Square sum and square difference labelings of semitotal-block graph for some class of graphs

Keerthi G. Mirajkar*
Priyanka G. Sthavarmath†

Abstract

A graph G is said to be square sum and square difference labeling, if there exists a bijection f from $V(G)$ to $\{1, 2, 3, ..., (p-1)\}$ which induces the injective function f^* from $E(G)$ to N, defined by $f^*(uv) = f(u)^2 + f(v)^2$ and $f^*(uv) = f(u)^2 - f(v)^2$ respectively, for each $uv \in E(G)$ and the resulting edges are distinctly labeled. G is said to be square sum and square difference graph, if it admits a square sum and square difference labeling respectively. The present work investigates, square sum and square difference labeling of semitotal-block graph for some class of graphs which are proved using number theory concept.

Keywords: Square Sum Labeling; Square Difference Labeling; Semitotal-Block Graph; Comb Graph; Crown Graph; and Friendship Graph.
2020 AMS subject classifications: 05C05, 05C38, 05C76, 05C78.[1]

---

*Associate Professor, Department of Mathematics, Karnatak University's Karnatak Arts College, Dharwad - 580 001, Karnataka, INDIA; keerthi.mirajkar@gmail.com
†Research Scholar, Department of Mathematics, Karnatak University's Karnatak Arts College, Dharwad - 580 001, Karnataka, INDIA; priyankasthavarmath1990@gmail.com
[1]Received on September 15, 2022. Accepted on December 15, 2022. Published on June 30, 2023. DOI: 10.23755/rm.v39i0.804. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Graph labeling is one of the fascinating and active area in the field of graph theory and it is the numbering of integers to the vertices or edges, or both under particular conditions and is mainly divided into two parts namely, vertex labeling function and edge labeling function. The numbers considered here are positive integers which was introduced by A. Rosa in 1967 by identifying four types of labelings namely, α, β, γ, and δ. β-labeling was later renamed as graceful and the name has been popular since. Graph labeling has a wide range of applications such as X-ray crystallography, coding theory, radar, astronomy, circuit design, network theory, communication networks and database management.

Nowadays, research in labeling of graph is increasingly expanding. In the previous 60 years, more than 200 kinds of labeling have been studied and nearly 2500 articles have been published. One such labeling is square sum labeling defined by V. Ajitha, S. Arumugam, and K. A. Germina in 2009. They investigated square sum labeling of trees, unicyclic graphs, mC_n, $m \geq 1$, cycle with a chord, the graph obtained by joining two copies of cycle C_n by a path P_k, and the graph defined by path union of k copies of C_n and also investigated that complete graph, when $n \geq 6$ is not square sum . R. Sebastian et al.[Sebastian and Germina, 2015] discussed on this concept by introducing square sum labeling of class of planar graphs. G.V. Ghodasara and M.J. patel [Ghodasara and Patel, 2017] introduced some bistar related square sum graphs. Afterwords, J. B. Babujee and S. Babitha [Babujee and Babitha, 2012] worked on some graphs which admits square sum labeling. One of the interesting labeling called square difference labeling which was first introduced by J. Shiama [Shiama, 2012] in 2012 and investigated the results on cycles, complete graphs, cycle cactus, ladder, lattice grids, wheels and quadrilateral snakes which admits square difference labeling. Also proved that the path graph is an odd square difference graph. In this article, square sum and square difference labelings of semitotal-block graphs for some class of graphs are obtained.

All the graphs mentioned in this article are finite, simple, undirected, and connected. For graph theoretic terms and system of symbols refer [F.Harary, 1969] and [Bondy and Murthy, 1976]. For latest and regular updates of different forms of labeling skills refer [Gallian, 2020] and for the terms related to number theory refer [Burton, 2006]

2 Preliminaries

Definition 2.1. *[Ghodasara and Patel, 2018] A (p, q) connected graph G is said to be **square sum labeling**(SSL) if there exists a bijection f from $V(G)$ to $\{0, 1, 2, \ldots, (p-1)\}$, which induces the injection f^* and values from $E(G)$ to N given by $f^*(uv) = \mid f(u)^2 + f(v)^2 \mid$ for all $e = uv$ are distinct.*

Definition 2.2. *[Shiama, 2012] A (p, q) connected graph G is said to be **square difference labeling**(SDL) if there exists a bijection f from $V(G)$ to $\{0, 1, 2, \ldots, (p-1)\}$, which induces the injection f^* and values from $E(G)$ to N given by $f^*(uv) = \mid f(u)^2 - f(v)^2 \mid$ for all $e = uv$ are distinct.*

