Further Diversification of Nano Binary Open Sets

J. Jasmine Elizabeth[1]
G. Hari Siva Annam[2]

Abstract

The purpose of this paper is to introduce and study the nano binary exterior, nano binary border and nano binary derived in nano binary topological spaces. Also studied their characterizations.

Keywords: N_B- Derived, N_B- Exterior, N_B- Border.

2010 AMS Subject Classification: 54A05, 54A99[3].

[1]Assistant Professor, Kamaraj College, Thoothukudi, (Part Time Research Scholar [Reg no. 19122102092008], Manonmaniam Sundaranar University, Tirunelveli-627012) Tamil Nadu, India. jasmineelizabeth89@gmail.com

[2]Assistant Professor, PG and Research Department of Mathematics, Kamaraj College, Thoothukudi-628003, Tamil Nadu, India. hsannam@yahoo.com. Affiliated to Manonmaniam Sundaranar University, Tirunelveli-627012, Tamil Nadu, India.

[3]Received on June 4th, 2022. Accepted on Sep 1st, 2022. Published on Nov 30th, 2022. doi: 10.23755/rm.v44i0.884. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

M. Lellis Thivagar [1] introduced the concept of nano topological space with respect to a subset X of a universe U. S. Nithyanantha Jothi and P. Thangavelu [2] introduced the concept of binary topological spaces. By combining these two concepts Dr. G. Hari Siva Annam and J. Jasmine Elizabeth [3] introduced nano binary topological spaces. In this paper we have introduced the nano binary border, nano binary derived and nano binary exterior in nano binary topological spaces. Also studied their properties and characterizations with suitable examples.

2. Preliminaries

Definition 2.1: [3] Let (U_1, U_2) be a non-empty finite set of objects called the universe and R be an equivalence relation on (U_1, U_2)named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(U_1, U_2, R)$is said to be the approximation space. Let $(X_1, X_2) \subseteq (U_1, U_2)$.

1. The lower approximation of (X_1, X_2) with respect to R is the set of all objects, which can be for certain classified as $(X_1, X_2)$with respect to R and it is denoted by $L_R(X_1, X_2)$.
That is,$L_R(X_1, X_2) = \cup_{(x_1,\ x_2)\epsilon(U_1,U_2)}\{R\,(x_1, x_2): R(x_1, x_2) \subseteq (X_1, X_2)\}$
Where $R(x_1, x_2)$denotes the equivalence class determined by (x_1, x_2).

2. The upper approximation of (X_1, X_2) with respect to R is the set of all objects, which can be possibly classified as $(X_1, X_2)$with respect to R and it is denoted by $U_R(X_1, X_2)$.
That is,$U_R(X_1, X_2) = \cup_{(x_1,x_2)\epsilon(U_1,U_2)}\{R\,(x_1, x_2): R(x_1, x_2) \cap (X_1, X_2) \neq \phi\}$.

3. The boundary region of (X_1, X_2) with respect to R is the set of all objects, which can be classified neither as (X_1, X_2) nor as not $-(X_1, X_2)$ with respect to R and it is denoted by $B_R(X_1, X_2)$.
That is, $B_R(X_1, X_2) = U_R(X_1, X_2) - L_R(X_1, X_2)$.

Definition 2.2: [3] Let (U_1, U_2) be the universe, R be an equivalence on (U_1, U_2) and $\tau_R(X_1, X_2) = \{(U_1, U_2), (\phi, \phi), L_R(X_1, X_2), U_R(X_1, X_2), B_R(X_1, X_2)\}$ where $(X_1, X_2) \subseteq (U_1, U_2)$. Then by the property R$(X_1, X_2)$satisfies the following axioms

1. $(U_1, U_2)$and $(\phi, \phi)\ \epsilon\ \tau_R(X_1, X_2)$.
2. The union of the elements of any sub collection of $\tau_R(X_1, X_2)$ is in $\tau_R(X_1, X_2)$.
3. The intersection of the elements of any finite sub collection of $\tau_R(X_1, X_2)$ is in $\tau_R(X_1, X_2)$.

That is,$\tau_R(X_1, X_2)$ is a topology on (U_1, U_2) called the nano binary topology on (U_1, U_2) with respect to(X_1, X_2).

We call $(U_1, U_2, \tau_R(X_1, X_2))$as the nano binary topological spaces. The elements of $\tau_R(X_1, X_2)$are called as nano binary open sets and it is denoted by N_B open sets. Their complement is called N_B closed sets.

Definition 2.3: [3] If $\big(U_1, U_2, \tau_R(X_1, X_2)\big)$is a nano binary topological spaces with respect to (X_1, X_2) and if $(H_1, H_2) \subseteq (U_1, U_2)$, then the nano binary interior of (H_1, H_2) is defined as the union of all $N_B$open subsets of (A_1, A_2) and it is defined by $N_B{}^{\circ}(H_1, H_2)$.

That is, $N_B{}^{\circ}(H_1, H_2)$ is the largest $N_B$open subset of(H_1, H_2). The nano binary closure of (H_1, H_2) is defined as the intersection of all $N_B$closed sets containing (H_1, H_2) and it is denoted by $\overline{N_B}(H_1, H_2)$.

