δ – function

Proposition:3.1 The δ – function on $\mathbb{R}^n$ is bounded.

Proof: Let$x, \epsilon \mathbb{R}^n$.

$\| \delta(x) \|_1 = \| \delta(x_1), \delta(x_2), \dots, \delta(x_n) \|_1$

$= \sum_{j=1}^{n} |x(j)|$

$\leq n$, since $|\delta(x_{(j)})| \leq 1$ for all j.

Hence δ is bounded.

Proposition:3.2 The δ – function on $\mathbb{R}^n$ is not continuous.

Proof: First we show that w_H is not continuous,

Let $x, \epsilon \mathbb{R}^n$.

Then $\left\| \frac{1}{n} x \right\|_1 = \frac{1}{n} \|x\|_1 \rightarrow 0$ as $n \rightarrow \infty$.

That is, $\frac{1}{n} x \rightarrow 0$ in $\mathbb{R}^n$ with $\|x\|_1$.

But $\left\| w_H\left(\frac{1}{n}\,x\right)\right\|_1 = \| w_H(x)\|$ for all n.

So, $w_H\left(\frac{1}{n}\,x\right) \nrightarrow 0$.

Hence w_H is not continuous.

Now $w_H(x) = \| \delta(x)\|_1$ for all $x \in \mathbb{R}^n$.

Thus, $w_H = \| \quad \|_1 \circ \delta$, where $\circ$ denotes the composition of functions.

$\| \quad \|_1$ is continuous [3].

Suppose δ is continuous.

Hence $w_H$is continuous, since the composition of two continuous functions is continuous.

This is not possible.

Hence $\delta$is not continuous ■

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