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On 2-Repeated Burst Codes

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Abstract. There are several kinds of burst errors for which error detecting and error correcting codes have been constructed. In this paper, we consider a new kind of burst error which will be termed as '2-repeated burst error of length b(fixed)'. Linear codes capable of detecting such errors have been studied. Further, codes capable of detecting and simultaneously correcting such errors have also been dealt with. The paper obtains lower and upper bounds on the number of parity-check digits required for such codes. An example of such a code has also been provided.

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1. Introduction

Investigations in coding theory have been made in several directions but one of the most important aspects considered has been the detection and correction of errors. The beginning was made with the detection and correction of random errors [refer Hamming (1950)] and thereafter the advent of BCH codes for multiple error correction was taken up. Though there is a long history concerning the growth of the subject and many of the codes developed have found applications in numerous areas of practical interest, one of the areas of practical importance in which a parallel growth of the subject took place is that of burst error detecting and correcting codes. It has also been observed that in many communication channels the likelihood of the occurrence of errors is more in adjacent digits rather than their occurrence in a random manner. Extending the work of Hamming (1950), Abramson (1959) developed codes which dealt with the correction of single and double adjacent errors. The work due to Fire (1959) depicted a more general concept of clustered errors which in the literature are known as 'burst errors'. A burst of length b may be defined as follows:

Definition 1. A burst of length b is a vector whose only non-zero components are among some b consecutive components, the first and the last of which is non-zero.

Fire (1959) considered two kinds of bursts viz. open-loop burst which are popularly refered to simply a burst (as in Definition 1) and the other is called as 'closed-loop burst' defined as follows:

Definition 2. Let b be an integer and $x = (\xi_1, \ldots, \xi_n)$ be a vector in $V^n(q)$, a vector space of n-tuples over $\mathrm{GF}(q)$. If $2 \leqslant b \leqslant \frac{n+1}{2}$, then x is called a 'closed-loop burst vector of length b' whenever there is an i such that $1 \leqslant i \leqslant b-1$, $\xi_i \cdot \xi_{n-b+i+1} \neq 0$, $\xi_{i+1} = \xi_{i+2} = \cdots = \xi_{n-b+i} = 0$.

Stone (1961), and Bridwell and Wolf (1970) considered multiple bursts. It was noted by Chien and Tang (1965) that in several channels errors occur in the form of a burst but not in the end digits of the burst. Channels due to Alexander, Gryb and Nast (1960) fall in this category. This prompted Chien and Tang to propose a modification in the definition of a burst and they defined a burst of length b which shall be called as CT burst of length b as follows:

Definition 3. A CT burst of length b is a vector whose only non-zero components are confined to some b consecutive positions, the first of which is non-zero.

This definition was further modified by Dass (1980) as follows:

Definition 4. A burst of length b(fixed) is an n-tuple whose only non-zero components are confined to b consecutive positions, the first of which is non-zero and the number of its starting positions in an n-tuple is among the first $n-b+1$ components.

It is clear that the nature of burst errors differ from channel to channel depending upon the behaviour of channels or the kind of errors which occur during the process of transmission. Also, in very busy communication channels, errors repeat themselves. So is a situation when errors occur in the form of a burst. In a way, we need to consider repeated bursts. Codes that detect and correct repeated open-loop bursts have been studied by Berardi, Dass and Verma (2009). In this paper, a 2-repeated burst (open-loop) of length b has been defined as follows:

Definition 5. A 2-repeated burst of length b is an n-tuple whose only non-zero components are confined to two distinct sets of b consecutive digits, the first and the last component of each set being non-zero.

The development of codes detecting and correcting repeated burst errors will economize in the number of parity-check digits required not only in comparison with codes dealing with detection and correction of the same number of random errors but also in comparison to the usual burst error detecting and correcting codes while considering such repeated bursts as single bursts.

In this paper, we introduce yet another kind of a repeated burst and define a '2-repeated burst of length b(fixed)' as follows:

Definition 6. A 2-repeated burst of length b(fixed) is an n-tuple whose only non-zero components are confined to two distinct sets of b consecutive digits, the first component of each set is non-zero and the number of its starting positions is among the first $n - 2b + 1$ components.

For example, (1000001000) is a 2-repeated burst of length up to 4(fixed) whereas (0000100100) is a 2-repeated burst of length at most 3(fixed).