That is, $\overline{N_B}(H_1, H_2)$ is the smallest $N_B$closed set containing(H_1, H_2).

3. Nano Binary Derived

Definition 3.1: A point $(x_1, x_2) \in (U_1, U_2)$ is said to be a N_B limit point of (A_1, A_2) if for each N_B-open set (K_1, K_2) containing (x_1, x_2) satisfies $(K_1, K_2) \cap \big((A_1, A_2) - (x_1, x_2)\big) \neq (\emptyset, \emptyset)$.

Definition 3.2: The set of all N_B limit points of (A_1, A_2) is said to be nano binary derived set and is denoted by $N_B\_D(A_1, A_2)$.

Theorem 3.3: In $(U_1, U_2, \tau_R(X_1, X_2))$, let (A_1, A_2) and (B_1, B_2) be two subsets of (U_1, U_2). Then the following holds:
1) $N_B\_D(\emptyset, \emptyset) = (\emptyset, \emptyset)$.
2) If $(x_1, x_2) \in N_B\_D(A_1, A_2)$ then $(x_1, x_2) \in N_B\_D((A_1, A_2) - (x_1, x_2))$.
3) If $(A_1, A_2) \subseteq (B_1, B_2)$, then $N_B\_D(A_1, A_2) \subseteq N_B\_D(B_1, B_2)$.
4) $N_B\_D(A_1, A_2) \cup N_B\_D(B_1, B_2) = N_B\_D((A_1, A_2) \cup (B_1, B_2))$.

Proof: 1) Let $(x_1, x_2) \in (U_1, U_2)$ and (G_1, G_2) be a N_B-open set containing (x_1, x_2). Then $\big((G_1, G_2) - (x_1, x_2)\big) \cap (\emptyset, \emptyset) = (\emptyset, \emptyset) \Rightarrow (x_1, x_2) \notin N_B\_D(\emptyset, \emptyset)$. Therefore, for any $(x_1, x_2) \in (U_1, U_2)$, (x_1, x_2) is not aN_B limit point of $(\emptyset, \emptyset)$. Hence $N_B\_D(\emptyset, \emptyset) = (\emptyset, \emptyset)$.
2)Let $(x_1, x_2) \in N_B\_D(A_1, A_2)$. Then $(G_1, G_2) \cap \big((A_1, A_2) - (x_1, x_2)\big) \neq (\emptyset, \emptyset)$, for every N_B-open set (G_1, G_2) containing (x_1, x_2) implies every N_B-open set (G_1, G_2) of (x_1, x_2), contains at least one point other than (x_1, x_2) of (A_1, A_2). Therefore $(x_1, x_2) \in N_B\_D((A_1, A_2) - (x_1, x_2))$.
3)Let $(x_1, x_2) \in N_B\_D(A_1, A_2)$. Then $(G_1, G_2) \cap \big((A_1, A_2) - (x_1, x_2)\big) \neq (\emptyset, \emptyset)$, for every N_B-open set (G_1, G_2) containing (x_1, x_2). Since $(A_1, A_2) \subseteq (B_1, B_2)$ implies $(G_1, G_2) \cap \big((B_1, B_2) - (x_1, x_2)\big) \neq (\emptyset, \emptyset) \Rightarrow (x_1, x_2) \in N_B\_D(B_1, B_2)$. Thus $(x_1, x_2) \in N_B\_D(A_1, A_2) \Rightarrow (x_1, x_2) \in N_B\_D(B_1, B_2)$. Therefore $N_B\_D(A_1, A_2) \subseteq N_B\_D(B_1, B_2)$.
4)Since $(A_1, A_2) \subseteq (A_1, A_2) \cup (B_1, B_2)$ and $(B_1, B_2) \subseteq (A_1, A_2) \cup (B_1, B_2)$. By (3), $N_B\_D(A_1, A_2) \subseteq N_B\_D\big((A_1, A_2) \cup (B_1, B_2)\big)$ and $N_B\_D(B_1, B_2) \subseteq N_B\_D((A_1, A_2) \cup$

$(B_1, B_2))$. Therefore, $N_B\_D(A_1, A_2) \cup N_B\_D(B_1, B_2) \subseteq N_B\_D((A_1, A_2) \cup (B_1, B_2))$…. (1). Let $(x_1, x_2) \notin N_B\_D(A_1, A_2) \cup N_B\_D(B_1, B_2)$. Then $(x_1, x_2) \notin N_{B_{D(A_1,A_2)}}$and $(x_1, x_2) \notin N_B\_D(B_1, B_2)$. Therefore, there exists N_B-open sets (G_1, G_2) and (H_1, H_2) containing (x_1, x_2) such that $(G_1, G_2) \cap ((A_1, A_2) - (x_1, x_2)) = (\emptyset, \emptyset)$ and $(H_1, H_2) \cap ((B_1, B_2) - (x_1, x_2)) = (\emptyset, \emptyset)$. Since $(G_1, G_2) \cap (H_1, H_2) \subseteq (G_1, G_2)$ and (H_1, H_2), $((G_1, G_2) \cap (H_1, H_2)) \cap ((A_1, A_2) - (x_1, x_2)) = (\emptyset, \emptyset)$ and $((G_1, G_2) \cap (H_1, H_2)) \cap ((B_1, B_2) - (x_1, x_2)) = (\emptyset, \emptyset)$. Also $(G_1, G_2) \cap (H_1, H_2)$ is a N_B-open set containing(x_1, x_2). Therefore, $((G_1, G_2) \cap (H_1, H_2)) \cap (((A_1, A_2) \cup (B_1, B_2)) - (x_1, x_2)) = (\emptyset, \emptyset)$. That is, (x_1, x_2) is not a N_B limit point of $(A_1, A_2) \cup (B_1, B_2)$. Hence $(x_1, x_2) \notin N_B\_D((A_1, A_2) \cup (B_1, B_2))$. Therefore, $N_B\_D((A_1, A_2) \cup (B_1, B_2)) \subseteq N_B\_D(A_1, A_2) \cup N_B\_D(B_1, B_2)$…. (2). From (1) and (2),$N_B\_D(A_1, A_2) \cup N_B\_D(B_1, B_2) = N_B\_D((A_1, A_2) \cup (B_1, B_2))$.

