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On the effect of doubling of intervals on the 0 - distributive property of the lattice of weak congruences of chains

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Abstract

Alan Day's doubling construction of intervals has been found to affect some properties of the lattice of weak congruences of chains. Here, in this paper, we study how it affects the property of 0-distributivity of the lattice of weak congruences of chains.

Keywords: doubling construction in lattices, weak congruence lattices, 0-distributive lattices.

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1. Introduction

Vojvodić G. and Šešelja B. initiated the study of the concept of weak congruences of a lattice in the year 1988 [12]. J. C. Varlet [9] was the first to introduce the concept of 0-distributive lattices. Several other researchers made various contributions in different aspects of 0-distributivity. For example, one can refer to [1], [8], [11]. A. Veeramani [10] in his thesis, has studied about the lattice of weak congruences of a finite chain, Boolean lattices and the lattices C_n, M_3, N_5 first by considering 0 and 1 as non-constants and then considering them as constants, again by considering the Boolean lattice as an algebra and he studied about some weaker properties like 0-distributivity, 0-modularity, consistency, etc.

G. Gratzer constructed a new lattice L^U from a given lattice L by adding an element a^U called the double of a $\neq 0$ or 1 in L where $L^U = L \cup \{a^U\}$ with a new order denoted by $\leq^U$ [6]. Following that construction, A Day [3] introduced a similar construction L[I] by doubling an interval I of a given lattice L. After that it witnessed many developments, e.g., see [4], [5], [7]. Alan Day in [2] proved that a distributive lattice remains distributive when it is doubled by either a lower interval or an upper interval. In our present study, we analyse the effect of doubling of intervals on the property of 0-distributivity in the lattice of weak congruences of chains.

2. Preliminaries

Definition 2.1 [6] A lattice L satisfying the following identities

- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

for all $x, y, z \in L$ is called a *distributive lattice*. If not, it is a non-distributive lattice.

Definition 2.2 [4] A lattice L is said to be *0 - distributive*, if for all $x, y, z \in L$, whenever $x \wedge y = 0$ and $x \wedge z = 0$, then $x \wedge (y \vee z) = 0$.

Lemma 2.3 [10] *If L_n is a chain of n elements, then $C_W(L_n)$ is 0 - distributive.*

Definition 2.4 [6] An equivalence relation θ on a lattice L is said to be a *congruence relation* on L, if it is compatible with both meet and join, that is, for all $a, b, c, d \in L$, $a \equiv b\ (\theta)$ and $c \equiv d\ (\theta)$ imply that $a \vee c \equiv b \vee d\ (\theta)$ and $a \wedge c \equiv b \wedge d\ (\theta)$.

Definition 2.5 [12] A *weak congruence relation* on an algebra A is a symmetric and transitive sub-universe of A^2.

Note 2.6 The lattice of all weak congruence relations of L including ϕ with respect to the relation $\subseteq$ is denoted by $C_W(L)$. We consider 0 and 1 of L as non-constants in this paper.

Remark 2.7 [12] In $C_W(L)$, we have

- $[\phi, \Delta] \cong Sub(L)$, the lattice of all sublattices of L.

- $[\Delta, \tau\,] \cong Con(L)$, the lattice of all congruences of L.

Definition 2.8 [6] Let $I = [a, b]$ be an interval of a lattice L. The set $I \times C_2$ is formed using the two-element chain $C_2 = \{0,1\}$. The set $L[I] = (L \setminus I) \cup (I \times C_2)$ is the lattice given by the ordering: for $x, y \in L[I]$ and $i, j \in C_2$;
$x \leq y$ *if* $x \leq y$ *in* L;
$(x, i) \leq y$ *if* $x \leq y$ *in* L;
$x \leq (y, j)$ *if* $x \leq y$ *in* L;
$(x, i) \leq (y, j)$ *if* $x \leq y$ *in* L *and* $i \leq j$ *in* C_2.
$L[I]$ is the lattice got by doubling of the interval Iin$L$. This is *Day's definition of doubling of intervals.*

2. Results and Discussions

In this section, we examine whether $[C_W(L_n)](I)$ is 0 - distributive or not. It turns out that $[C_W(L_n)](I)$ remains 0-distributive in case of lower and upper intervals in $C_W(L_n)$, whereas in the case of an intermediate interval, 0 – distributivity gets affected.

