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Detour Global Domination for Degree Splitting Graphs of Some Graphs

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Abstract

In this paper, we introduced the new concept detour global domination number for degree splitting graph of standard graphs. A set S is called a *detour global dominating set* of $G = (V, E)$ if S is both detour and global dominating set of G. The *detour global domination number* is the minimum cardinality of a detour global dominating set in G. Let $V(G)$ be $S_1 \cup S_2 \cup \cdots \cup S_t \cup T$, where S_i is the set having at least two vertices of same degree and $T = V(G) - \cup S_i$, where $1 \leq i \leq t$. The *degree splitting graph* $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \cdots, w_t$ and joining w_i to each vertex of S_i for $i = 1, 2, \cdots, t$. In this article we recollect the concept of degree splitting graph of a graph and we produced some results based on the detour global domination number for degree splitting graph of path graph, cycle graph, star graph, bistar graph, complete bipartite graph and complete graph. **Keywords**: Detour set, Dominating set, Detour Domination, Global Domination, Detour Global Domination, Degree Splitting graphs.
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1 Introduction

By a graph $G = (V, E)$ we mean a finite, connected, undirected graph with neither loops nor multiple edges. The *order* $|V|$ and *size* $|E|$ of *G* are denoted by *p* and *q* respectively. For graph theoretic terminology we refer to West[1]. For vertices x and y in a connected graph G, the *detour distance* $D(x, y)$ is the length of a longest $x - y$ path in G[2]. An $x - y$ path of length $D(x, y)$ is called an $x - y$ detour. The closed interval $I_D[x, y]$ consists of all vertices lying on some $x - y$ detour of G. For $S \subseteq V, I_D[S] = \cup_{x,y \in S} I_D[x, y]$. A set S of vertices is a detour set if $I_D[S] = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. A detour set of cardinality $dn(G)$ is called a *minimum detour set* [3].

A set $S \subseteq V(G)$ in a graph G is a *dominating set* of G if for every vertex v in V-S, there exists a vertex $u \in S$ such that v is adjacent to u. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G[4]. The complement $\overline{G}$ of a graph G also has $V(G)$ as its point set, but two points are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A set $S \subseteq V(G)$ is called a *global dominating set* of G if it is a dominating set of both G and $\overline{G}$[5].

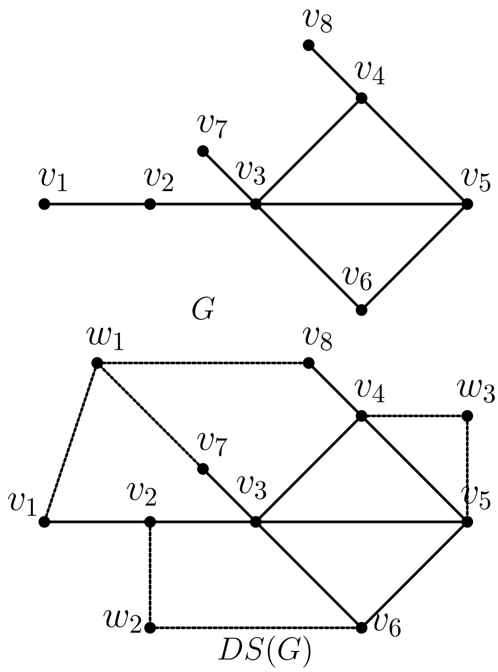
A vertex of degree 0 is called an *isolated vertex* and a vertex of degree 1 is called an *end vertex* or a *pendant vertex*. A vertex that is adjacent to a pendant vertex is called a *support vertex*.