Theorem 3.4: Let $(A_1, A_2)$and (B_1, B_2) be two subsets of N_B topological space $(U_1, U_2, \tau_R(X_1, X_2))$.Then $N_B\_D((A_1, A_2) \cap (B_1, B_2)) \subseteq N_B\_D(A_1, A_2) \cap N_B\_D(B_1, B_2)$.

Proof: Since $(A_1, A_2) \cap (B_1, B_2) \subseteq (A_1, A_2)$and $(A_1, A_2) \cap (B_1, B_2) \subseteq (B_1, B_2)$. By the previous theorem,$N_B_D((A_1, A_2) \cap (B_1, B_2)) \subseteq N_B_D(A_1, A_2)$and $N_B\_D((A_1, A_2) \cap (B_1, B_2)) \subseteq N_B\_D(B_1, B_2)$. Therefore,$N_B\_D((A_1, A_2) \cap (B_1, B_2)) \subseteq N_B\_D(A_1, A_2) \cap N_B\_D(B_1, B_2)$.

Remark 3.5: The reverse inclusion may not true as shown in the following example.

Example 3.6: $U_1 = \{a, b, c\}$, $U_2 = \{1, 2\}$ with$(U_1, U_2)/R = \{(\{a, b\}, \{2\}), (\{c\}, \{1\})\}$. Let (X_1, X_2) =$(\{b\}, \{2\})$. Then $\tau_R(X_1, X_2)$ =$\{(\Phi, \Phi), (U_1, U_2), (\{a, b\}, \{2\})\}$. Here $(A_1, A_2) = (\{a, b\}, \{1\})$and $(B_1, B_2) = (\{b, c\}, \{1,2\})$, $(A_1, A_2) \cap (B_1, B_2)$=$(\{b\}, \{1\})$ and hence $N_B\_D((A_1, A_2) \cap (B_1, B_2))$=$\{(\{a\}, \{1\})$, $(\{a\}, \{2\})$, $(\{b\}, \{2\})$, $(\{c\}, \{1\}), (\{c\}, \{2\})\}$. Also $N_B\_D(A_1, A_2)$=$\{(\{a\}, \{1\})$, $(\{a\}, \{2\})$, $(\{b\}, \{1\})$, $(\{b\}, \{2\}), (\{c\}, \{1\}), (\{c\}, \{2\})\}$ and $N_B\_D(B_1, B_2)$=$\{(\{a\}, \{1\})$, $(\{a\}, \{2\})$, $(\{b\}, \{1\})$, $(\{c\}, \{1\}), (\{c\}, \{2\})\}$. But $N_B\_D(A_1, A_2) \cap N_B\_D(B_1, B_2)$=$\{(\{a\}, \{1\})$, $(\{a\}, \{2\})$, $(\{b\}, \{1\})$, $(\{c\}, \{1\}), (\{c\}, \{2\})\}$.
Thus, $N_B\_D(A_1, A_2) \cap N_B\_D(B_1, B_2) \nsubseteq N_B\_D((A_1, A_2) \cap (B_1, B_2))$.

Theorem 3.7: $\overline{N_B}(A_1, A_2) = (A_1, A_2) \cup N_B\_D(A_1, A_2)$, where $(A_1, A_2) \subseteq (U_1, U_2)$.
Proof: If $(x_1, x_2) \in (A_1, A_2) \cup N_B\_D(A_1, A_2)$, Then $(x_1, x_2) \in (A_1, A_2)$ or $(x_1, x_2) \in N_B\_D(A_1, A_2)$. Let $(x_1, x_2) \notin (A_1, A_2)$. Then $(x_1, x_2) \in N_B\_D(A_1, A_2)$. Therefore, for every N_B-open set (G_1, G_2)containing (x_1, x_2), $(G_1, G_2) \cap ((A_1, A_2) - (x_1, x_2)) \neq (\emptyset, \emptyset)$. Since $(x_1, x_2) \notin (A_1, A_2), (G_1, G_2) \cap (A_1, A_2) \neq (\emptyset, \emptyset)$. Therefore, $(x_1, x_2) \in \overline{N_B}(A_1, A_2)$. Therefore, $(A_1, A_2) \cup N_B\_D(A_1, A_2) \subseteq \overline{N_B}(A_1, A_2)$…. (1). Let $(x_1, x_2) \in \overline{N_B}(A_1, A_2)$and $(x_1, x_2) \in (A_1, A_2)$. Then the result is obvious. If $(x_1, x_2) \in \overline{N_B}(A_1, A_2)$and $(x_1, x_2) \notin (A_1, A_2)$. Therefore, $(G_1, G_2) \cap (A_1, A_2) \neq (\emptyset, \emptyset)$ for every N_B-open set (G_1, G_2) containing (x_1, x_2) and hence $(G_1, G_2) \cap ((A_1, A_2) - (x_1, x_2)) \neq$