Theorem 3.1 If L_n is a chain of n elements, then $[C_W(L_n)](I)$ is 0 – distributive where I is a lower interval in $C_W(L_n)$.
Proof. Let $L_n = \{ 0 \prec x_1 \prec x_2 \prec I \prec x_{n-1} = 1 \}$ be a chain of n elements. Let $C_W(L_n)$ be the lattice of all weak congruences of L_n. Let $I = [\phi, \theta]$ where θ is a proper congruence relation of L_n. Let $[C_W(L_n)](I)$ be the doubling of $C_W(L_n)$ by the interval I.
Let $A, B, C \in [C_W(L_n)](I)$, where I is a lower interval of $C_W(L_n)$ such that
$A \wedge B = (\phi, 0)$ and $A \wedge C = (\phi, 0)$.
To prove that, $[C_W(L_n)](I)$ is 0 – distributive. That is, we have to prove that
$A \wedge (B \vee C) = (\phi, 0)$ (3.1)
Suppose$(\{(x_i, x_i)\}, 0) \leq A \wedge (B \vee C) \Rightarrow (\{(x_i, x_i)\}, 0) \leq A$and
$(\{(x_i, x_i)\}, 0) \leq B \vee C \Rightarrow (\{(x_i, x_i)\}, 0) \leq A$ *and* $(\{(x_i, x_i)\}, 0) \leq B$ or
$(\{(x_i, x_i)\}, 0) \leq C$ or$(\{(x_i, x_i)\}, 0) \leq$ both B and C or incomparable with both.
Now,$A, B, C \in [C_W(L_n)](I) \Rightarrow A \in C_W(L_n) \setminus I$ or $A \in I \times C_2, B \in C_W(L_n) \setminus I$ or $B \in I \times C_2, C \in C_W(L_n) \setminus I$ or$C \in I \times C_2$.

The following cases arise:

i. $A, B, C \in C_W(L_n) \setminus I$
ii. $A, B \in C_W(L_n) \setminus I$ *and* $C \in I \times C_2$
iii. $A \in C_W(L_n) \setminus I$ *and* $B, C \in I \times C_2$
iv. $A, C \in C_W(L_n) \setminus I$ *and* $B \in I \times C_2$
v. $A, B, C \in I \times C_2$
vi. $A, B \in I \times C_2$ *and* $C \in C_W(L_n) \setminus I$
vii. $A, C \in I \times C_2$ *and* $B \in C_W(L_n) \setminus I$
viii. $A \in I \times C_2$ *and* $B, C \in C_W(L_n) \setminus I$

Case (i): Let $A, B, C \in C_W(L_n) \setminus I$. $A, B, C \in C_W(L_n) \setminus I$ implies that $A, B, C \in C_W(L_n)$. Therefore, (3.1) follows as $C_W(L_n)$ is 0- distributive.

Case (ii): Let $A, B \in C_W(L_n) \setminus I$ and $C \in I \times C_2$, that is, $C = (\theta_3, j)$ where j = either 0 or 1.
(I) Let$(\{(x_i, x_i)\}, 0) \leq B \Rightarrow \{(x_i, x_i)\} \leq B \Rightarrow (x_i, x_i) \in B \Rightarrow (x_i, x_i) \in A \cap B$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge B$. This is a contradiction, since $A \wedge B = (\phi, 0)$.
(II) Let $(\{(x_i, x_i)\}, 0) \leq C = (\theta_3, j) \Rightarrow \{(x_i, x_i)\} \subseteq \theta_3$.
Also, $\{(x_i, x_i)\} \subseteq A \Rightarrow \{(x_i, x_i)\} \subseteq A \wedge \theta_3 \Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \cap \theta_3, j) = A \wedge C$.
This is a contradiction, since $A \wedge C = (\phi, 0)$.
(III) Let $(\{(x_i, x_i)\}, 0) \leq B$ and $C \Rightarrow (x_i, x_i) \in B$ and$(x_i, x_i) \in \theta_3 \Rightarrow (x_i, x_i) \in A \cap B$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge B$. This is a contradiction, since $A \wedge B = (\phi, 0)$.
(IV) $(\{(x_i, x_i)\}, 0)$ is incomparable with both B and C.
But, $(\{(x_i, x_i)\}, 0) \leq B \vee C = B \vee (\theta_3, 0)$ implies $(x_i, x_i) \in B \vee \theta_3 \Rightarrow$ there exists a $x_k \neq x_i$ such that $(x_i, x_k) \in B \; or \; (x_i, x_k) \in \theta_3$. By symmetry and transitivity,
$(x_i, x_i) \in B$ or$(x_i, x_i) \in C$.Again we have $A \wedge B \neq (\phi, 0)$or $A \wedge C \neq (\phi, 0)$.This contradiction proves (3.1).

