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Fixed point theorems in uniformly convex Banach spaces

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Abstract

In this article, we establish a concept of fixed point result in Uniformly convex Banach space. Our main finding uses the Ishikawa iteration technique in uniformly convex Banach space to demonstrate strong convergence. Additionally, we use our primary result to demonstrate some corollaries.

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1 Introduction

Mann [Mann [1953]] defined mean value methods in an iterative scheme in 1953, and Ishikawa [Ishikawa [1974]] established fixed points using a new iteration method technique in 1954. Takahashi [Takahashi [1970]] introduced the idea of convexity in metric spaces and non-expansive mappings in 1970. Then Machado [Machado [1973]] went on to discuss about a classification of convex subsets of normed spaces. After that, Luis Bernal-Gonzalez [Bernal-Gonzalez [1996]] discussed convex domain in uniformly Banach spaces. Berinde [Berinde [2004]] investigates iterative scheme to finding fixed points using quasi contractive mappings in uniformly convex Banach spaces(stands for CBS), extended to uniformly convex Banach spaces(CBS).

Throughout this paper, we use strong convergence of ishikawa iterations to prove such fixed point results in uniformly CBS using different type of contractions.

2 Preliminaries

Definition 2.1. *Let (X, d) be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$*

$$d\,(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space (X, d) together with a convex structure W is called a convex metric space,which is denoted by (X, d, W).

Definition 2.2. *Let (X, d, W) be a convex metric space. A nonempty subset C of X is said to be convex if $W(x, y, \lambda) \in C$ whenever $(x, y, \lambda) \in C \times C \times I$.*

Definition 2.3. *Let $f : X \to X$. A point $x \in X$ is called a fixed point of f if $f(x) = x$.*

Definition 2.4. *Let E be a uniformly Banach space and $T : \mathsf{E} \to \mathsf{E}$ a map for which there is a real constant $k_1 \in (0, 1/5)$ such that each pair $u, v \in X$,*

$$\|Tu - Tv\| \leq k_1\{\|u - v\| + \|u - Tu\| + \|v - Tv\| + \|u - Tv\| + \|v - Tu\|\}.$$

Then, T has a fixed point by the approximation of Picard.

Definition 2.5. *Let E be a uniformly Banach space and $T : \mathsf{E} \to \mathsf{E}$ a map for which there is a real constant $k_2 \in (0, 1/3)$ such that for each pair $u, v \in X$,*

$$\|Tu - Tv\| \leq k_2\{\|u - v\| + \frac{\|u - Tu\| + \|v - Tv\|}{2} + \frac{\|u - Tv\| + \|v - Tu\|}{2}\}.$$

Then, T has a fixed point by the approximation of Picard.

Definition 2.6. *Let* E *be a uniformly Banach space and* $T : \mathsf{E} \to \mathsf{E}$ *a given operator. Let* $u_0 \in \mathsf{E}$ *be arbitrary and* $\{\alpha_n\} \subset [0,1]$ *a sequence of real numbers. The sequence* $\{ u_n\} \subset \mathsf{E}$ *defined by*

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n, n = 0, 1, 2, \cdots \tag{1}$$

is called the Mann iteration.

Definition 2.7. *Let* E *be a uniformly Banach space and* $T : \mathsf{E} \to \mathsf{E}$ *a given operator. Let* $u_0 \in \mathsf{E}$ *be arbitrary,* $\{\alpha_n\}$ *and* $\{\beta_n\} \subset [0,1]$ *a sequence of real numbers. The sequence* $\{ u_n\} \subset \mathsf{E}$ *defined by*

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T v_n, n = 0, 1, 2, \cdots \tag{2}$$

$$v_n = (1-\beta_n)u_n + \beta_n T u_n, n = 0, 1, 2, \cdots . \tag{3}$$

Then $\{u_n\}$ *is called Ishikawa iteration.*

Result 2.1. *Berinde [2004] The condition of Mann and Ishikawa iteration for strong convergence are given below*

(a) Let K *be a closed convex subset of a uniformly Banach space* E *and* $T : \mathsf{K} \to \mathsf{K}$ *as an operator satisfying contraction. Let* $\{u_n\}$ *be defined by Definition 2.6 and* $x_0 \in \mathsf{K}$*, with* $\{\alpha_n\} \in [0,1]$ *satisfying*

$$\sum_{n=0}^{\infty} \alpha_n = \infty. \tag{4}$$

Then, $\{u_n\}$ *converges strongly to a fixed point.*

(b) Let K *be a closed convex subset of a uniformly Banach space* E *and* $T : \mathsf{K} \to \mathsf{K}$ *as an operator satisfying the contraction. Let* $\{u_n\}$ *be defined by Definition 2.7 and* $u_0 \in \mathsf{K}$*, with* $\{\alpha_n\}, \{\beta_n\} \in [0,1]$ *satisfying*

$$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty. \tag{5}$$

Then, $\{u_n\}$ *strongly converges to a fixed point.*□

3 Main Results

Theorem 3.1. *Let* K *be a closed convex subset of a uniformly Banach space* E *and* $T : \mathsf{K} \to \mathsf{K}$ *an operator satisfying equation*

$$\|Tu - Tv\| \leq k_1\{\|u-v\| + \|u-Tu\| + \|v-Tv\| + \|u-Tv\| + \|v-Tu\|\}. \tag{6}$$