Definition 2.3. *[Kulli, 1976] **Semitotal-block graph** $T_b(G)$ of a graph G is the graph whose set of vertices is the union of the set of vertices and blocks of G and in which two points are adjacent iff the corresponding vertices of G are adjacent or the corresponding blocks are incident.*

Definition 2.4. *[Shiama, 2012] The **comb graph**$(P_n \odot K_1)$ is a graph aquired by joining a pendant edge to each vertex of a path graph P_n with $2n$ vertices and $(2n-1)$ edges.*

Definition 2.5. *[Govindan and Dhivya, 2019] The **crown graph** $(C_n \odot K_1)$ is a graph aquired by joining a pendant edge to each vertex of a cycle C_n.*
*The **friendship graph** F_n [Govindan and Dhivya, 2019] can be aquired by joining n copies of the cycle graph C_3 with a common vertex.*

The following theorem is used for an immediate results.
Theorem A: [Kulli, 1976] If G is connected graph with p vertices and q edges and if b_r is the number of blocks to which vertex v_i belongs in G, then the semitotal-block graph $T_b(G)$ has $\sum b_i + 1$ vertices and $q + \sum b_i$ edges.

3 Main Results

Theorem 3.1. *For $n \geq 4$, $T_b(F_n)$ admits SSL.*

Proof. By Theorem A, the vertex set of $T_b(F_n)$ be $V(T_b(F_n)) = 3n + 1$ and the edge set be $E(T_b(F_n)) = 6n$. Let $B = b_1, b_2, b_3, ..., b_r$ be the blocks of F_n and the total number of blocks are equal to n copies of C_3 in F_n i.e., $\mid B(F_n) \mid = n$. There are four cases to be consider
Case 1. $n \equiv 0 \ (mod 4)$

Define a function $f : V(T_b(F_n) \to \{0, 1, 2, \ldots, 3n\}$ thus, the vertices and ten classes of edges, which are distinctly labeled as follows,

$$\begin{aligned} f(b_i) &= 2i + 1 \;\; for \;\; 1 \le i \le n, \;\; f(u) = 1 \\ f(u_i) &= 2i, \;\; for \;\; 0 \le i \le \frac{3n-2}{2} \\ f(u_{\frac{3n-2}{2}+1}) &= 3n, \;\; f(u_{\frac{3n}{2}+i}) = (3n-1) - 2(i-1) \;\; for \;\; 1 \le i \le \frac{n}{2} - 1. \end{aligned}$$

E_1: Edges connecting consecutive distinct even numbers and edge labels $f^*\{u_{2i}u_{2i+1}; 0 \le i \le \frac{3n-4}{4}\}$ are in increasing sequence of even order.
E_2: $f^*(u_{\frac{3n}{2}}u_{\frac{3n+2}{2}}) = (18n^2 - 6n + 1)$.
E_3: Edges connecting consecutive distinct odd numbers and edge labels $f^*\{u_{\frac{3n+2}{2}+i}\, u_{\frac{3n+2}{2}+j}; 1 \le i \le \frac{n-4}{4}$ $where \; i \; is \; odd \; and \; 2 \le j \le \frac{n}{4} \; where \; j \; is \; even\}$ are in decreasing sequence of even order.
E_4: Edges are connected by apex vertex 1 and consecutive distinct odd numbers and the edge labels $f^*\{ub_j; 1 \le j \le n\}$ are in increasing order of even numbers of the form $(4K + 4m + 2)$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_5: The edge labels $f^*\{uu_i; 0 \le i \le \frac{3n}{2}\}$ are in increasing order of the form $4K$, where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex vertex 1 are distinct.
E_6: The edge labels $f^*\{uu_{\frac{3n}{2}+i}; 1 \le i \le \frac{n-2}{2}\}$ are in decreasing order of the form $(9n^2 - 12nm + 6n + 4K - 4m + 2)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the edges joining even numbers and apex vertex 1 are distinct.
E_7: The edge labels $f^*\{u_{2i}b_j; 0 \le i \le \frac{3n}{4} \; and \; 1 \le j \le \frac{3n+4}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2$, $L = j^2$, $m = i$, and $l = j$.
E_8: The edge labels $f^*\{u_{2i-1}b_j; 1 \le i \le \frac{3n}{4} \; and \; 1 \le j \le \frac{3n}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2$,$L = j^2$, $m = i$, and $l = j$.
E_9: Edges joining distinct odd numbers and the edge labels, $f^*\{u_{\frac{3n}{2}+i}\, b_{\frac{n+4}{2}+(j-1)}; 2 \le i \le \frac{n}{2} \; i \; is \; even \; and \; 1 \le j \le \frac{n}{4}\}$, are in descending order.