Case (iii): Let $A \in C_W(L_n) \setminus I$ and $B, C \in I \times C_2$.Let $B = (\theta_2, j)$ and $C = (\theta_3, j)$ where $j =$ either 0 or 1.
(I) Let $(\{(x_i, x_i)\}, 0) \leq B = (\theta_2, j) \Rightarrow \{(x_i, x_i)\} \subseteq \theta_2$.
Also, $(\{(x_i, x_i)\}, 0) \leq A \Rightarrow \{(x_i, x_i)\} \subseteq A \cap \theta_2$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \cap \theta_2, j) = A \wedge B$. This is a contradiction, since $A \wedge B = (\phi, 0)$.
(II) Let $(\{(x_i, x_i)\}, 0) \leq C = (\theta_3, j) \Rightarrow \{(x_i, x_i)\} \subseteq \theta_3$.Also, $(\{(x_i, x_i)\}, 0) \leq A$
$\Rightarrow \{(x_i, x_i)\} \subseteq A \cap \theta_3 \Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \cap \theta_3, j) = A \wedge C$.
This is a contradiction, since $A \wedge C = (\phi, 0)$.
(III) Let $(\{(x_i, x_i)\}, 0) \leq B$ and $C \Rightarrow \{(x_i, x_i)\} \subseteq \theta_2$and $\{(x_i, x_i)\} \subseteq \theta_3$
$\Rightarrow \{(x_i, x_i)\} \subseteq A \cap \theta_2$ and$\{(x_i, x_i)\} \subseteq A \cap \theta_3$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \cap \theta_2, 0) = A \wedge B$ and $(\{(x_i, x_i)\}, 0) \leq (A \cap \theta_3, j) = A \wedge C$
This is a contradiction, since $A \wedge B = (\phi, 0)$ and $A \wedge C = (\phi, 0)$.
(IV) $(\{(x_i, x_i)\}, 0)$ is incomparable with both B and C.
But, $(\{(x_i, x_i)\}, 0) \leq B \vee C = (\theta_2 \vee \theta_3, j)$ implies $(x_i, x_i) \in \theta_2 \vee \theta_3$
$\Rightarrow$ there exists a $x_k \neq x_i$such that $(x_i, x_k) \in \theta_2 \; or \; (x_i, x_k) \in \theta_3$
$\Rightarrow (x_i, x_i) \in \theta_2 \; or \; (x_i, x_i) \in \theta_3$, by symmetry and transitivity in θ_2 and θ_3.
$\Rightarrow (\{(x_i, x_k)\}, 0) \leq (\theta_2, 0) \; or \; (\{(x_i, x_k)\}, 0) \leq (\theta_3, 0)$.
That is, $(\{(x_i, x_k)\}, 0) \leq B \; or \; (\{(x_i, x_k)\}, 0) \leq C$.
Therefore, we have $(\{(x_i, x_i)\}, 0) \leq B \; or \; (\{(x_i, x_i)\}, 0) \leq C$.
This is a contradiction to our assumption. Therefore, (3.1) holds.