Let $\{u_n\}$ be the Ishikawa iteration and $u_0 \in K$, where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ with $\{\alpha_n\}$ satisfying equation $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{u_n\}$ converges strongly to a fixed point of T.

Proof. Suppose T has a fixed point p in K. Consider $u, v \in \mathsf{K}$ and T is an operator satisfies above equation,

$$\begin{aligned}
\|Tu - Tv\| &\leq k_1\{\|u - v\| + \|u - Tu\| + \|v - Tv\| + \|u - Tv\| \\
&\quad + \|v - Tu\|\} \\
&= k_1\{2\|u - v\| + 2\|u - Tv\| + \|u - Tu\| + \|v - Tu\|\} \\
&\leq k_1\{2\|u - v\| + 2[\|u - Tu\| + \|Tu - Tv\|] + \|u - Tu\| \\
&\quad + \|v - Tu\|\} \\
(1 - 2k_1)\|Tu - Tv\| &\leq 2k_1\|u - v\| + 3k_1\|u - Tu\| + k_1\|v - Tu\| \\
\|Tu - Tv\| &\leq \frac{2k_1}{1 - 2k_1}\|u - v\| + \frac{3k_1}{1 - 2k_1}\|u - Tu\| + \frac{k_1}{1 - 2k_1}\|v - Tu\|.
\end{aligned}$$

Take $\delta = \frac{k_1}{1-2k_1}$, then we have $\delta \in [0,1)$, it result that the inequality

$$\|Tu - Tv\| \leq 2\delta\|u - v\| + 3\delta\|u - Tu\| + \delta\|v - Tu\| \forall u, v \in \mathsf{K}. \tag{7}$$

Now let $\{u_n\}_{n=0}^{\infty}$ be Ishikawa iteration defined on Definition 2.7 and $u_0 \in \mathsf{K}$ arbitrary then

$$\begin{aligned}
\|u_{n+1} - p\| &= \|(1 - \alpha_n)u_n + \alpha_n Tv_n - (1 - \alpha_n + \alpha_n)p\| \\
&= \|(1 - \alpha_n)(u_n - p) + \alpha_n(Tv_n - p)\|
\end{aligned}$$

$$\|u_{n+1} - p\| \leq (1 - \alpha_n)\|u_n - p\| + \alpha_n\|Tv_n - p\|. \tag{8}$$

In equation (7), put $u = p$ and $v = v_n$, we have

$$\begin{aligned}
\|Tv_n - p\| &\leq 2\delta\|p - v_n\| + 3\delta\|p - Tp\| + \delta\|v_n - Tp\| \\
&= 2\delta\|p - v_n\| + \delta\|v_n - p\|
\end{aligned}$$

$$\|Tv_n - p\| \leq 3\delta\|v_n - p\|. \tag{9}$$

Furthermore, $\|v_n - p\| = \|(1 - \beta_n)u_n + \beta_n Tu_n - (1 - \beta_n + \beta_n)p\|$

$$= \|(1 - \beta_n)(u_n - p) + \beta_n(Tu_n - p)\|$$

$$\|v_n - p\| \leq (1 - \beta_n)\|u_n - p\| + \beta_n\|Tu_n - p\|. \tag{10}$$

Again in equation (7), put $u = p$ and $v = u_n$, we get

$$\begin{aligned}
\|Tu_n - p\| &\leq 2\delta\|p - u_n\| + 3\delta\|p - Tp\| + \delta\|u_n - p\| \\
&= (2\delta + \delta)\|u_n - p\|
\end{aligned}$$