Definition 1.1. Let $G = (V, E)$ be a connected graph with atleast two vertices. A set $S \subseteq V(G)$ is said to be a *detour global dominating set* of G if S is both detour and global dominating set of G. The *detour global domination number*, denoted by $\bar{\gamma_d}(G)$ is the minimum cardinality of a detour global dominating set of G and the detour global dominating set with cardinality $\bar{\gamma_d}(G)$ is called the $\bar{\gamma_d}$-set of G or $\bar{\gamma_d}(G)$-set.[6]

In [7], R. Ponraj and S. Somasundaram have initiated a study on degree splitting graph $DS(G)$ of a graph G which is defined as follows:

Definition 1.2. Let $G = (V, E)$ be a graph with $V(G) = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$, where S_i is the set having at least two vertices of same degree and $T = V(G) - \cup S_i$, where $1 \le i \le t$. The degree splitting graph $DS(G)$ is obtained from G by adding vertices $w_1, w_2, \cdots, w_t$ and joining w_i to each vertex of S_i for $i = 1, 2, \cdots, t$.

Example 1.1. *In Figure 1.1, a graph G and its degree splitting graph $DS(G)$ are shown.*



$Figure\ 1.1$

Here, $S_1 = \{v_1, v_7, v_8\}, S_2 = \{v_2, v_6\}, S_3 = \{v_4, v_5\}$ *and* $T = \{v_3\}$.

Remark 1.1. *If* $V(G) = \cup S_i, 1 \leq i \leq t$, *then* $T = \phi$.

2 Some basic results

In this section, we recall some basic results of detour global domination number of a graph which will be used throughout the paper.

Theorem 2.1. *[6] Every isolated vertex of G belongs to every detour global dominating set of G.*

Theorem 2.2. *[6] Every full vertex of a connected graph G of order p belongs to every detour global dominating set of G.*

Theorem 2.3. *[6] For the path graph* $P_p, (p \geq 4), \bar{\gamma}_d(P_p) = \gamma_d(P_p) = \lceil \frac{p+2}{3} \rceil$.

Theorem 2.4. *[6] For any star graph* $K_{1,n-1}, (n \geq 2), \bar{\gamma}_d(K_{1,n-1}) = n$.

Theorem 2.5. *[6] For any bistar graph* $B_{m,n}$, $m, n \geq 1$, $\bar{\gamma}_d(B_{m,n}) = m + n$.

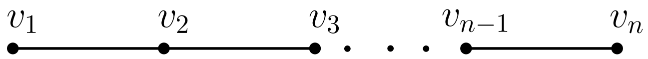
Theorem 2.6. *[6] For* $m, n \geq 2$, $\bar{\gamma}_d(K_{m,n}) = 2$.

3 Degree Splitting graphs of known graphs and their Detour Global Domination number

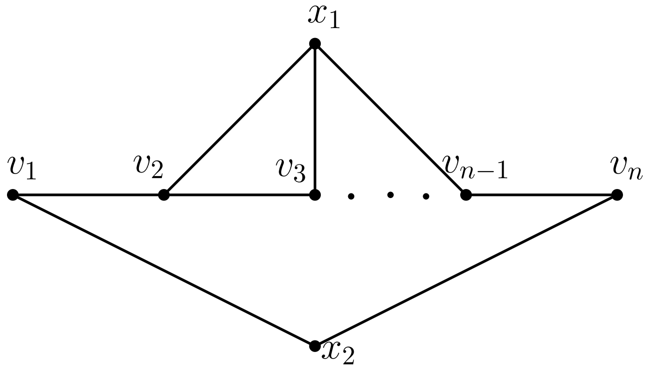
Let us find detour global domination number of degree splitting graph $DS(G)$ of the graphs path, cycle, star, bistar, complete bipartite and complete graph.

Theorem 3.1. *For any integer* $n \geq 3, \bar{\gamma}_d(DS(P_n)) = 2$.

Proof. Let $v_1 v_2 \cdots v_n$ be the path P_n with partitions $S_1 = \{v_2, v_3, \cdots, v_{n-1}\}$ and $S_2 = \{v_1, v_n\}$. To obtain $DS(P_3)$ from P_3 we add x, which corresponds to S_2 also P_3 is isomorphic to C_4 and to obtain $DS(P_n)$ for $n \geq 4$ we add x_1 and x_2, which corresponds to S_1 and S_2, respectively. As a result, $V(DS(P_3)) = \{x, v_1, v_2, v_3\}$ and $V(DS(P_n)) = \{x_1, x_2, v_1, v_2, \cdots, v_n\}$, where $|V(DS(P_n))| = n + 2$ for $n \geq 4$.