$(\emptyset,\emptyset)$. Therefore, $(x_1,x_2) \in N_B\_D(A_1,A_2)$ and hence $(x_1,x_2) \in (A_1,A_2) \cup N_B\_D(A_1,A_2)$. Therefore, $\overline{N_B}(A_1,A_2) \subseteq (A_1,A_2) \cup N_B\_D(A_1,A_2)$… (2). From (1) and (2), $\overline{N_B}(A_1,A_2) = (A_1,A_2) \cup N_B\_D(A_1,A_2)$.

Result 3.8: ${N_B}^o(A_1,A_2) = (A_1,A_2) - N_B\_D[(\mathrm{U}_1,\mathrm{U}_2) - (A_1,A_2)]$, where $(A_1,A_2) \subseteq (\mathrm{U}_1,\mathrm{U}_2)$.
Proof: By the previous theorem, $\overline{N_B}(A_1,A_2) = (A_1,A_2) \cup N_B\_D(A_1,A_2) \Rightarrow (\mathrm{U}_1,\mathrm{U}_2) - \overline{N_B}(A_1,A_2) = \big((\mathrm{U}_1,\mathrm{U}_2) - (A_1,A_2)\big) \cap \big((\mathrm{U}_1,\mathrm{U}_2) - N_B\_D(A_1,A_2)\big) \Rightarrow (\mathrm{U}_1,\mathrm{U}_2) - \overline{N_B}(A_1,A_2) = \big((\mathrm{U}_1,\mathrm{U}_2) - (A_1,A_2)\big) - N_B\_D(A_1,A_2) \Rightarrow {N_B}^o\big((\mathrm{U}_1,\mathrm{U}_2) - (A_1,A_2)\big) = \big((\mathrm{U}_1,\mathrm{U}_2) - (A_1,A_2)\big) - N_B\_D(A_1,A_2)$. By replacing $(\mathrm{U}_1,\mathrm{U}_2) - (A_1,A_2)$ by (A_1,A_2) and $(A_1,A_2)$by $(\mathrm{U}_1,\mathrm{U}_2) - (A_1,A_2)$, ${N_B}^o(A_1,A_2) = (A_1,A_2) - N_B\_D[(\mathrm{U}_1,\mathrm{U}_2) - (A_1,A_2)]$.

4. Nano Binary Exterior

Definition 4.1: For a subset$(A_1,A_2) \subseteq (U_1,U_2)$, the nano binary exterior of (A_1,A_2) is defined as${N_B}^o((U_1,U_2) - (A_1,A_2))$. It is denoted by $N_B\_E(A_1,A_2)$.

Definition 4.2: For a subset$(A_1,A_2) \subseteq (U_1,U_2)$, the nano binary border of (A_1,A_2) is defined as $(A_1,A_2) - {N_B}^o(A_1,A_2)$. It is denoted by $N_B\_B(A_1,A_2)$.

Theorem 4.3: Let $(A_1,A_2)$and (B_1,B_2) be two subsets of N_B topological space $(U_1,U_2,\tau_R(X_1,X_2))$. Then the following holds:
1) If $(A_1,A_2) \subseteq (B_1,B_2)$, then $N_B\_E(B_1,B_2) \subseteq N_B\_E(A_1,A_2)$.
2) $N_B\_E\big((A_1,A_2) \cup (B_1,B_2)\big) \subseteq N_B\_E(A_1,A_2) \cup N_B\_E(B_1,B_2)$.
3) $N_B\_E(A_1,A_2) \cap N_B\_E(B_1,B_2) \subseteq N_B\_E\big((A_1,A_2) \cap (B_1,B_2)\big)$.
Proof: 1) If $(A_1,A_2) \subseteq (B_1,B_2)$then $(U_1,U_2) - (B_1,B_2) \subseteq (U_1,U_2) - (A_1,A_2) \Rightarrow {N_B}^o\big((U_1,U_2) - (B_1,B_2)\big) \subseteq {N_B}^o\big((U_1,U_2) - (A_1,A_2)\big) \Rightarrow N_B\_E(B_1,B_2) \subseteq N_B\_E(A_1,A_2)$.
2) Since $(A_1,A_2) \subseteq (A_1,A_2) \cup (B_1,B_2)$and $(B_1,B_2) \subseteq (A_1,A_2) \cup (B_1,B_2)$. By (1), $N_B_E\big((A_1,A_2) \cup (B_1,B_2)\big) \subseteq N_B_E(A_1,A_2)$and $N_B\_E\big((A_1,A_2) \cup (B_1,B_2)\big) \subseteq N_B\_E(B_1,B_2)$. Therefore, $N_B\_E\big((A_1,A_2) \cup (B_1,B_2)\big) \subseteq N_B\_E(A_1,A_2) \cup N_B\_E(B_1,B_2)$.
3) Since $(A_1,A_2) \cap (B_1,B_2) \subseteq (A_1,A_2)$and $(A_1,A_2) \cap (B_1,B_2) \subseteq (B_1,B_2)$. By (1) $N_B_E(A_1,A_2) \subseteq N_B_E\big((A_1,A_2) \cap (B_1,B_2)\big)$and $N_B\_E(B_1,B_2) \subseteq N_B\_E\big((A_1,A_2) \cap (B_1,B_2)\big)$. Therefore, $N_B\_E(A_1,A_2) \cap N_B\_E(B_1,B_2) \subseteq N_B\_E\big((A_1,A_2) \cap (B_1,B_2)\big)$.