$$\|Tu_n - p\| \leq 3\delta\|u_n - p\|. \tag{11}$$

Using equation (9),(10),(11) in equation (8), we get

$$\begin{aligned}
\|u_{n+1} - p\| &\leq (1-\alpha_n)\|u_n - p\| + \alpha_n\|Tv_n - p\| \\
&\leq (1-\alpha_n)\|u_n - p\| + 3\alpha_n\delta\|v_n - p\| \\
&\leq (1-\alpha_n)\|u_n - p\| + 3\alpha_n\delta[(1-\beta_n)\|u_n - p\| + \beta_n\|Tu_n - p\|] \\
&= (1-\alpha_n)\|u_n - p\| + 3\alpha_n\delta(1-\beta_n)\|u_n - p\| + 3\alpha_n\delta\beta_n\|Tu_n - p\| \\
&\leq (1-\alpha_n)\|u_n - p\| + 3\alpha_n\delta(1-\beta_n)\|u_n - p\| + 9\alpha_n\delta^2\beta_n\|u_n - p\| \\
&= [1 - \alpha_n + 3\alpha_n\delta - 3\alpha_n\beta_n\delta + 9\alpha_n\delta^2\beta_n]\|u_n - p\| \\
&= 1 - \alpha_n(1-3\delta) - 3\alpha_n\beta_n\delta(1-3\delta)]\|u_n - p\| \\
&= [1 - (1-3\delta)\alpha_n(1+3\delta\beta_n)]\|u_n - p\|
\end{aligned}$$

which by the inequality, $1 - (1-3\delta)\alpha_n(1+3\delta\beta_n) \leq 1 - (1-3\delta)^2\alpha_n$

$$\Rightarrow \|u_{n+1} - p\| \leq [1 - (1-3\delta)^2\alpha_n]\|u_n - p\|, n = 0, 1, 2, ... \tag{12}$$

by equation (8), we obtain

$$\|u_{n+1} - p\| \leq \prod_{k=0}^{n}[1 - (1-3\delta)^2\alpha_k]\|u_0 - p\|. \tag{13}$$

Where, $\delta \in (0,1), \alpha_k, \beta_n \in [0,1]$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$ by result (a), we get $\lim_{n\to\infty}\prod_{k=0}^{n}[1-(1-3\delta)^2\alpha_k] = 0$. By equation (13) which implies $\lim_{n\to\infty}\|u_{n+1} - p\| = 0$. Therefore, $\{u_n\}_{n=0}^{\infty}$ converges strongly to p. □

Theorem 3.2. *Let* K *be a closed convex subset of a uniformly Banach space* E *and* $T : \mathsf{K} \to \mathsf{K}$ *an operator satisfying equation*

$$\|Tu - Tv\| \leq k_2\{\|u - v\| + \frac{\|u - Tu\| + \|v - Tv\|}{2} + \frac{\|u - Tv\| + \|v - Tu\|}{2}\}.$$

Let $\{u_n\}$ *be the Ishikawa iteration and* $u_0 \in \mathsf{K}$, *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$ *with* $\{\alpha_n\}$ *satisfying equation* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *Then* $\{u_n\}$ *converges strongly to a fixed point of* T.

Proof. Consider T has a fixed point p in K. Consider $u, v \in \mathsf{K}$ and T is an operator satisfies equation

$$\begin{aligned}
\|Tu - Tv\| &\leq k_2\{\|u - v\| + \tfrac{\|u-Tu\|+\|v-Tv\|}{2} + \tfrac{\|u-Tv\|+\|v-Tu\|}{2}\} \\
&\leq k_2\{\|u - v\| + \tfrac{\|u-v\|+\|v-Tu\|+\|v-Tv\|}{2} + \tfrac{\|u-Tv\|+\|v-Tu\|}{2}\} \\
&= k_2\{\tfrac{3}{2}\|u - v\| + \|v - Tu\| + \tfrac{1}{2}\|v - Tv\| + \tfrac{\|u-Tv\|}{2}\} \\
(1-k_2)\|Tu - Tv\| &\leq \tfrac{3}{2}k_2\{\|u - v\| + \|v - Tv\|\} + \tfrac{k_2}{2}\|u - Tv\| \\
\|Tu - Tv\| &\leq \tfrac{3k_2}{2(1-k_2)}\{\|u - v\| + \|v - Tv\|\} + \tfrac{k_2}{2(1-k_2)}\|u - Tv\|
\end{aligned}$$

$$Take\ \delta = \frac{k_2}{1-k_2} \in [0,1). \|Tu - Tv\| \leq \frac{3}{2}\delta\{\|u-v\| + \|v - Tv\|\} + \frac{\delta}{2}\|u - Tv\|. \tag{14}$$