E_{10}: Edges joining distinct odd numbers and the edge labels
$f^*\{u_{\frac{3n}{2}+i} b_{\frac{3n+4}{4}+(j-1)}$
$; 1 \leq i \leq \frac{n-2}{2}\ i\ is\ odd\ and\ 1 \leq j \leq \frac{n}{4}\}$, are in descending order.

Case 2. $n \equiv 1\ (mod 4)$

Define a function $f : V(T_b(F_n)) \to \{0, 1, 2, \ldots, 3n\}$ so, the vertices are labeled as,

$$\begin{aligned} f(b_i) &= 2i+1\ \ for\ \ 1 \leq i \leq n,\ \ f(u) = 1 \\ f(u_i) &= 2i,\ \ for\ \ 0 \leq i \leq \frac{3n-1}{2} \\ f(u_{\frac{3n-1}{2}+i}) &= 3n - 2(i-1)\ \ for\ \ 1 \leq i \leq \frac{n-1}{2}. \end{aligned}$$

The induced edge labeling function is injective and labeled values are from $E(G)$ to N. There are nine classes of edges, which are distinctly labeled as follows,

E_1: Edges connecting consecutive distinct even numbers and edge labels, $f^*\{u_{2i}u_{2i+1}; 0 \leq i \leq \frac{3(n-1)}{4}\}$ are in increasing sequence of even order.

E_2: Edges connecting consecutive distinct odd numbers and edge labels $f^*\{u_{\frac{3n-1}{2}+i} u_{\frac{3n-1}{2}+j}; 1 \leq i \leq \frac{n-3}{2}\ where\ i\ is\ odd\ and\ 2 \leq j \leq \frac{n-1}{2}\ where\ j\ is\ even\}$ are in decreasing sequence of even order.

E_3: Edges are connected by apex vertex 1 and consecutive distinct odd numbers and the edge labels $f^*\{ub_j; 1 \leq j \leq n\}$ are in increasing order of the form $(4K + 4m + 1)$, where $K, m \in N$ and $K = i^2$, $m = i$.

E_4: The edge labels $f^*\{uu_i; 0 \leq i \leq \frac{3n-1}{2}\}$ are in increasing order of the form $4K$, where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex vertex 1 are distinct.

E_5: The edge labels $f^*\{uu_{(\frac{3n-1}{2})+i}; 1 \leq i \leq \frac{n-1}{2}\}$ are in decreasing order of the form $(9n^2 - 12nm + 12n + 4K - 8m + 5)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the edges joining even numbers and apex vertex 1 are distinct.

E_6: The edge labels $f^*\{u_{2i}b_j; 0 \leq i \leq \frac{3(n-1)}{4}\ and\ 1 \leq j \leq \frac{3n+1}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2$, $L = j^2$, $m = i$, and $l = j$.

E_7: The edge labels $f^*\{u_{2i-1}b_j; 1 \leq i \leq \frac{3n+1}{4}\ and\ 1 \leq j \leq \frac{3n+1}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2$, $L = j^2$, $m = i$, and $l = j$.

E_8: Edges joining distinct odd numbers and the edge labels,
$f^*\{u_{\frac{3(n-1)}{2}+i}\, b_{\frac{3n+5}{4}+(j-1)}; 2 \leq i \leq \frac{n-1}{2}\ i\ is\ even\ and\ 1 \leq j \leq \frac{n-1}{4}\}$,
edge labels are in descending order of distinct even numbers.
E_9: Edges joining distinct odd numbers and the edge labels
$f^*\{u_{\frac{3(n-1)}{2}+i}\, b_{\frac{3n+5}{4}+(j-1)}; 3 \leq i \leq \frac{n+1}{2}\ i\ is\ odd\ and\ 1 \leq j \leq \frac{n-1}{4}\}$,
edge labels are in descending order of distinct even numbers.
Case 3. $n \equiv 2\ (mod 4)$
Define a function $f : V(T_b(F_n)) \rightarrow \{0, 1, 2, \ldots, 3n\}$ such that, vertices are labeled as,

$$\begin{aligned} f(b_i) &= 2i+1 \ \ for \ \ 1 \leq i \leq n, \ \ f(u) = 1, \ \ f(u_i) = 2i \ \ for \ \ 0 \leq i \leq \frac{3n}{2} \\ f(u_{\frac{3n}{2}+i}) &= (3n-1) - 2(i-1) \ \ for \ \ 1 \leq i \leq \frac{n}{2} - 1. \end{aligned}$$