Case (iv): Let $A, C \in C_W(L_n) \setminus I$ and $B \in I \times C_2$, that is, $B = (\theta_2, j)$ where j = either 0 or 1.
(I) Let $(\{(x_i, x_i)\}, 0) \leq B = (\theta_2, j) \Rightarrow \{(x_i, x_i)\} \subseteq \theta_2$.
Also, $\{(x_i, x_i)\} \subseteq A \Rightarrow \{(x_i, x_i)\} \subseteq A \cap \theta_2 \Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \cap \theta_2, j) = A \wedge B$.
This is a contradiction, since $A \wedge B = (\phi, 0)$.
(II) Let $(\{(x_i, x_i)\}, 0) \leq C \Rightarrow \{(x_i, x_i)\} \leq C \Rightarrow (x_i, x_i) \in C \Rightarrow (x_i, x_i) \in A \cap C$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge C$. This is a contradiction, since $A \wedge C = (\phi, 0)$.
(III) Let $(\{(x_i, x_i)\}, 0) \leq B$ and $C \Rightarrow (x_i, x_i) \in \theta_2$and $(x_i, x_i) \in C \Rightarrow (x_i, x_i) \in A \cap C$
$\Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge C$. This is a contradiction, since $A \wedge C = (\phi, 0)$.

(IV) $(\{(x_i,x_i)\},0)$ is incomparable with both B and C.
But, $(\{(x_i,x_i)\},0) \le B \vee C = (\theta_2 \vee C, 0)$ implies $(x_i,x_i) \in \theta_2 \vee C \Rightarrow$ there exists a $x_k \neq x_i$ such that $(x_i,x_k) \in \theta_2$ *or* $(x_i,x_k) \in C$. By symmetry and transitivity, $(x_i,x_i) \in B$ *or* $(x_i,x_i) \in C$. This is a contradiction to our assumption. Therefore, (3.1) holds.

Case (v): Let $A,B,C \in I \times C_2$, that is, $A = (\theta_1, j)$, $B = (\theta_2, j)$, $C = (\theta_3, j)$ where j = either 0 or 1.
(I) Let $(\{(x_i,x_i)\},0) \le B = (\theta_2, j) \Rightarrow \{(x_i,x_i)\} \subseteq \theta_2$.
Also, $\{(x_i,x_i)\} \subseteq A \Rightarrow \{(x_i,x_i)\} \subseteq A \wedge \theta_2 \Rightarrow (\{(x_i,x_i)\},0) \le (A \wedge \theta_2, j) = A \wedge B$.
This is a contradiction, since $A \wedge B = (\phi, 0)$.
(II) Let $(\{(x_i,x_i)\},0) \le C = (\theta_3, j) \Rightarrow \{(x_i,x_i)\} \subseteq \theta_3$.
Also, $\{(x_i,x_i)\} \subseteq A \Rightarrow \{(x_i,x_i)\} \subseteq A \cap \theta_3 \Rightarrow (\{(x_i,x_i)\},0) \le (A \cap \theta_3, j) = A \wedge C$.
This is a contradiction, since $A \wedge C = (\phi, 0)$.
(III) Let $(\{(x_i,x_i)\},0) \le B$ *and* $C \Rightarrow \{(x_i,x_i)\} \subseteq \theta_2$ *and* $\{(x_i,x_i)\} \subseteq \theta_3$
$\Rightarrow \{(x_i,x_i)\} \subseteq A \cap \theta_2$ and $\{(x_i,x_i)\} \subseteq A \cap \theta_3$
$\Rightarrow (\{(x_i,x_i)\},0) \le (A \cap \theta_2, 0) = A \wedge B$ and $(\{(x_i,x_i)\},0) \le (A \cap \theta_3, j) = A \wedge C$.
This is a contradiction, since $A \wedge B = (\phi, 0)$ *and* $A \wedge C = (\phi, 0)$.
(IV) $(\{(x_i,x_i)\},0)$ is incomparable with both B and C.
But, $(\{(x_i,x_i)\},0) \le B \vee C = (\theta_2 \vee \theta_3, 0)$ implies $(x_i,x_i) \in \theta_2 \vee \theta_3$
$\Rightarrow$ there exists a $x_k \neq x_i$ such that $(x_i,x_k) \in \theta_2$ *or* $(x_i,x_k) \in \theta_3$
$\Rightarrow (\{(x_i,x_k)\},0) \le (\theta_2, 0)$ *or* $(\{(x_i,x_k)\},0) \le (\theta_3, 0)$.
That is, $(\{(x_i,x_k)\},0) \le B$ *or* $(\{(x_i,x_k)\},0) \le C$. By symmetry and transitivity in B and C, we have $(\{(x_i,x_i)\},0) \le B$ *or* $(\{(x_i,x_i)\},0) \le C$. This is a contradiction to our assumption. Therefore, (3.1) holds.