Now, let $\{u_n\}$ be Ishikawa iteration defined on Definition 2.7 and $u_0 \in \mathsf{K}$ then, $\|u_{n+1} - p\| = \|(1-\alpha_n)u_n + \alpha_n T v_n - (1-\alpha_n+\alpha_n)p\|$

$$\|u_{n+1} - p\| \leq (1-\alpha_n)\|u_n - p\| + \alpha_n\|Tv_n - p\| \tag{15}$$

In equation (14), put $v = p$ and $u = u_n$, $\|Tu_n - p\| \leq \frac{3\delta}{2}\|u_n - p\| + \frac{\delta}{2}\|u_n - p\|$

$$\|Tu_n - p\| \leq 2\delta\|u_n - p\| \tag{16}$$

In equation (14), put $u = v_n$ and $v = p$, $\|Tv_n - p\| \leq \frac{3\delta}{2}\|v_n - p\| + \frac{\delta}{2}\|v_n - p\|$

$$\|Tv_n - p\| \leq 2\delta\|v_n - p\| \tag{17}$$

$$\|v_n - p\| = \|(1-\beta_n)u_n + \beta_n T u_n - (1-\beta_n+\beta_n)p\|$$
$$\|v_n - p\| \leq (1-\beta_n)\|u_n - p\| + \beta_n\|Tu_n - p\| \tag{18}$$

Using equation (16), (17), (18) in equation (15), we get

$$\begin{aligned} \|u_{n+1} - p\| &\leq (1-\alpha_n) + 2\alpha_n\delta\|v_n - p\| \\ &\leq (1-\alpha_n) + 2\delta\alpha_n[(1-\beta_n)\|u_n - p\| + \beta_n\|Tu_n - p\|] \\ &\leq (1-\alpha_n) + 2\delta\alpha_n(1-\beta_n)\|u_n - p\| + 4\delta^2\alpha_n\beta_n\|u_n - p\| \\ &= [1 - \alpha_n(1+2\delta\beta_n)(1-2\delta)]\|u_n - p\| \end{aligned}$$

Which by inequality, $1 - \alpha_n(1+2\delta\beta_n)(1-2\delta) \leq 1 - (1-2\delta)^2\alpha_n$

$$\Rightarrow \|u_{n+1} - p\| \leq [1 - (1-2\delta)^2\alpha_n]\|u_n - p\| \tag{19}$$

By equation (19), we obtain

$$\|u_{n+1} - p\| \leq \prod_{k=0}^{n}[1 - (1-2\delta)^2\alpha_k]\|u_0 - p\| \tag{20}$$

where $\delta \in (0,1), \alpha_k, \beta_n \in [0,1]$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$ by result (a), we get,

$\lim_{n\to\infty}\prod_{k=0}^{n}[1-(1-2\delta)^2\alpha_k] = 0$. By equation (20), $\Rightarrow \lim_{n\to\infty}\|u_{n+1} - p\| = 0$. Therefore $\{u_n\}_{n=0}^{\infty}$ converges strongly to p.□

Corolary 3.1. *Let* K *be a closed convex subset of a uniformly Banach space* E *and* $T : \mathsf{K} \to \mathsf{K}$ *an operator satisfying equation* $\|Tu - Tv\| \leq \frac{k}{4}\{\|u - Tu\| + \|v - Tv\| + \|u - Tv\| + \|v - Tu\|\}$. *Let* $\{u_n\}$ *be the Ishikawa iteration and* $u_0 \in \mathsf{K}$, *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences of positive numbers in* $[0,1]$ *with* $\{\alpha_n\}$ *satisfying equation* $\sum_{n=0}^{\infty}\alpha_n = \infty$. *Then* $\{u_n\}$ *converges strongly to a fixed point of* T.

Corolary 3.2. *Let* E *be an uniformly Banach space,* K *is a closed convex subset of* E *and* $T : \mathsf{K} \to \mathsf{K}$ *an operator satisfying equation*

$$\|Tu - Tv\| \leq k\|u - v\| \tag{21}$$

Let $\{u_n\}_{n=0}^{\infty}$ *be the Ishikawa iteration and* $u_0 \in \mathsf{K}$*, where* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are sequences of positive numbers in* $[0, 1]$ *with* $\{\alpha_n\}_{n=0}^{\infty}$ *satisfying* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Then* $\{u_n\}_{n=0}^{\infty}$ *converges strongly to a fixed point of* T*.*

Corolary 3.3. *Let* E *be an uniformly Banach space,* K *is a closed convex subset of* E *and* $T : \mathsf{K} \to \mathsf{K}$ *an operator satisfying equation*

$$\|Tu - Tv\| \leq \frac{k}{2}\{\|u - Tu\| + \|v - Tv\|\} \tag{22}$$

Let $\{u_n\}_{n=0}^{\infty}$ *be the Ishikawa iteration and* $u_0 \in \mathsf{K}$*, where* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are sequences of positive numbers in* $[0, 1]$ *with* $\{\alpha_n\}_{n=0}^{\infty}$ *satisfying* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Then* $\{u_n\}_{n=0}^{\infty}$ *converges strongly to a fixed point of* T*.*

4 Conclusions

In this work, we presented the result on strong convergence of fixed point of T. We developed different kinds of contractive conditions to prove strong convergence using Ishikawa iterative method in uniforly convex Banach space. Our main result may be vision for other authors using different contraction to prove several converging fixed point result.

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