Remark 4.4: The inclusion may be strict. We can see in the following example.

Example 4.5: Let $\mathrm{U}_1 = \{a,b,c\}, \mathrm{U}_2 = \{1,2\}$ with $(\mathrm{U}_1,\mathrm{U}_2)/_\mathrm{R}\{(\{\mathrm{a,b}\},\{2\}),(\{\mathrm{c}\},\{1\})\}$. Let $(\mathrm{X}_1,\mathrm{X}_2)$ =$(\{\mathrm{b}\},\{2\})$. Then $\tau_\mathrm{R}(\mathrm{X}_1,\mathrm{X}_2)$ =$\{(\Phi,\Phi),(\mathrm{U}_1,\mathrm{U}_2),(\{\mathrm{a,b}\},\{2\})\}$.
2) Take $(A_1,A_2) = (\{a,b\},\{2\})$ *and* $(B_1,B_2) = (\{\mathrm{c}\},\{1\})$. $N_B\_E(\{a,b\},\{2\}) \cup N_B\_E(\{\mathrm{c}\},\{1\}) = {N_B}^o(\{\mathrm{c}\},\{1\}) - {N_B}^o(\{a,b\},\{2\}) = (\Phi,\Phi) - (\{a,b\},\{2\}) = (\{a,\ b\},\{2\})$. Also, $N_B\_E((\{a,b\},\{2\}) \cup (\{\mathrm{c}\},\{1\})) = N_B\_E(\mathrm{U}_1,\mathrm{U}_2) = {N_B}^o(\Phi,\Phi) =$

(Φ,Φ). Therefore, $(\{\mathrm{a,b}\},\{2\}) \nsubseteq (\Phi,\Phi)$ and hence $N_B\_E\big((A_1,A_2)\cup(B_1,B_2)\big) \subset N_B\_E(A_1,A_2)\cup N_B\_E(B_1,B_2)$.

3) Take $(A_1,A_2) = (\{c\},\{1,2\})$ *and* $(B_1,B_2) = (\{\mathrm{a,c}\},\{1\})$. $N_B_E((\{c\},\{1,2\})\cap(\{\mathrm{a,c}\},\{1\})) = N_B_E(\{\mathrm{c}\},\{1\}) = {N_B}^o(\{a,b\},\{2\}) = (\{a,b\},\{2\})$and $N_B\_E(\{c\},\{1,2\})\cap N_B\_E(\{\mathrm{a,c}\},\{1\}) = {N_B}^o(\{\mathrm{a,b}\},\{\emptyset\})\cap {N_B}^o(\{\mathrm{b}\},\{2\}) = (\Phi,\Phi)\cap(\Phi,\Phi) = (\Phi,\Phi)$. Therefore, $(\{\mathrm{a,b}\},\{2\}) \nsubseteq (\Phi,\Phi)$ and hence $N_B\_E(A_1,A_2)\cap N_B\_E(B_1,B_2) \subset N_B\_E((A_1,A_2)\cap(B_1,B_2))$.

Theorem 4.6: Let (A_1,A_2) and (B_1,B_2) be two subsets of N_B topological space $(U_1,U_2,\tau_R(X_1,X_2))$. Then the following holds:
1) $N_B\_E(A_1,A_2) = (U_1,U_2) - \overline{N_B}(A_1,A_2)$
2)$N_B\_E(N_B\_E(A_1,A_2)) = {N_B}^o\left(\overline{N_B}(A_1,A_2)\right)$
3) $N_B_E(U_1,U_2) = (\emptyset,\emptyset)andN_B_E(\emptyset,\emptyset) = (U_1,U_2)$
4) $N_B\_E(A_1,A_2) = N_B\_E[(U_1,U_2) - N_B\_E(A_1,A_2)]$
5) ${N_B}^o(A_1,A_2) \subseteq N_B\_E(N_B\_E(A_1,A_2)$
6) ${N_B}^o(A_1,A_2), N_B\_E(A_1,A_2), N_B\_F(A_1,A_2)$ are mutually disjoint and $(U_1,U_2) = {N_B}^o(A_1,A_2)\cup N_B\_E(A_1,A_2)\cup N_B\_F(A_1,A_2)$.
7) $(A_1,A_2)\cap N_B\_E(A_1,A_2) = (\emptyset,\emptyset)$
8) $N_B\_E(A_1,A_2) \subseteq (U_1,U_2) - (A_1,A_2)$
Proof: 1)$N_B\_E(A_1,A_2) = {N_B}^o\big((U_1,U_2) - (A_1,A_2)\big) = (U_1,U_2) - \overline{N_B}(A_1,A_2)$. Hence (1) is proved.