Case (vi): Let $A,B \in I \times C_2$, that is, $A = (\theta_1, j)$, $B = (\theta_2, j)$ where j = either 0 or 1 and $C \in C_W(L_n) \setminus I$.
(I) Let $(\{(x_i,x_i)\},0) \le B = (\theta_2, i) \Rightarrow \{(x_i,x_i)\} \subseteq \theta_2$.
Also, $\{(x_i,x_i)\} \subseteq A \Rightarrow \{(x_i,x_i)\} \subseteq A \wedge \theta_2 \Rightarrow (\{(x_i,x_i)\},0) \le (A \wedge \theta_2, j) = A \wedge B$.
This is a contradiction, since $A \wedge B = (\phi, 0)$.
(II) Let $(\{(x_i,x_i)\},0) \le C \Rightarrow \{(x_i,x_i)\} \le C \Rightarrow (x_i, x_i) \in C \Rightarrow (x_i,x_i) \in A \cap C$
$\Rightarrow (\{(x_i,x_i)\},0) \le A \wedge C$. This is a contradiction, since $A \wedge C = (\phi, 0)$.
(III) Let $(\{(x_i,x_i)\},0) \le B$ *and* $C \Rightarrow (x_i,x_i) \in \theta_2$ *and* $(x_i,x_i) \in C$
$\Rightarrow \{(x_i,x_i)\} \le A \cap C \Rightarrow (\{(x_i,x_i)\},0) \le A \wedge C$.
This is a contradiction, since $A \wedge C = (\phi, 0)$.
(IV) $(\{(x_i,x_i)\},0)$ is incomparable with both B and C.
But, $(\{(x_i,x_i)\},0) \le B \vee C = (\theta_2 \vee C, 0)$ implies $(x_i,x_i) \in \theta_2 \vee C \Rightarrow$ there exists a $x_k \neq x_i$ such that $(x_i,x_k) \in \theta_2$ *or* $(x_i,x_k) \in C$. By symmetry and transitivity, we have $(x_i,x_i) \in \theta_2$ *or* $(x_i,x_i) \in C$. This is a contradiction to our assumption.Therefore, (3.1) holds.

Case (vii): Let $A,C \in I \times C_2$, that is, $A = (\theta_1, j)$, $C = (\theta_3, j)$ where j = either 0 or 1 and $B \in C_W(L_n) \setminus I$.
(I) Let $(\{(x_i,x_i)\},0) \le B \Rightarrow \{(x_i,x_i)\} \le B \Rightarrow (x_i, x_i) \in B \Rightarrow (x_i,x_i) \in A \cap B$

$\Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge B$. This is a contradiction, since $A \wedge B = (\phi, 0)$.
(II) Let $(\{(x_i, x_i)\}, 0) \leq C = (\theta_3, j) \Rightarrow \{(x_i, x_i)\} \subseteq C$.
Also, $\{(x_i, x_i)\} \subseteq A \Rightarrow \{(x_i, x_i)\} \subseteq A \cap \theta_3 \Rightarrow (\{(x_i, x_i)\}, 0) \leq (A \wedge \theta_3, j) = A \wedge C$.
This is a contradiction, since $A \wedge C = (\phi, 0)$.
(III) Let $(\{(x_i, x_i)\}, 0) \leq B \; and \; C \Rightarrow (x_i, x_i) \in B \; and \; (x_i, x_i) \in \theta_3$
$\Rightarrow (x_i, x_i) \in A \cap B \Rightarrow (\{(x_i, x_i)\}, 0) \leq A \wedge B$. This is a contradiction, since $A \wedge B = (\phi, 0)$.
(IV) $(\{(x_i, x_i)\}, 0)$ is incomparable with both B and C.
But, $(\{(x_i, x_i)\}, 0) \leq B \vee C = (B \vee \theta_3, 0)$ implies $(x_i, x_i) \in B \vee \theta_3. \Rightarrow$ there exists a $x_k \neq x_i$ such that $(x_i, x_k) \in B \; or \; (x_i, x_k) \in \theta_3$. By symmetry and transitivity, we have $(x_i, x_i) \in B \; or \; (x_i, x_i) \in C$. This is a contradiction to our assumption. Therefore, (3.1) holds.

