Existence and Policy Effectiveness in Feedback Nash LQ-Games*

Nicola Acocella

University of Rome *La Sapienza*

and

Giovanni Di Bartolomeo

University of Teramo

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Abstract. This paper illustrates how the classical theory of economic policy can profitably be used to verify some properties of the Linear Nash Feedback Equilibrium in difference LQ-games. In particular, we find that both a necessary condition for the equilibrium existence and a sufficient condition for policy ineffectiveness can be defined in the terms of the simple Tinbergen counting rule.

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1. Introduction

One fundamental question of many economic debates is that of effectiveness of public or private agents' action. The most prominent examples are the long debates on the neutrality of monetary and fiscal policy. However, the issue is not exclusive of economic policy; on the contrary, ineffectiveness has to be interpreted in a broader sense. For instance, debates on the effectiveness of advertising strategies in marketing or firm's price policies are related to a similar general problem.

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In so far as the decision-maker's action does not take account of the reactions of other agents (parametric approach), the debate is a purely empiric issue. However, especially after Lucas' (1976) criticism, a parametric approach can hardly be justified and a strategic one seems to be more coherent in almost all economic applications. In that case, the need for describing the mechanisms leading to ineffectiveness in general terms becomes a crucial theoretical modeling issue.

Some recent studies in that direction provide some general conditions for policy ineffectiveness and equilibrium existence in static LQ-games (see Acocella and Di Bartolomeo, 2004, 2006). This new approach shows how the classical theory of economic policy can profitably be used to define some general properties of policy games. This paper is one of the first attempts to extend this approach to a context of dynamic games.[1] More specifically, we consider feedback equilibrium in LQ-difference games and illustrate how a necessary condition for the equilibrium existence and a sufficient condition for policy ineffectiveness can be defined in terms of a simple counting rule of targets and instruments.

In a companion paper, Acocella *et al*. (2007) have investigated the same problem in the case of sparse-matrix dynamic systems, when, written in its structural form, a large dynamic system governing the economy relates each endogenous variable to just a few other endogenous variables and a small number of lagged endogenous variables, control variables, or predetermined variables. In such a context, Acocella *et al*. (2007) provide an example to illustrate the usefulness of this line of research by considering a model incorporating a Taylor rule, a description of expectations formation and a relation that can be interpreted as either a dynamic open-economy Phillips curve or a New-Keynesian IS curve with dynamics. Small variations in the model specification can bring, or take away, policy effectiveness – allowing the policy makers the possibility to disagree on none, one or several of the target values in their (common) targets – and possibly make institutional and policy independence counterproductive.

This paper focuses on a case opposite to that of a sparse matrix system, i.e. a case of full rank matrices, where all the economic variables are highly interdependent.

[1] Difference and differential games are extensively studied and used in many economic applications. For applications to macroeconomic theory, see among others Levine and Brociner (1994), Neck and Dockner (1995), Aarle *et al*. (1997), Başar *et al*. (1988), Levine and Smith (2000), Engwerda *et al*. (2002), Pappa (2004), Di Bartolomeo *et al*. (2006), Plasmans *et al*. (2005). For a more complete overview considering also different areas, see Dockner *et al*. (2000). See Başar and Olsder (1995), Dockner *et al*. (2000), Engwerda (2000a, 2000b) for some methodological aspects.

The outline of the paper is as follows. Section 2 defines basic concepts and introduces a formal framework to describe LQ-difference games. Section 3 derives two theorems, stating a sufficient condition for policy ineffectiveness and a necessary condition for the equilibrium existence in the traditional Tinbergen's terms. The paper ends with some concluding remarks.

2. Controllability and LQ-Policy Games

2.1 Some Preliminary Definitions

In order to apply the traditional theory of economic policy to study the properties of Nash feedback equilibrium, we first recall the traditional Tinbergen's *golden rule*.

Definition (*Golden Rule*): *A policymaker satisfies the golden rule of economic policy if the number of its independent instruments equals the number of its independent targets.*

Second, we need to redefine policy ineffectiveness, since its classical definition[2] cannot be maintained in the realm of policy games as policy instruments are endogenous variables, whose values really depend on the preferences of the decision-makers. The following definition of ineffectiveness can be accepted instead.[3]

Definition (ineffectiveness): *A policy is ineffective if the equilibrium values of the targets are never affected by changes in the parameters of its criterion.*

We are now ready to introduce our policy game.