2)$N_B\_E(N_B\_E(A_1,A_2)) = N_B\_E\big[{N_B}^o\big((U_1,U_2) - (A_1,A_2)\big)\big] = N_B\_E\big[(U_1,U_2) - \overline{N_B}(A_1,A_2)\big] = {N_B}^o\left((U_1,U_2) - [(U_1,U_2) - \overline{N_B}(A_1,A_2)]\right) = {N_B}^o\left(\overline{N_B}(A_1,A_2)\right)$.Therefore, $N_B\_E(N_B\_E(A_1,A_2)) = {N_B}^o\left(\overline{N_B}(A_1,A_2)\right)$.

3)$N_B_E(U_1,U_2) = {N_B}^o((U_1,U_2) - (U_1,U_2)) = {N_B}^o(\emptyset,\emptyset) = (\emptyset,\emptyset)$and $N_B\_E(\emptyset,\emptyset) = {N_B}^o((U_1,U_2) - (\emptyset,\emptyset)) = {N_B}^o(U_1,U_2) = (U_1,U_2)$. Therefore, $N_B_E(U_1,U_2) = (\emptyset,\emptyset)$and $N_B\_E(\emptyset,\emptyset) = (U_1,U_2)$.

4)$N_B\_E[(U_1,U_2) - N_B\_E(A_1,A_2)] = {N_B}^o\big((U_1,U_2) - [(U_1,U_2) - N_B\_E(A_1,A_2)]\big) = {N_B}^o(N_B\_E(A_1,A_2)) = {N_B}^o[{N_B}^o\big((U_1,U_2) - (A_1,A_2)\big)] = {N_B}^o\big((U_1,U_2) - (A_1,A_2)\big) = N_B\_E(A_1,A_2)$. Therefore, $N_B\_E(A_1,A_2) = N_B\_E[(U_1,U_2) - N_B\_E(A_1,A_2)]$.

5)Since $(A_1,A_2) \subseteq \overline{N_B}(A_1,A_2) \Rightarrow {N_B}^o(A_1,A_2) \subseteq {N_B}^o\left(\overline{N_B}(A_1,A_2)\right) = {N_B}^o\big((U_1,U_2) - {N_B}^o[(U_1,U_2) - (A_1,A_2)]\big) = N_B\_E({N_B}^o[(U_1,U_2) - (A_1,A_2)]) = N_B\_E(N_B\_E(A_1,A_2))$. Therefore, ${N_B}^o(A_1,A_2) \subseteq N_B\_E(N_B\_E(A_1,A_2))$.

6)Assume that $N_B\_E(A_1,A_2)\cap {N_B}^o(A_1,A_2) \neq (\emptyset,\emptyset)$. Then there exists $(x_1,x_2) \in N_B\_E(A_1,A_2)\cap {N_B}^o(A_1,A_2) \Rightarrow (x_1,x_2) \in N_B\_E(A_1,A_2)$ and $(x_1,x_2) \in {N_B}^o(A_1,A_2) \Rightarrow (x_1,x_2) \in (U_1,U_2) - (A_1,A_2)$and $(x_1,x_2) \in (A_1,A_2)$, which is not possible. Hence our assumption is wrong. Therefore, $N_B\_E(A_1, A_2) \cap N_B{}^o(A_1, A_2) = (\emptyset, \emptyset)$. In the same way we can prove the others. $N_B\_E(A_1, A_2) = (U_1, U_2) - \overline{N_B}(A_1, A_2) = (U_1, U_2) - \overline{N_B}(A_1, A_2) = (U_1, U_2) - [N_B{}^o(A_1, A_2) \cup N_B\_F(A_1, A_2)]$. Therefore, $(U_1, U_2) = N_B\_E(A_1, A_2) \cup N_B{}^o(A_1, A_2) \cup N_B\_F(A_1, A_2)$.

7)$(A_1, A_2) \cap N_B\_E(A_1, A_2) = (A_1, A_2) \cap N_B{}^o((U_1, U_2) - (A_1, A_2)) \subseteq (A_1, A_2) \cap ((U_1, U_2) - (A_1, A_2)) = (\emptyset, \emptyset)$. Therefore, $(A_1, A_2) \cap N_B\_E(A_1, A_2) = (\emptyset, \emptyset)$.

8)$N_B\_E(A_1, A_2) = N_B{}^o((U_1, U_2) - (A_1, A_2)) \subseteq (U_1, U_2) - (A_1, A_2)$.

Note 4.7: If (A_1, A_2) is N_B closed, then equality holds in (5).