The induced edge labeling function is injective and labeled values are from $E(G)$ to N. There are nine classes of edges, which are distinctly labeled as follows,
E_1: Edges connecting consecutive distinct even numbers and edge labels,
$f^*\{u_{2i}u_{2i+1}; 0 \leq i \leq \frac{3n-2}{4}\}$ are in increasing sequence of even order.
E_2: Edges connecting consecutive distinct odd numbers and edge labels,
$f^*\{u_{\frac{3n-2}{2}+i}\, u_{\frac{3n-2}{2}+j}; 1 \leq i \leq \frac{n-2}{2}\ where\ i\ is\ even\ and\ 3 \leq j \leq \frac{n}{2}\ where\ j\ is\ odd\}$ are in decreasing sequence of even numbers.
E_3: Edges are connected by apex vertex 1 and consecutive distinct odd numbers and the edge labels $f^*\{ub_j; 1 \leq j \leq n\}$ are in increasing order of the form $(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_4: The edge labels $f^*\{uu_i; 0 \leq i \leq \frac{3n}{2}\}$ are in increasing order of the form $4K$, where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex vertex 1 are distinct.
E_5: The edge labels $f^*\{uu_{(\frac{3n}{2})+i}; 1 \leq i \leq \frac{n-2}{2}\}$ are in decreasing order of the form $(9n^2 - 12nm + 6n + 4K - 4m + 2)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the edges joining even numbers and apex vertex 1 are distinct.
E_6: The edge labels $f^*\{u_{2i}b_j; 0 \leq i \leq \frac{3n-2)}{4}\ and\ 1 \leq j \leq \frac{3n+2}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and

$K = i^2$, $L = j^2$, $m = i$, and $l = j$.
E_7: The edge labels $f^*\{u_{2i-1}b_j; 1 \leq i \leq \dfrac{3n+2}{4}\ and\ 1 \leq j \leq \dfrac{3n+2}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2$, $L = j^2$, $m = i$, and $l = j$.
E_8: Edges joining distinct odd numbers and the edge labels
$f^*\{u_{\frac{3n-2}{2}+i}\, b_{\frac{3(n+2)}{4}+(j-1)}; 2 \leq i \leq \dfrac{n-2}{2},\ i\ is\ even\ and\ 1 \leq j \leq \dfrac{n-2}{4}\}$
are in descending sequence of distinct even numbers.
E_9: Edges joining distinct odd numbers and the edge labels
$f^*\{u_{\frac{3n-2}{2}+i}\, b_{\frac{3(n+2)}{4}+(j-1)}; 3 \leq i \leq \dfrac{n}{2},\ i\ is\ odd\ and\ 1 \leq j \leq \dfrac{n-2}{4}\}$, are in descending order of distinct even numbers.
Case 4. $n \equiv 3\ (mod 4)$
Define a function $f : V(T_b(F_n)) \rightarrow \{0, 1, 2, \ldots, 3n\}$ such that, vertices are labeled as,

$$\begin{aligned} f(b_i) &= 2i + 1\ \ for\ \ 1 \leq i \leq n,\ \ f(u) = 1 \\ f(u_i) &= 2i,\ \ for\ \ 0 \leq i \leq \frac{3(n-1)}{2} \\ f(u_{\frac{3(n-1)}{2}+i}) &= 3n - 2(i-1)\ \ for\ \ 1 \leq i \leq \frac{n-1}{2}. \end{aligned}$$