2.2 A formal framework

We consider the problem where n players try each to minimize their respective quadratic performance criterion. The game is played for T periods, where T may be arbitrarily large.
Each player controls a different set of inputs to a single system, which is described by the following difference equation:

(1) $$x(t+1) = Ax(t) + \sum_{i \in N} B_i u_i(t)$$

[2] The classical definition of policy ineffectiveness implies that autonomous changes in policymaker's instruments have no influence on the targets

[3] See Gylfason and Lindbeck (1994).

where N is the set of the players; $x \in \mathbb{R}^{M}$ is the vector of the states of the system; $u_i \in \mathbb{R}^{m(i)}$ is the (control) variable vector that player i can manipulate; $A \in \mathbb{R}^{M \times M}$ and $B_i \in \mathbb{R}^{M \times m(i)}$ are full rank matrices describing the constant system parameters.

The performance criterion player $i \in N$ aims to minimize is

$$(2) \qquad J_i\left(u_1, u_2, \ldots, u_n\right) = \sum_{t=0}^{T} \left(x(t) - \overline{x}_i\right)' Q_i \left(x(t) - \overline{x}_i\right)$$

where $\overline{x}_i \in \mathbb{R}^{M}$ is a vector of given target values and Q is an appropriate constant symmetric positive definite matrix of weights. We assume that $x_i \neq x_j$ for all $i \in N \neq j \in N$. For reasons that we shall clarify we keep targets and instruments formally separate. However, in order to take account of the costs or limits in the use of some instruments, we could simply introduce an additional loss into equation (2) due to deviations of the instruments from the vector of their target values or equality (static) constraints concerning the instruments into equation (1).

3. Controllability, Ineffectiveness and Equilibrium Existence

Controllability, in the terms of the *Golden Rule* of economic policy, ineffectiveness and Linear Feedback Nash equilibrium existence are related together by the following two theorems.

Theorem 1 (ineffectiveness): *If one (and only one) player satisfies the golden rule, all the other players' policies are ineffective.*

Theorem 2 (non-existence): *No Linear Feedback Nash Equilibrium*[4] *of the policy game described exists if at least two players satisfy the Golden Rule (unless they share the same target values).*

The proofs of the theorems are simple and can be combined into a single one.

Proof. We start by *guessing* that the policymakers' value functions are quadratic,[5] $V_i\left(x\right) = \left(x(t) - \overline{x}_i\right)' P_i \left(x(t) - \overline{x}_i\right)$, where P_i are positive definite symmetric matrices (for the

[4] We restrict ourselves to the case of the Feedback Nash equilibrium based on a linear rule as defined in Engwerda (2006). In principle our results can be extended to the case of non-linear feedback rules by assuming an affine function as value function. However, the task is not without computational costs because the difficulties of computing non linear rules, which usually are infinite (see Tsutsui and Mino, 1990). See also Dockner *et al*. (2000) and Fujiwara and Matsueda (2007), who also show how to derive the Markov Feedback Nash equilibrium from a linear rule. In addition notice that the linear rule result to be optimal for the case of a finite period LQ games (see Başar *et al*., 1988; or Başar and Olsder, 1995, chapter 6).

sake of simplicity, time indexes will be omitted). By using the transition law to eliminate the next period state, the *n* Bellman equations become:

(3) $$(x-\bar{x}_i)' P_i (x-\bar{x}_i) = \min_{u_i} \left\{ (x-\bar{x}_i)' Q_i (x-\bar{x}_i) + \left(Ax + \sum_{i \in N} B_i u_i - \bar{x}_i \right)' P_i \left(Ax + \sum_{i \in N} B_i u_i - \bar{x}_i \right) \right\}$$

A Linear Feedback Nash Equilibrium must satisfy the first-order conditions:

(4) $$B_i' P_i B_i u_i = -B_i' P_i (Ax - \bar{x}_i) - B_i' P_i \sum_{j \in N/i} B_j u_j$$

to which the following policy rule corresponds:

(5) $$u_i = -(B_i' P_i B_i)^{-1} B_i' P_i (Ax - \bar{x}_i) - (B_i' P_i B_i)^{-1} B_i' P_i \sum_{j \in N/i} B_j u_j$$