P_n



$DS(P_n)$
$Figure\ 3.1$

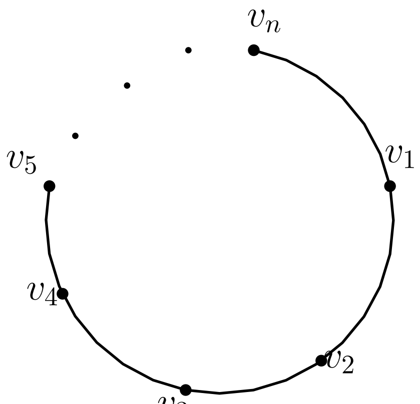
Consider $I_D[x, v_1]$ for $n = 3$, which has only one $x - v_1$ detour path of length 3 that contains all the vertices of $DS(P_3)$. As a result, $S = \{x, v_1\}$ is a minimum cardinality detour set. Also, x dominates S_2 and v_2 is dominated by v_1. Thus, S is a minimum detour dominating set of $DS(P_3)$. Moreover in $\overline{DS(P_3)}$, x dominates v_2 and v_3 is dominated by v_1. Hence, S is a minimum detour global dominating set of $DS(P_3)$ and so $\bar{\gamma}_d(DS(P_3)) = 2$.

Consider $I_D[x_1, x_2]$ for $n \geq 4$, which has two distinct $x_1 - x_2$ detour path of length n that contains all the vertices of $DS(P_n)$. As a result, $S = \{x_1, x_2\}$ is a minimum cardinality detour set. Also, x_1 dominates S_2 and x_2 dominates

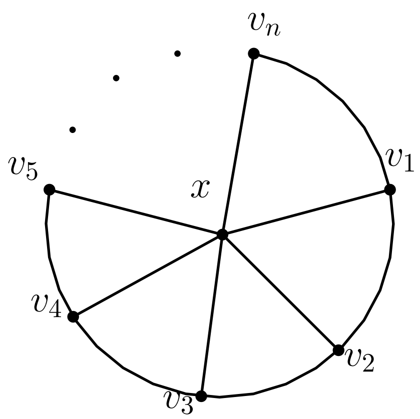
S_1. Thus, S is a detour dominating set of $DS(P_n)$. Moreover in $\overline{DS(P_n)}$, x_1 dominates S_1 and x_2 dominates S_2 and hence S is a detour global dominating set $DS(P_n)$. Hence, we conclude that for $n \geq 3, \bar{\gamma_d}(DS(P_n) = 2$. □

Theorem 3.2. *For any integer* $n \geq 3$, $\bar{\gamma_d}(DS(C_n)) = 3$.

Proof. Let $v_1 v_2 \cdots v_n$ be the cycle C_n. To obtain $DS(C_n)$ for $n \geq 3$ we add a vertex x which is adjacent to every vertices in C_n. As a result, $V(DS(C_n)) = \{x, v_1, v_2, \cdots, v_n\}$, where $|V(DS(C_n))| = n + 1$ for $n \geq 3$. Clearly, $DS(C_n)$ is isomorphic to the wheel graph W_n.



C_n



$DS(C_n)$

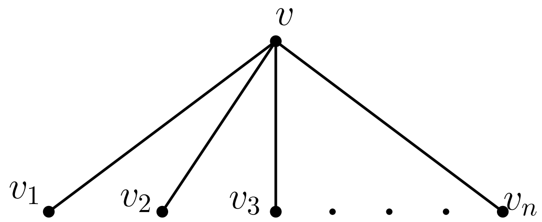
$Figure\ 3.2$

Since, $DS(C_n)$ is a connected graph, then $\bar{\gamma_d}(DS(C_n)) \geq 2$. Consider the set $S = \{x, v_i\}$, where $1 \leq i \leq n$. Clearly, the $x - v_i$ detour path of $DS(C_n)$ contains all the vertices of $DS(C_n)$ and also, x dominates every vertices of C_n. Hence, S is a minimum detour dominating set of $DS(C_n)$. But in $\overline{DS(C_n)}$, the neighbours of v_i are not dominated by any vertices of S. So, choose v_{i+1} from