The induced edge labeling function is injective and labeled values are from $E(G)$ to N. There are ten classes of edges, which are distinctly labeled as follows,
E_1: Edges connecting consecutive distinct even numbers and edge labels $f^*\{u_{2i}u_{2i+1}; 0 \leq i \leq \dfrac{3n-5}{4}\}$ are in increasing sequence of even order.
E_2: $f^*(u_{\frac{3n-1}{2}}u_{\frac{3n+1}{2}}) = (18n^2 - 6n + 1)$.
E_3: Edges connecting consecutive distinct odd numbers and edge labels $f^*\{u_{\frac{3n+1}{2}+i}\, u_{(\frac{3n+1}{2}+j}; 1 \leq i \leq \dfrac{n-5}{2}\ where\ i\ is\ odd\ and\ 2 \leq j \leq \dfrac{n-3}{2}\ where\ j\ is\ even\}$ are in decreasing sequence of even order.
E_4: Edges are connected by apex vertex 1 and consecutive distinct odd numbers and the edge labels $f^*\{ub_j; 1 \leq j \leq n\}$ are in increasing order of the form $(4K + 4m + 1)$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_5: The edge labels $f^*\{uu_i; 0 \leq i \leq \dfrac{3n-1}{2}\}$ are in increasing order of the form $4K$,where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex

vertex 1 are distinct.
E_6: The edge labels $f^*\{uu_{\frac{3n-1}{2}+i}; 1 \leq i \leq \frac{n-1}{2}\}$ are in decreasing order of the form $(9n^2 - 12nm + 6n + 4K - 8m + 5)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the edges joining even numbers and apex vertex 1 are distinct.
E_7: The edge labels $f^*\{u_{2i}b_j; 0 \leq i \leq \frac{3n-1}{4}\ and\ 1 \leq j \leq \frac{3(n+1)}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$,where $K, L, m, l \in N$ and $K = i^2$, $L = j^2$, $m = i$, and $l = j$.
E_8: The edge labels $f^*\{u_{2i-1}b_j; 1 \leq i \leq \frac{3n-1}{4}\ and\ 1 \leq j \leq \frac{3n-1}{4}\}$ are in increasing order of the form $(4K + 4L + 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2$,$L = j^2$, $m = i$, and $l = j$.
E_9: Edges joining distinct odd numbers and the edge labels
$f^*\{u_{\frac{3(n-1)}{2}+i}b_{\frac{3n-1)}{4}+j}; 2 \leq i \leq \frac{n+1}{2},\ i\ is\ even\ and\ 1 \leq j \leq \frac{n+1}{4}\}$ are in descending order of distinct even numbers.
E_{10}: Edges joining distinct odd numbers and the edge labels
$f^*\{u_{\frac{3(n-1)}{2}+i}b_{\frac{3n-1)}{4}+j}; 1 \leq i \leq \frac{n-1}{2},\ i\ is\ odd\ and\ 1 \leq j \leq \frac{n+1}{4}\}$ are in descending order of distinct even numbers. Clearly, all the ten classes of edges are distinctly labeled. Hence, for $n \geq 4$, $T_b(F_n)$ admits SSL. $\square$.

Example 3.1: The square sum labeling of semitotal-block graph of F_4, F_5, F_6 and F_7 are shown in below figure.