Case (viii): Let $A \in I \times C_2$, B and $C \in C_W(L_n) \setminus I$. Let $A = (\theta_1, j)$ where $j =$ either 0 or 1. This case follows, since $B \; and \; C \in C_W(L_n) \setminus I$ implies $B \; and \; C \in C_W(L_n)$ which is 0-distributive.
Hence, $[C_W(L_n)](I)$ is 0-distributive whenever I is a lower interval of $C_W(L_n)$.

Example 3.2 Consider the 0-distributive lattice $C_W(L_4)$ where L_4 is $\{0 \prec a \prec b \prec 1\}$.

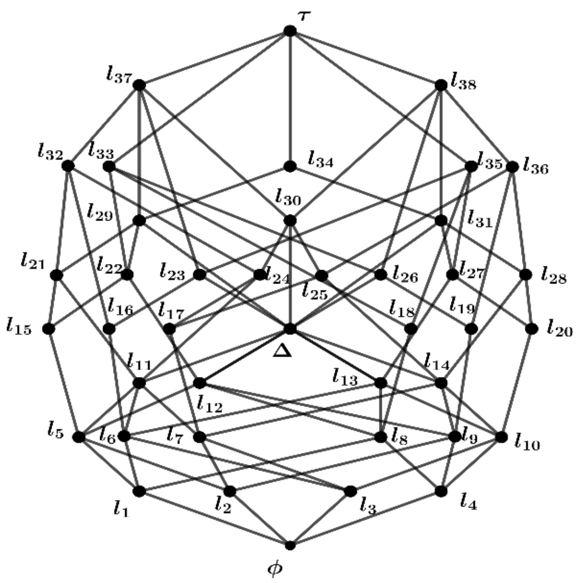

Figure 1. $C_W(L_4)$.

Consider the interval $I = [\phi, l_{32}]$ in the above lattice *Figure 1*. Let $C_2 = \{0,1\}$ be the two-element chain. We can form the new lattice $[C_W(L_4)](I) = \{C_W(L_4) \setminus I\} \cup (I \times C_2)$ given in *Figure 2*.

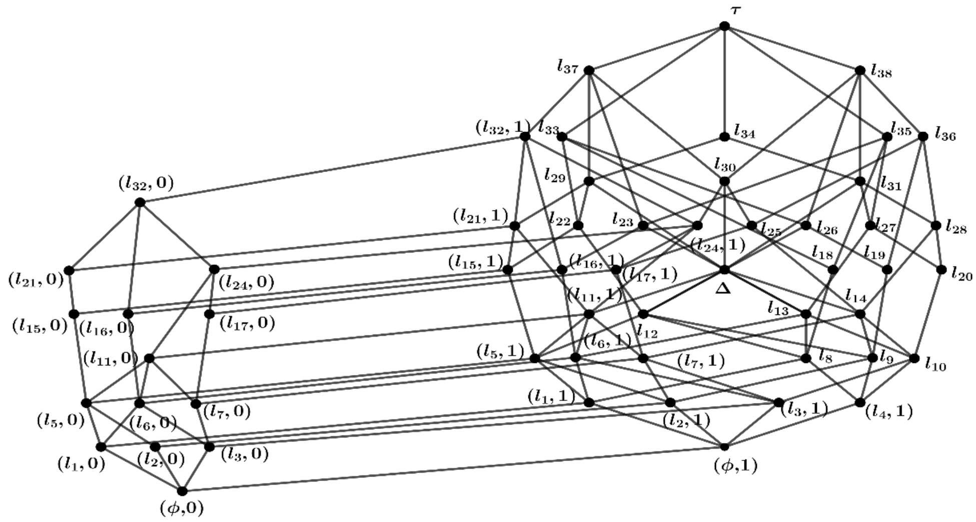

Figure 2. $[C_W(L_4)](I)$ where $I = [\phi, l_{32}]$.