Theorem 4.8: In $(U_1, U_2, \tau_R(X_1, X_2))$, $(A_1, A_2)$and (B_1, B_2) be two subsets of (U_1, U_2). Then the following holds:

1) $N_B{}^o(A_1, A_2) \cap N_B\_B(A_1, A_2) = (\emptyset, \emptyset)$
2) (A_1, A_2) is N_B-open if and only if $N_B\_B(A_1, A_2) = (\emptyset, \emptyset)$
3) $N_B{}^o(N_B\_B(A_1, A_2)) = (\emptyset, \emptyset)$
4) $N_B\_B(N_B{}^o(A_1, A_2)) = (\emptyset, \emptyset)$
5) $N_B\_B(N_B\_B(A_1, A_2)) = N_B\_B(A_1, A_2)$
6) $N_B\_B(A_1, A_2) = (A_1, A_2) - N_B{}^o(A_1, A_2) = (A_1, A_2) \cap \overline{N_B}((U_1, U_2) - (A_1, A_2))$.
7) If $(A_1, A_2) \subseteq (B_1, B_2)$, then $N_B\_B(B_1, B_2) \subseteq N_B\_B(A_1, A_2)$.
8) $N_B\_B((A_1, A_2) \cup (B_1, B_2)) \subseteq N_B\_B(A_1, A_2) \cup N_B\_B(B_1, B_2)$.
9) $N_B\_B(A_1, A_2) \cap N_B\_B(B_1, B_2) \subseteq N_B\_B((A_1, A_2) \cap (B_1, B_2))$.
10) $N_B_B(A_1, A_2) = N_B_D((U_1, U_2) - (A_1, A_2))$and $N_B\_D(A_1, A_2) = N_B\_B((U_1, U_2) - (A_1, A_2))$.
11) $(A_1, A_2) = N_B{}^o(A_1, A_2) \cup N_B\_B(A_1, A_2)$.

Proof: 1) $N_B{}^o(A_1, A_2) \cap N_B\_B(A_1, A_2) = N_B{}^o(A_1, A_2) \cap [(A_1, A_2) - N_B{}^o(A_1, A_2)] = N_B{}^o(A_1, A_2) \cap [(A_1, A_2) \cap ((U_1, U_2) - N_B{}^o(A_1, A_2))] = N_B{}^o(A_1, A_2) \cap ((U_1, U_2) - N_B{}^o(A_1, A_2)) \cap (A_1, A_2) = (\emptyset, \emptyset) \cap (A_1, A_2) = (\emptyset, \emptyset)$.Therefore, $N_B{}^o(A_1, A_2) \cap N_B\_B(A_1, A_2) = (\emptyset, \emptyset)$

2)Any subset (A_1, A_2) of N_B topological space $(U_1, U_2, \tau_R(X_1, X_2))$ is N_B-open $\Leftrightarrow (A_1, A_2) = N_B{}^o(A_1, A_2) \Leftrightarrow (A_1, A_2) - N_B{}^o(A_1, A_2) = (\emptyset, \emptyset) \Leftrightarrow N_B\_B(A_1, A_2) = (\emptyset, \emptyset)$.

3)$N_B{}^o(N_B\_B(A_1, A_2)) = N_B{}^o((A_1, A_2) - N_B{}^o(A_1, A_2)) = N_B{}^o[(A_1, A_2) \cap \left((U_1, U_2) - N_B{}^o(A_1, A_2)\right)] \subseteq N_B{}^o(A_1, A_2) \cap N_B{}^o\left((U_1, U_2) - N_B{}^o(A_1, A_2)\right) \subseteq N_B{}^o(A_1, A_2) \cap \left((U_1, U_2) - N_B{}^o(A_1, A_2)\right) = (\emptyset, \emptyset)$. Therefore, $N_B{}^o(N_B\_B(A_1, A_2)) = (\emptyset, \emptyset)$.

4)$N_B\_B(N_B{}^o(A_1, A_2)) = N_B{}^o(A_1, A_2) - N_B{}^o(N_B{}^o(A_1, A_2)) = N_B{}^o(A_1, A_2) - N_B{}^o(A_1, A_2) = (\emptyset, \emptyset)$. Therefore, $N_B\_B(N_B{}^o(A_1, A_2)) = (\emptyset, \emptyset)$.

5)$N_B\_B(N_B\_B(A_1, A_2)) = N_B\_B(A_1, A_2) - N_B{}^o(N_B\_B(A_1, A_2)) = N_B\_B(A_1, A_2) - (\emptyset, \emptyset)$ (By (3)) $= N_B\_B(A_1, A_2)$. Therefore, $N_B\_B(N_B\_B(A_1, A_2)) = N_B\_B(A_1, A_2)$.

6)$N_B\_B(A_1,A_2) = (A_1,A_2) - N_B{}^o(A_1,A_2) = (A_1,A_2) \cap ((U_1,U_2) - N_B{}^o(A_1,A_2)) = (A_1,A_2) \cap \overline{N_B}((U_1,U_2) - (A_1,A_2))$. Therefore, $N_B\_B(A_1,A_2) = (A_1,A_2) \cap \overline{N_B}((U_1,U_2) - (A_1,A_2))$.