These 2-repeated burst patterns of length b(fixed) include several 2-repeated bursts of length b or less in an obvious manner. Moreover, these are four times in number than the 2-repeated burst patterns of the same length in the binary case, and in the q-nary case these are $\frac{q^2}{(q-1)^2}$-times the number of 2-repeated bursts. It is clear from the fact that the number of 2-repeated burst vectors of length b is $(q-1)^4(q)^{2(b-2)}$ and the number of 2-repeated burst vectors of length b(fixed) is $(q-1)^2(q)^{2(b-1)}$ giving the ratio as $\frac{q^2}{(q-1)^2}$.

In section 2, we obtain bounds for codes detecting 2-repeated bursts of length b(fixed). Section 3 presents a bound for codes which can detect and simultaneously correct such 2-repeated bursts. In what follows a linear code will be considered as a subspace of the space of all n-tuples over GF(q). The distance between two vectors shall be considered in the Hamming sense.

2. 2-repeated burst error detecting codes

In this section, we consider linear codes that are capable of detecting any 2-repeated burst of length b(fixed). Clearly, the patterns to be detected should not be code words. In other words we consider codes that have no 2-repeated burst of length b(fixed) as a code word. Firstly, we obtain a lower bound over the number of parity-check digits required for such a code.

Theorem 1. *Any (n,k) linear code over* GF(q) *that detects any 2-repeated burst of length b*(fixed) *must have at least $2b$ parity-check digits.*

Proof. The result will be proved on the basis that no detectable error vector can be a code word.

Let V be an (n,k) linear code over GF(q). Consider a set X that has all those vectors which have their non-zero components confined to some two fixed distinct b consecutive components in the first $n-b+1$ components.

We claim that no two vectors of the set X can belong to the same coset of the standard array, else a code word shall be expressible as a sum or difference of two error vectors.

Assume on the contrary that there is a pair, say x_1, x_2 in X belonging to the same coset of the standard array. Their difference viz. $x_1 - x_2$ must be a code vector. But $x_1 - x_2$ is a vector all of whose non-zero components are confined to the same two fixed b consecutive components and so is a member of X, i.e., $x_1 - x_2$ is a 2-repeated burst of length b(fixed), which is a contradiction. Thus all the vectors in X must belong to distinct cosets of the standard array. The number of such vectors over GF(q) is clearly q^{2b}. The theorem follows since there must be at least this number of cosets. □

Remark 1. Incidentally, this result coincides with [Theorem 1, Berardi, Dass and Verma (2009)] when bursts considered are open-loop bursts.

An upper bound on the number of check digits required for the construction of a linear code is provided in the following theorem. This bound assures the existence of a linear code that can detect all 2-repeated bursts of length b(fixed). The bound has been obtained by first constructing a matrix under certain constraints and then by reversing the order of its columns altogether giving rise to a parity-check matrix for the requisite code, a technique given by Dass (1980).

Theorem 2. *There exists an* (n, k) *linear code that has no* 2*-repeated burst of length* b(fixed) *as a code word provided that*

$$q^{n-k} > q^{b-1}[1 + (n - 2b + 1)(q - 1)q^{b-1}]\,. \tag{1}$$

Proof. The existence of such a code will be shown by constructing an appropriate $(n-k) \times n$ parity-check matrix H. Firstly, we construct a matrix H' from which the requisite parity-check matrix H shall be obtained by reversing the order of its columns altogether. Any non-zero $(n-k)$-tuple is chosen as the first column h_1 of H'. Subsequent columns are added to H' such that after having selected the first $j-1$ columns $h_1, h_2, \ldots, h_{j-1}$, j-th column h_j is added provided that

$$\begin{aligned} h_j \;\neq\; & (\alpha_{j-b+1}h_{j-b+1} + \alpha_{j-b+2}h_{j-b+2} + \cdots + \alpha_{j-1}h_{j-1}) \\ & + (\beta_i h_i + \beta_{i+1}h_{i+1} + \cdots + \beta_{i+b-1}h_{i+b-1}) \end{aligned} \tag{2}$$

where either all β_i are zero or if β_t is the last nonzero coefficient then $b \leqslant t \leqslant j-b$, α_j's and β_i's in $\mathrm{GF}(q)$. This condition ensures that no 2-repeated burst of length b(fixed) will be a code word. The number of ways in which the coefficients α_j can be selected is clearly q^{b-1}. To enumerate the coefficients β_i is equivalent to enumerate the number of bursts of length b(fixed) amongst the first $j-b$ components. This number, including the vector of all zeros, is [Theorem 1, Dass (1980)]

$$1 + (j - 2b + 1)(q - 1)q^{b-1}\,.$$

Thus, the total number of possible combinations that h_j can not be equal to, is

$$q^{b-1}[1+(j-2b+1)(q-1)q^{b-1}]\,. \tag{3}$$

At worst, all these linear combinations might yield a distinct sum. Therefore a column h_j can be added to H' provided that

$$q^{n-k} > (3)\,.$$

The required parity-check matrix $H=[H_1H_2\ldots H_n]$ can be obtained from H' by reversing the order of its columns altogether $(h_i \rightarrow H_{n-i+1})$. For a code of length n, replacing j by n gives the result. □