Theorem 3.3 $[C_W(L_n)](I)$ *is 0 - distributive, when I is an upper interval of* $C_W(L_n)$.
Proof. Let $I = [\{(1,1)\}, \tau]$. Let $A, B, C \in [C_W(L_n)](I)$ such that $A \wedge B = \phi$, $A \wedge C = \phi$.
Claim: $A \wedge (B \vee C) = \phi$. (3.2)
There can be two possibilities, that is, A maybe in $I \times C_2$ or A maynot be in $I \times C_2$.
The following cases arise:

i. $A \in I \times C_2$ *and* $B, C \in C_W(L_n) \setminus I$
ii. $A \in C_W(L_n) \setminus I$ *and* $B, C \in C_W(L_n) \setminus I$
iii. $A \in C_W(L_n) \setminus I$ *and* $B, C \in I \times C_2$
iv. $A, B \in C_W(L_n) \setminus I$ *and* $C \in I \times C_2$

Case (i): Let $A \in I \times C_2$ and $B, C \in C_W(L_n) \setminus I$.
We note that either $A \wedge (B \vee C) \in I \times C_2$ or $A \wedge (B \vee C) \notin I \times C_2$.
Suppose $A \wedge (B \vee C) \neq \phi$. Therefore, there exists $\{(x_i, x_i)\} \leq A \wedge (B \vee C)$
$\Rightarrow \{(x_i, x_i)\} \leq A$ and $\{(x_i, x_i)\} \leq B \vee C \Rightarrow \{(x_i, x_i)\} \leq A$ and $\{(x_i, x_i)\} \leq B$ or $\{(x_i, x_i)\} \leq C$ or $\{(x_i, x_i)\} \leq$ both B and C or incomparable with both.
$\Rightarrow \{(x_i, x_i)\} \leq A$ *and* $\{(x_i, x_i)\} \leq B$ which is a contradiction, since $A \wedge B \neq \phi$.
Similarly, we get a contradiction, when $\{(x_i, x_i)\} \leq A$ *and* $\{(x_i, x_i)\} \leq C$ and when $\{(x_i, x_i)\} \leq A$ and $\{(x_i, x_i)\} \leq B$ & C.So, $A \wedge (B \vee C) = \phi$.
When $\{(x_i, x_i)\} \nleq both\ B$ & C, then there exists x_k such that $\{(x_i, x_k)\} \leq B$ or $\{(x_i, x_k)\} \leq C \Rightarrow (x_i, x_k) \in B\ or\ (x_i, x_k) \in C$. So, by symmetry and transitivity, we have $(x_i, x_i) \in B$.So, $A \wedge B \geq \{(x_i, x_i)\}$, a contradiction again. So, (3.2) holds.

Case (ii): Let $A \in C_W(L_n) \setminus I$ and $B, C \in C_W(L_n) \setminus I$. As $C_W(L_n)$ is 0 - distributive, it follows that $A \wedge (B \vee C) = \phi$.

Case (iii): Let $A \in C_W(L_n) \setminus I$ and $B, C \in I \times C_2$. Suppose $A \wedge (B \vee C) \neq \phi$.

Therefore, there exists $(x_i, x_i) \leq A \wedge (B \vee C)$.As in Case *(i)*,$A \wedge (B \vee C) = \phi$ follows.

Case (iv): Let $A, B \in C_W(L_n) \setminus I$ and $C \in I \times C_2$. Suppose $A \wedge (B \vee C) \neq \phi$.
Therefore, there exists $(x_i, x_i) \leq A \wedge (B \vee C)$. As in Case *(i)*,$A \wedge (B \vee C) = \phi$ follows.
Hence, for an upper interval I, $[C_W(L_n)](I)$ is 0-distributive.