Figure a : F_4

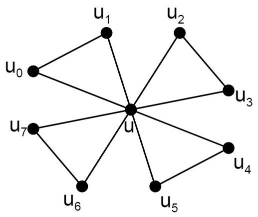


Figure A : $T_b(F_4)$
Case 1 : $n \equiv 0(mod4)$

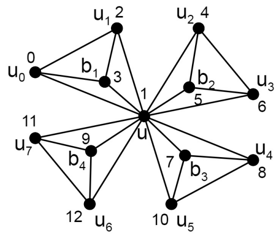


Figure b : F_5

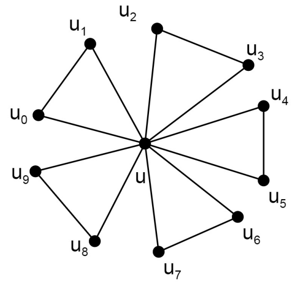


Figure B : $T_b(F_5)$
Case 2 : $n \equiv 1(mod4)$

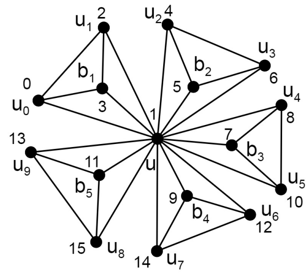


Figure c : F_6

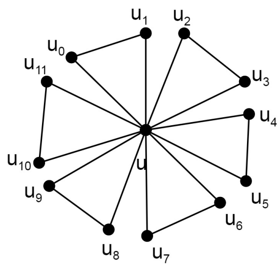


Figure C : $T_b(F_6)$
Case 3 : $n \equiv 2(mod4)$

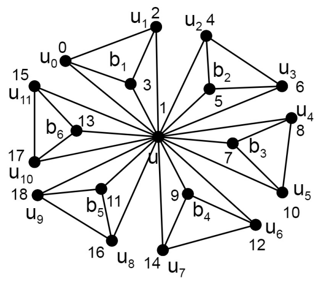


Figure d : F_7

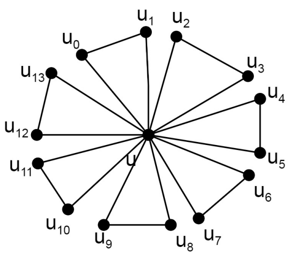
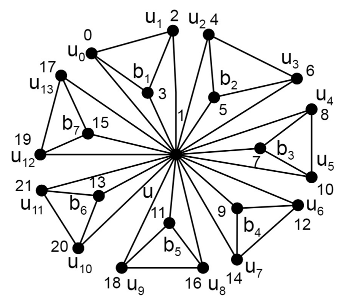


Figure D : $T_b(F_7)$
Case 4 : $n \equiv 3(mod4)$



Theorem 3.2. *For $n \geq 3$, $T_b(P_n \odot K_1)$ admits SSL.*

Proof. By Theorem A, the vertex set of $T_b(P_n \odot K_1)$ be $V(T_b(P_n \odot K_1)) = 2n + r$ and the edge set be $E(T_b(P_n \odot K_1)) = 6n - 3$. Let $B = b_1, b_2, b_3, ..., b_r$ be the blocks of $(P_n \odot K_1)$ and the total number of blocks (r) are equal to $\mid E(P_n \odot K_1) \mid$ i.e., $\mid B(P_n \odot K_1) \mid$=2n-1.

Define a function $f : V(T_b(P_n \odot K_1)) \to \{0, 1, 2, \ldots, (2n + r - 1)\}$ thus, the vertices are assigned as,

$$\begin{aligned}
if\ n = 3\ \ f(u_i) &= 2i\ \ for\ \ 1 \le i \le n \\
f(b_i) &= 2i + 1\ \ for\ \ 1 \le i \le n,\ \ f(b_{n+1}) = 0,\ \ f(b_r) = 1 \\
f(v_i) &= (2n + 1) + i\ \ for\ \ 1 \le i \le n \\
if\ n > 3\ \ f(u_i) &= 2i\ \ for\ \ 1 \le i \le n \\
f(b_i) &= 2i + 1\ \ for\ \ 1 \le i \le n,\ \ f(b_{n+1}) = 0,\ \ f(b_{n+2}) = 1 \\
f(v_i) &= (2n + 1) + i\ \ for\ \ 1 \le i \le n \\
f(b_{(n+3)+(i-1)}) &= (3n + 1) + i\ \ for\ \ 1 \le i \le (n - 3).
\end{aligned}$$

The induced edge labeling function is injective and values are from $E(G)$ to N. There are seven classes of edges, which are distinctly labeled as follows,
E_1: Edges joining naturally distinct consecutive even numbers and the edge labels $f^*\{u_iu_{i+1}; 1 \le i \le (n - 1)\}$ are in strictly increasing sequence.
E_2: The labeling of edges $f^*\{u_iv_i; 1 \le i \le n\}$ are in ascending order of the form $(4^2 + 4n + 4nm + 5K + 2m + 1)$, where $K, m \in N$ and $K = i^2, m = i$, which are distinctly labeled.
E_3: Edges connecting consecutive distinct even and odd numbers and the edge labels $f^*\{u_ib_i; 1 \le i \le n\}$ are in increasing sequence of the form $(8K + 4m + 1)$, where $K, m \in N$ and $K = i^2, m = i$.
E_4: The edge labels $f^*\{v_ib_i; 1 \le i \le n\}$ are in ascending order of the form $(4n^2 + 4n + 4nm + 5K + 6m + 2)$, where $K, m \in N$ and $K = i^2,\ m = i$, which are distinctly labeled.
E_5: The edge labels $f^*\{u_ib_{r-1}; 1 \le i \le (n - 1)\}$ are in ascending order of the form $4K$, where $K \in N$ and $K = i^2$ which are distinctly labeled.
E_6: The edge labels $f^*\{u_ib_r; 2 \le i \le n\}$ are in ascending order of the form $(4K + 1)$, where $K \in N$ and $K = i^2$, which are distinctly labeled.
E_7: If $n > 3$ the edge labels $f^*\{u_ib_{(n+3)+(i-1)}; 3 \le i \le (n - 1)\ and\ 1 \le j \le (n-3)\}$ and $f^*\{u_ib_{(n+3)+(i-1)}; 4 \le i \le n\ and\ 1 \le j \le (n-3)\}$ are in ascending orders of the form $(9n^2 + 6nl + 6n + 4K + L + 2l + 1)$, where $K, L, m, l \in N$ and $K = i^2,\ m = i,\ L = j^2,\ l = j$, which are distinctly labeled. Clearly, the assignment pattern of all the classes of edge are distinct. So, f is SSL and $T_b(P_n \odot K_1)$ admits SSG. □