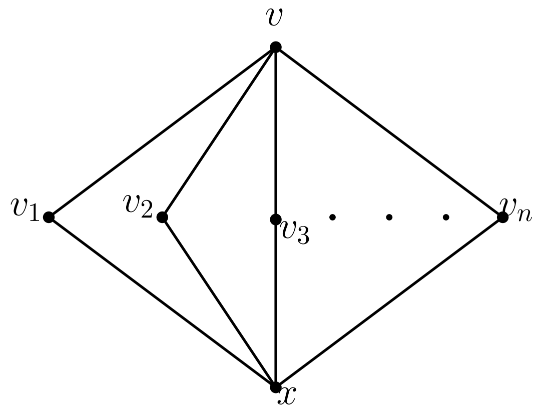
$\overline{DS(C_n)}$ then v_i, v_{i+1} and x dominates every vertices from $\overline{DS(C_n)}$. Therefore, $S_1 = S \cup \{v_{i+1}\} = \{v, v_i, v_{i+1}\}$ is a minimum detour global dominating set of $DS(C_n)$ and hence $\bar{\gamma}_d(DS(C_n)) = 3$. $\square$

Theorem 3.3. *For any integer* $n \geq 2, \bar{\gamma}_d(DS(K_{1,n})) = 2$.

Proof. Let $v_1, v_2, \cdots, v_{n-1}$ are the end vertices and v is the full vertex of the star $K_{1,n-1}$ and x be the corresponding vertex which is added to obtain the graph $DS(K_{1,n})$. Then $V(DS(K_{1,n})) = \{v, v_1, v_2, \cdots, v_n, x\}$. Clearly, $|V(DS(K_{1,n-1})| = n + 2$.



$K_{1,n}$

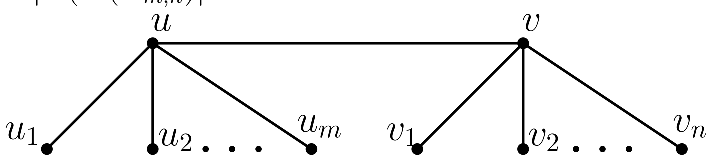


$DS(K_{1,n})$
$Figure$ 3.3

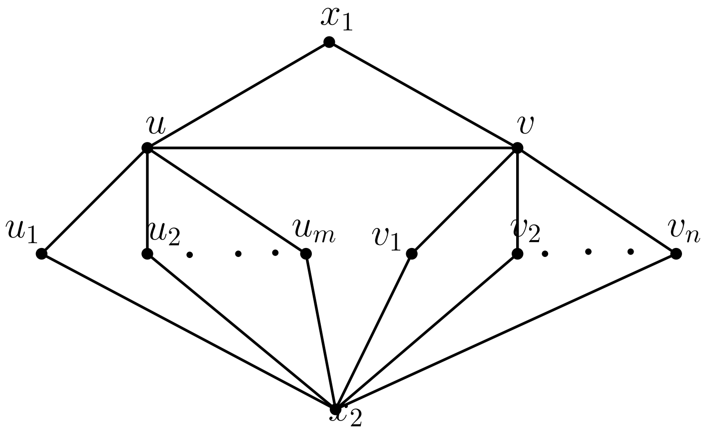
Since, $DS(K_{1,n})$ is connected, then $2 \leq \bar{\gamma}_d(DS(K_{1,n})) \leq n + 1$. Consider $S = \{v, v_i\}$ for some $i, 1 \leq i \leq n$. Then there are $n-$ detour path which travels between v and v_i that includes all the vertices of $DS(K_{1,n})$. Therefore, $S = \{v, v_i\}$ is a detour set of minimum cardinality. Moreover, v dominates every v_i and v itself in $DS(K_{1,n})$ and v_i dominates x in $DS(K_{1,n})$ for $1 \leq i \leq n$. Now consider in $\overline{DS(K_{1,n})}$ where v dominates x and v_i dominates all other v_j for $1 \leq j \neq i \leq n$. This concludes that S is a minimum detour global dominating set of $DS(K_{1,n})$ and hence $\bar{\gamma}_d(DS(K_{1,n})) = 2$. $\square$