7)If $(A_1,A_2) \subseteq (B_1,B_2)$, then $N_B{}^o(A_1,A_2) \subseteq N_B{}^o(B_1,B_2) \Rightarrow (U_1,U_2) - N_B{}^o(B_1,B_2) \subseteq (U_1,U_2) - N_B{}^o(A_1,A_2) \Rightarrow (A_1,A_2) \cap ((U_1,U_2) - N_B{}^o(B_1,B_2)) \subseteq (A_1,A_2) \cap ((U_1,U_2) - N_B{}^o(A_1,A_2)) \Rightarrow (A_1,A_2) - N_B{}^o(B_1,B_2) \subseteq (A_1,A_2) - N_B{}^o(A_1,A_2) \Rightarrow N_B\_B(B_1,B_2) \subseteq N_B\_B(A_1,A_2)$ (By (6))

8)Since $(A_1,A_2) \subseteq (A_1,A_2) \cup (B_1,B_2)$and $(B_1,B_2) \subseteq (A_1,A_2) \cup (B_1,B_2)$. By (4) $N_B_B((A_1,A_2) \cup (B_1,B_2)) \subseteq N_B_B(A_1,A_2)$and $N_B\_B((A_1,A_2) \cup (B_1,B_2)) \subseteq N_B\_B(B_1,B_2)$. Therefore, $N_B\_B((A_1,A_2) \cup (B_1,B_2)) \subseteq N_B\_B(A_1,A_2) \cup N_B\_B(B_1,B_2)$.

9)Since $(A_1,A_2) \cap (B_1,B_2) \subseteq (A_1,A_2)$and $(A_1,A_2) \cap (B_1,B_2) \subseteq (B_1,B_2)$. By (4), $N_B_B(A_1,A_2) \subseteq N_B_B((A_1,A_2) \cap (B_1,B_2))$and $N_B\_B(B_1,B_2) \subseteq N_B\_B((A_1,A_2) \cap (B_1,B_2))$. Therefore, $N_B\_B(A_1,A_2) \cap N_B\_B(B_1,B_2) \subseteq N_B\_B((A_1,A_2) \cap (B_1,B_2))$.

10)$N_B\_B(A_1,A_2) = (A_1,A_2) - N_B{}^o(A_1,A_2)$. By result 3.8, $(A_1,A_2) - N_B{}^o(A_1,A_2) = (A_1,A_2) - [(A_1,A_2) - N_B_D((U_1,U_2) - (A_1,A_2))] = N_B_D((U_1,U_2) - (A_1,A_2))$By replacing $(A_1,A_2)$by $(U_1,U_2) - (A_1,A_2)$, $N_B\_D(A_1,A_2) = N_B\_B((U_1,U_2) - (A_1,A_2))$.

11)$N_B{}^o(A_1,A_2) \cup N_B\_B(A_1,A_2) = N_B{}^o(A_1,A_2) \cup ((A_1,A_2) - N_B{}^o(A_1,A_2)) = N_B{}^o(A_1,A_2) \cup ((A_1,A_2) \cap ((U_1,U_2) - N_B{}^o(A_1,A_2))) = (N_B{}^o(A_1,A_2) \cup (A_1,A_2)) \cap (N_B{}^o(A_1,A_2) \cup ((U_1,U_2) - N_B{}^o(A_1,A_2))) = (A_1,A_2) \cap (U_1,U_2) = (A_1,A_2)$. Therefore, $(A_1,A_2) = N_B{}^o(A_1,A_2) \cup N_B\_B(A_1,A_2)$.

5. Conclusion

Nano binary derived, nano binary border and nano binary exterior in nano binary topological spaces were introduced and their properties were discussed. In future we will discuss generalized closed sets in nano binary topological spaces.

References

[1] Lellis Thivagar. M and Carmel Richard, On Nano forms of weakly open sets, International Journal of Mathematics and Statistics Invention, 1(1) 2013, 31-37.

[2] Nithyanantha Jothi. S and P. Thangavelu, On Binary Topological Spaces, Pacific-Asian Journal of Mathematics.

[3] Hari Siva Annam. G and J. Jasmine Elizabeth, Cognition of Nano Binary Topological Spaces, Global Journal of Pure and Applied Mathematics, 6(2019), 1045-1054.

[4] Jasmine Elizabeth. J and G. Hari Siva Annam, Some notions on nano binary continuous, Malaya Journal of Matematik, Vol. 9, N0. 1,592-597, 2021.

[5] Jasmine Elizabeth. J and G. Hari Siva Annam, Nano binary contra continuous functions in nano binary topological spaces, J. Math. Comput. Sci. 11 (2021), No. 4, 4994-5011.

[6] Lellis Thivagar. M and Sutha Devi. V, On Multi-granular Nano Topology, Accepted (2015), South East Asian Bulletin of Mathematics, Springer Verlag.

[7] C. Janaki and A. Jeyalakshmi, A new form of generalized closed sets via regular local function in ideal topological spaces, Malaya Journal of Matematik, S (1) (2015), 1-9.

[8] O. Njastad, On some classes of nearly open sets, Pacific J. Math.,15 (1965), 961-970.

[9] E. F. Lashin and T. Medhat, Topological reduction of information systems, Chaos, Solitions and Fractals, 25 (2015), 277-286.

[10] C. Richard, Studies on nano topological spaces, Ph.D. Thesis, Madurai Kamaraj University, India (2013).