Remark 2. In view of the fact that the result obtained in Theorem 2 is the same as the result for the correction of bursts of length b(fixed), such a code can serve dual purpose viz. it can either be used to correct bursts of length b(fixed) or can be used to detect 2-repeated bursts of length b(fixed).

3. Simultaneous detection and correction of repeated burst errors

In this section we determine extended Reiger's bound [Reiger (1960); also refer Theorem 4.15, Peterson and Weldon (1972)] for simultaneous detection and correction of 2-repeated bursts of length b(fixed). The following theorem gives a bound on the number of parity-check digits for a linear code that simultaneously detects and corrects 2-repeated bursts of length b(fixed).

Theorem 3. *An* (n, k) *linear code over* $\mathrm{GF}(q)$ *that corrects all* 2*-repeated bursts of length* b(fixed) *must have at least* $4b$ *parity-check digits. Further, if the code corrects all* 2*-repeated bursts of length* b(fixed) *and simultaneously detects* 2*-repeated bursts of length* d(fixed) $(d \geqslant b)$ *then the code must have at least* $2(b + d)$ *parity-check digits.*

Proof. We first prove the first part. Consider a burst of length $4b$(fixed) in the first $n - b + 1$ components. Such a vector is expressible as a sum or difference of two vectors, each of which is a 2-repeated burst of length b(fixed). These component vectors must belong to different cosets of the standard array because both such errors are correctable errors. Accordingly, such a vector viz. burst of length $4b$(fixed) can not be a code vector. In view of Theorem 1, such a code must have at least $4b$ parity-check digits.

Further, consider a burst of length $2(b+d)$(fixed), the burst confining to the first $n - b + 1$ components. Such a vector is expressible as a sum or difference of two vectors, one of which is a 2-repeated burst of length b(fixed) and the other is a 2-repeated burst of length d(fixed). Both such component vectors, one being a detectable error and the other being a correctable error, can not belong to the same coset of the standard array. Therefore such a vector can not be a code vector, i.e., a burst of length $2(b+d)$(fixed) can not be a code vector. Hence the code must have at least $2(b + d)$ parity-check digits. □

Remark 3. Incidentally, this result coincides with [Theorem 3, Berardi, Dass and Verma (2009)], when bursts considered are open-loop bursts.

Example. We conclude the paper with an example.

Consider a $(7,2)$ binary code with parity check matrix

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This matrix has been constructed by the synthesis procedure, outlined in the proof of Theorem 2, by taking $b = 3$. It can be seen from Table 1 that the syndromes of the different 2-repeated bursts of length 3(fixed) are nonzero, showing thereby that the code that is the null space of this matrix can detect all bursts of length 3(fixed).

Table 1

Error vectors	Syndromes
1000000	00001
1001000	01001
1001100	11001
1001010	01110
1001110	11110
1000100	10001
1000110	10110
1000101	11111
1000111	11000
1100000	00010
1101000	01010
1101100	11010
1101010	01101

(Contd.)

Error vectors	Syndromes
1101110	11101
1100100	10010
1100110	10101
1100101	11100
1100111	11011
1010000	00101
1011000	01101
1011100	11101
1011010	01010
1011110	11010
1010100	10101
1010110	10010
1010101	11011
1010111	11100
1110000	00110
1111000	01110
1111100	11110
1111010	01001
1111110	11001
1110100	10110
1110110	10001
1110101	11000
1110111	11111
0100000	00011
0100100	10011
0100110	10100
0100101	11101
0100111	11010
0110000	00111
0110100	10111
0110110	10000
0110101	11001
0110111	11110
0101000	01011
0101100	11011

(*Contd.*)

Error vectors	Syndromes
0101110	11100
0101101	10101
0101111	10010
0111000	01111
0111100	11111
0111110	11000
0111101	10001
0111111	10110
0010000	00100
0011000	01100
0010100	10100
0011100	11100
0001000	01000
0001100	11000
0001010	01111
0001110	11111
0000100	10000
0000110	10111
0000101	11110
0000111	11001

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