Example 3.2: The square sum labeling of semitotal-block graph of $P_4 \odot K_1$ is shown in below figure.

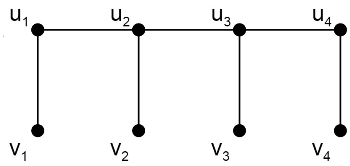


Figure: $P_4 \odot K_1$

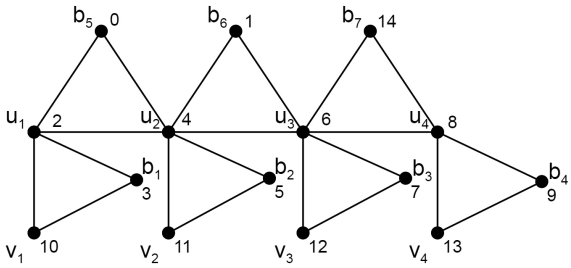


Figure: $T_b\ (P_4 \odot K_1)$

Theorem 3.3. *For $n \geq 3$, $T_b(C_n \odot K_1)$ admits SSL except $n \equiv 0\ (mod 8)$.*

Proof. The vertex set of $T_b(C_n \odot K_1)$ be $V(T_b(C_n \odot K_1)) = 3n + 1$ and the edge set be $E(T_b(C_n \odot K_1)) = 5n$. Let $B = b_1, b_2, b_3, ..., b_r$ be the blocks of $(C_n \odot K_1)$ and the total number of blocks in $(C_n \odot K_1)$ are equal to $(n + 1)$ i.e., $\mid B(C_n \odot K_1) \mid$=n+1.
Define a function $f : V(T_b(C_n \odot K_1)) \to \{0, 1, 2, \ldots, 3n\}$ thus, the vertices are assigned as,

$$\begin{aligned}
f(b_r) &= 3n,\ \ f(u_n) = 2n,\ \ f(v_n) = n,\ \ f(b_{r-1}) = 0 \\
f(u_i) &= f(b_r) - i\ \ for\ \ 1 \leq i \leq n \\
f(v_i) &= f(u_n) - i\ \ for\ \ 1 \leq i \leq n \\
f(b_i) &= f(v_n) - i,\ \ for\ \ 1 \leq i \leq n.
\end{aligned}$$

The induced edge labeling function is injective and values are from $E(G)$ to N. There are five classes of edges and all edge labels are in the form of decreasing sequence, which are distinctly labeled as follows,
E_1: The edge assignments $f^*\{b_r u_i; 1 \leq i \leq n\}$ are in the form $(18n^2 + K - 6nm)$, where $K, m \in N$ and $K = i^2,\ m = i$.
E_2: The edge assignments $f^*\{u_i v_i; 1 \leq i \leq n\}$ are in the form $(2K + 13n^2 - 10nm)$, where $K, m \in N$ and $K = i^2,\ m = i$.
E_3: The edge assignments $f^*\{v_i b_i; 1 \leq i \leq n\}$ are in the form $(2K + 9n^2 - 8nm)$, where $K, m \in N$ and $K = i^2, m = i$.
E_4: The edge assignments $f^*\{u_i u_{j+1}; 1 \leq i \leq (n-1)\ and\ 1 \leq j \leq (n-1)\}$ are in the form $(K + 9n^2 - 6nm) + (L + 9n^2 - 6nl)$, where $K, L, m, n \in N$ and $K = i^2,\ L = j^2; m = i,\ l = j$.
E_5: $f^*(u_n u_1) = (13n^2 - 6n + 1)$. Clearly, the labeling pattern of all the classes

of edges are not being same. So, the $T_b(C_n \odot K_1)$, is SSL. □

Example 3.3: The square sum labeling of semitotal-block graph of $C_5 \odot K_1$ is shown in below figure.