Example 3.4 Consider the interval $[l_1, \tau]$ in *Figure 1*. The lattice formed by the doubling of the interval is given in *Figure 3*.

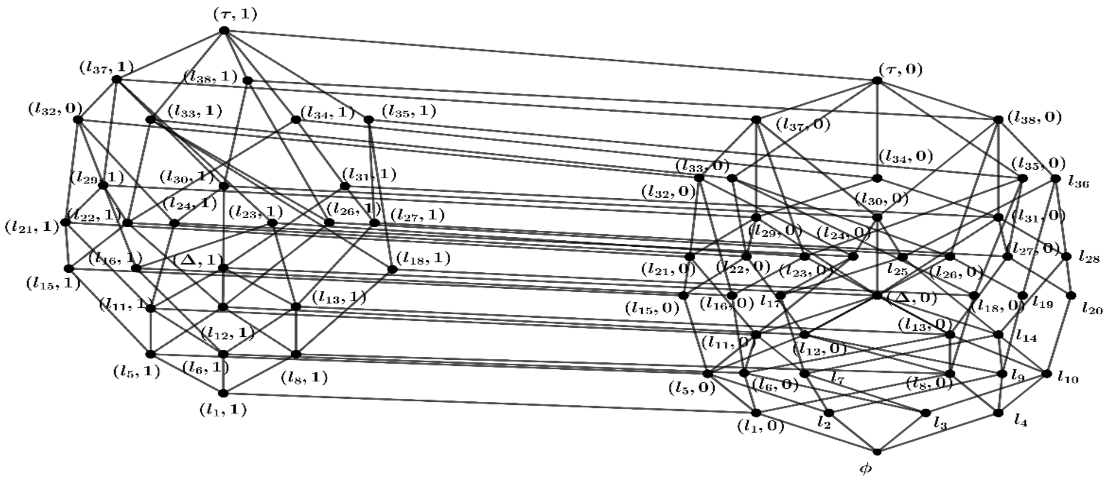

Figure 3. $[C_W(L_4)](I)$ where $I = [l_1, \tau]$.

Remark 3.5 The property of 0 - distributivity doesn't hold if we consider an intermediate interval I of $[C_W(L_n)]$.

Example 3.6 Consider the intermediate interval $I = [l_4, l_{20}]$ in Figure 1. Consider the elements $(l_4, 1)$ and $(l_{10}, 0)$ disjoint with l_3, that is, $l_3 \wedge (l_4, 1) = \phi$ and $l_3 \wedge (l_{10}, 0) = \phi$. Now, $l_3 \wedge [(l_4, 1) \vee (l_{10}, 0)] = l_3 \wedge (l_{10}, 1) = l_3 \neq \phi$.

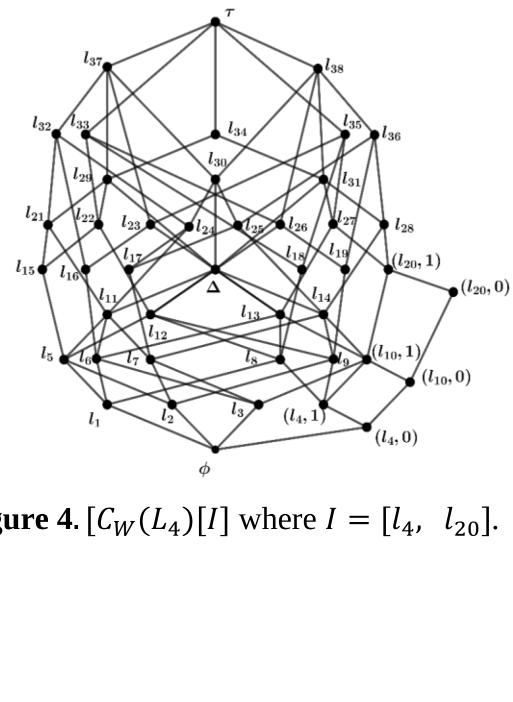

Figure 4. $[C_W(L_4)[I]$ where $I = [l_4, \ l_{20}]$.

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