Theorem 3.4. *For the bistar graph* $B_{m,n}$, $\bar{\gamma}_d(DS(B_{m,n}) = 2$.

Proof. Consider the bistar graph $B_{m,n}$ with $V(B_{m,n}) = \{u, v, u_i, v_j/1 \leq i \leq m; 1 \leq j \leq n\}$. Here, u_i and v_j are the vertices adjacent with u and v respectively. Let x_1 and x_2 be the corresponding vertices which are added to obtain $DS(B_{m,n})$. Then $V(DS(B_{m,n})) = \{u, v, u_i, v_j, x_1, x_2/1 \leq i \leq m; 1 \leq j \leq n\}$ and so $|V(S'(B_{m,n})| = m + n + 4$.



$B_{m,n}$



$DS(B_{m,n})$

$Figure\ 3.4$

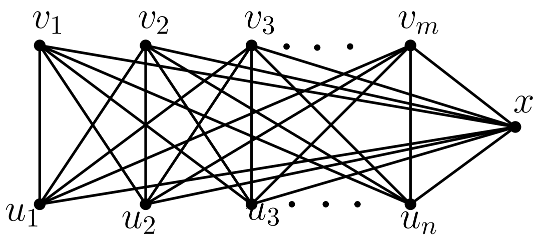
Since G is a connected graph, $\bar{\gamma_d}(DS(B_{m,n})) \geq 2$. Consider $I_D[x_1, x_2]$ which has $m+n$ transversal detour path of length four between x_1 and x_2 which include all the vertices of $DS(B_{m,n})$. Therefore, $S = \{x_1, x_2\}$ is a detour set of minimum cardinality. Also, x_1 dominates u and v and x_2 dominates all u_i and v_j for $1 \leq i \leq n$ and $1 \leq j \leq n$ in $DS(B_{m,n})$. It follows that S is a detour dominating set of $DS(B_{m,n})$. Moreover, in $\overline{DS(B_{m,n})}$, x_2 dominates x_1, u and v; x_1 dominates $u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_m$. Clearly, S is a minimum detour global dominating set of $DS(B_{m,n})$ and hence $\bar{\gamma_d}(DS(B_{m,n})) = 2$. $\square$

Theorem 3.5. *For any integer* $m, n \geq 2$, $\bar{\gamma_d}(DS(K_{m,n})) = \begin{cases} 3 & \text{if } m = n \\ 2 & \text{if } m \neq n \end{cases}$

Proof. Consider $K_{m,n}$ with $V(K_{m,n}) = \{u_i, v_j/1 \leq i \leq m, 1 \leq j \leq n\}$ with partition $V_1 = \{v_1, v_2, \cdots, v_m\}$ and $V_2 = \{u_1, u_2, \cdots, u_n\}$. Now we consider the following two cases.

Case (1) $m = n$

In this case each vertex is of same degree and so let x be the added vertex which is adjacent to every u_i and v_j, $1 \leq i \leq m$ and $1 \leq j \leq n$. Thus, we obtain the graph $DS(K_{m,n})$. Then $V(DS(K_{m,n})) = \{u_i, v_j, x/1 \leq i \leq m; 1 \leq j \leq n\}$ and so $|V(DS(K_{m,n}))| = m + n + 1$.