Now, to demonstrate Theorem 1, we focus on player 1 without loss of generality. If player 1 satisfies the *Golden Rule*, then $m(1) = M$ and $B_1 \in \mathbb{R}^{M \times M}$ is a square matrix. Equation (5) becomes:

(6) $$u_1 = -B_1^{-1} (Ax - \bar{x}_1) - B_1^{-1} \sum_{j=2}^{n} B_j u_j$$

which implies:

(7) $$x(t+1) = \bar{x}_1 \text{ for } t \in [0, +\infty]$$

Thus, if a Linear Feedback Nash Equilibrium exists, the value of the target variables only depends on the preferences of player 1, since, in this case, condition (6) must be verified for all $t \in [0, +\infty]$. This completes the proof of Theorem 1. As to Theorem 2, we only need to show that, if also another player (e.g. player 2) satisfies its Golden Rule, the equilibrium does not exist.

As above, if also player 2 satisfies its Golden Rule, the following reaction function must hold in equilibrium:

(8) $$u_2 = -B_2^{-1} (Ax - \bar{x}_2) - B_2^{-1} B_1 u_1 - B_2^{-1} \sum_{j=3}^{n} B_j u_j$$

By plugging equation (8) into equation (6), it is clear that they cannot be mutually satisfied unless:

(9) $$\bar{x}_1 - \bar{x}_2 = 0$$

[5] Recall that we are looking for the Linear Feedback Nash equilibrium. See Engwerda (2006: Section 4) for more technical details. See also Sargent (1987: 42-48) or Ljungqvist and Sargent (2004: Chapter 5) for the single policymaker case.

which is not true in general, since $\bar{x}_1 \neq \bar{x}_2$. ■

Theorem 1 gives a sufficient condition for policy ineffectiveness, but this does not assure the equilibrium existence, which may fail to occur. By contrast, Theorem 2 gives a necessary condition for the equilibrium existence since it states a sufficient condition for non-existence. However, it may be not sufficient.[6] Note that if Theorem 1 is satisfied, Theorem 2 is not. This directly derives from the sentence in bracket in Theorem 1.

3. The instrument cost issue

It is useful to compare our results to a well-known theorem of existence of Nash equilibrium, Dasgupta and Maskin's (1986), which relates existence to the costs of the instruments since, in a similar manner, we have expressed the necessary condition for existence in terms of an instruments/targets counting rule.

Dasgupta and Maskin (1986) shows that a sufficient condition for the Nash equilibrium existence is that the space of strategies of each player is convex and compact. If the players' controls are unbounded, the Nash equilibrium may not exist. In static linear quadratic games, the introduction of quadratic instrument costs would make them bounded, thus assuring equilibrium existence.

In our terms, the introduction of quadratic instrument costs would imply that the dimensions of matrices Q_i become $M + m(i)$. Thus, the number of instruments would always be less than that of targets, the system would be not controllable by any player and equilibrium would exist. It is worth noticing, however that Theorem 2 is more general than the mentioned theorem of existence, since that of instrument costs is a particular case.

4. Concluding Remarks

This paper represents a first attempt to generalize some recent results developed in static policy games to a dynamic context. In the fashion of the classical theory of economic policy, we have shown that if one player satisfies the well-known simple Tinbergen's counting rule

[6] Existence is a rather complex matter in this context. Moreover, being in a dynamic system also stability is required. See Engwerda (2000a, 2000b).

(Golden Rule), either all the other players' policies are ineffective or no Linear Nash Feedback Equilibrium exists, without exact agreement on all the (dynamic) target values, (i.e., unless all the players satisfying the Golden Rule have equal targets values). In doing that, since difference games imply many technical complications, we have introduced a number of simplifications. Some of them are not crucial. Others cannot be easily relaxed. For instance, discounting, a finite time, and additive uncertainty can be easily introduced. Of course, in the case of stochastic games we should discuss our results in terms of expected values. We have also assumed that all the policy-makers share all the targets. However, our theorems are probably weak, in the sense that they are limit cases based on the strong concept of static controllability. It is well known, in fact, that in general fewer instruments than targets are needed to control a dynamic system. Once the theorems are reformulated in terms of dynamic controllability, by following Preston (1974) and Aoki (1975), it may be possible to define more general and less stringent conditions. This seems to be a promising line for future research.

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