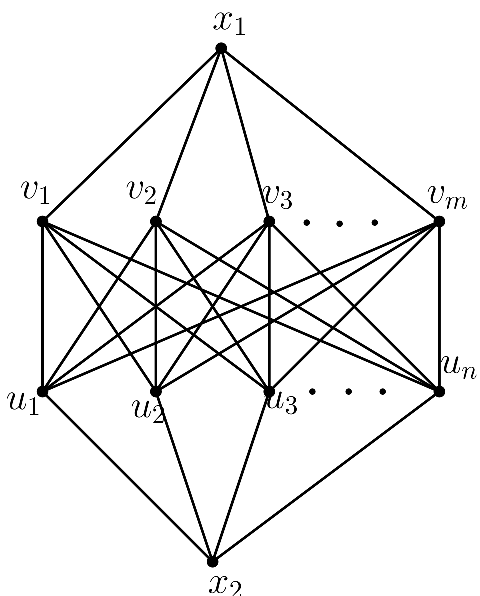
$DS(K_{m,m})$

$Figure\ 3.5$

Consider the set $S = \{v_i, u_j\}$ for some i, j where $1 \leq i \leq m$ and $1 \leq j \leq n$. Then $I_D[v_i, u_j] = V(DS(K_{m,n}))$ and also v_i dominates every u_j and x; u_j dominates every v_i. Hence, $S = \{v_i, u_j\}$ is a minimum detour dominating set. Now consider $\overline{DS(K_{m,n})}$, where x is an isolated vertex and not dominated by any vertex of S. This shows that S is not a detour global dominating set of $DS(K_{m,n})$. Hence, we include x in S such that $S = \{u_i, v_j, x\}$ is a detour global dominating set of $DS(K_{m,n})$ for $1 \leq i \leq m; 1 \leq j \leq n$. Therefore, for $m = n$, $\bar{\gamma}_d(DS(K_{m,n})) = 3$.

Case (2) $m \neq n$

In this case each vertex u_i is of same degree and each vertex v_j is of same degree where $deg(u_i) \neq deg(v_j)$, $1 \leq i \leq m; 1 \leq j \leq n$ so let x_1 and x_2 be the added vertex where x_1 is adjacent to every u_i and x_2 is adjacent to every v_j. Thus, we obtain the graph $DS(K_{m,n})$. Then $V(DS(K_{m,n})) = \{u_i, v_j, x_1, x_2/1 \leq i \leq m; 1 \leq j \leq n\}$ and so $|V(DS(K_{m,n}))| = m + n + 2$.



$DS(K_{m,m})$
$Figure\ 3.6$

Consider the set $S = \{x_1, x_2\}$, where $I_D[x_1, x_2] = V(DS(K_{m,n}))$ and also x_1 dominates every v_i and x_2 dominates every u_j. Hence, $S = \{x_1, x_2\}$ is a minimum detour dominating set. Now consider $\overline{DS(K_{m,n})}$, where x_1 dominates every u_i for $1 \leq i \leq n$ and x_2 dominates every v_j for $1 \leq j \leq n$. Clearly, $S = \{x_1, x_2\}$ is a minimum detour global dominating set and so for $m \neq n$, $\bar{\gamma_d}(DS(K_{m,n})) = 2$. $\square$

Theorem 3.6. *For any integer $n \geq 2$, $\bar{\gamma_d}(DS(K_n)) = n$.*

Proof. Here, $DS(K_n)$ is isomorphic to K_{n+1}. We know that all the vertices are isolated vertices in the complement graph of $DS(K_n)$. Therefore, the detour global dominating set must contain all the vertices of $DS(K_n)$ and so, for $n \geq 2$, $\bar{\gamma_d}(DS(K_n)) = n$. $\square$

4 Conclusion

Inspired by the global dominating set and detour set we introduce the detour global dominating set for degree splitting graph. We have determined the detour global domination number for degree splitting graph of path graph, cycle graph, star graph, bistar graph, complete bipartite graph and complete graph. Furthermore our results are also justified with suitable examples. The detour global domination number can also be obtained for many more graphs.

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