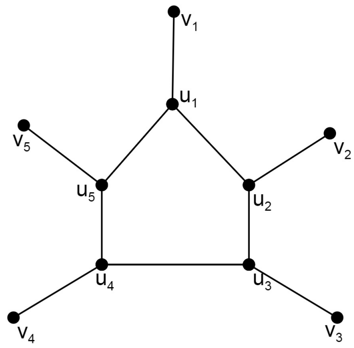


Figure: $C_5 \odot K_1$

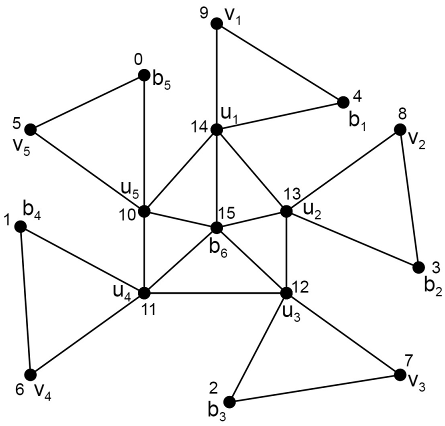


Figure: $T_b(C_5 \odot K_1)$

Theorem 3.4. *For $n \geq 3$, $T_b(F_n)$ admits SDL except $n \equiv 1\ (mod 4)$.*

Proof. By Theorem A, the vertex set of $T_b(F_n)$ be $V(T_b(F_n)) = 3n+1$ and the edge set be $E(T_b(F_n)) = 6n$. Let $B = b_1, b_2, b_3, ..., b_r$ be the blocks of F_n and the total number of blocks are equal to n copies of C_3 in F_n i.e., $\mid B(F_n) \mid$=n.

We consider three cases

Case 1. $n \equiv 0\ (mod 4)$

Define a function $f : V(T_b(F_n)) \to \{0, 1, 2, \ldots, 3n\}$ thus, the vertices are assigned as,

$$\begin{aligned} f(b_i) &= 2i-1 \ \ for \ \ 1 \leq i \leq n, \ \ f(u) = 0 \\ f(u_i) &= 2i \ \ for \ \ 0 \leq i \leq \frac{3n}{2} \\ f(u_{\frac{3n}{2}+i}) &= (3n-1) - 2(i-1) \ \ for \ \ 1 \leq i \leq \frac{n}{2}. \end{aligned}$$

The induced edge labeling function is injective and labeled values are from $E(G)$ to N. There are nine classes of edges, which are distinctly labeled as follows,

E_1: Edges connecting consecutive distinct even numbers and edge labels $f^*\{u_{2i-1}u_{2i}; 1 \leq i \leq \frac{3n}{4}\}$ are in increasing sequence of even order.

E_2: Edges connecting consecutive distinct odd numbers and edge labels, $f^*\{u_{\frac{3n}{2}+i}\ u_{\frac{3n+2}{2}+i}; 1 \leq i \leq \frac{n}{4}\}$ are in decreasing sequence of even order.

E_3: The edge labels $f^*\{u_0b_i; 1 \leq i \leq n\}$ are in increasing order of the form $(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the edges joining odd numbers and apex vertex 0 are distinct.

E_4: The edge labels $f^*\{u_0u_i; 1 \leq i \leq \frac{3n}{2}\}$ are in increasing order of the form $4K$, where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex vertex 0 are distinct.

E_5: The edge labels $f^*\{u_0u_{\frac{3n}{2}+i}; 1 \leq i \leq \frac{n}{2}\}$ are in increasing order of the form $(9n^2 + 4K - 12nm + 6n - 4m + 1)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the edges joining even numbers and apex vertex 0 are distinct.

E_6: The edge labels $f^*\{u_{2i-1}b_j; 1 \leq i \leq \frac{3n}{4}\ and\ 1 \leq j \leq \frac{3n}{4}\}$ are in increasing order of the form $(4K + 4L - 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2$,$L = j^2$, $m = i$, and $l = j$.

E_7: The edge labels $f^*\{u_{2i}b_j; 1 \leq i \leq \frac{3n}{4} and 1 \leq j \leq \frac{3n}{4}\}$ are in increasing order of the form $(4K + 4L - 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2$, $L = j^2$, $m = i$, and $l = j$.

E_8: Edges joining distinct odd numbers and the edge labels $f^*\{u_{\frac{3n}{2}+i}\, b_{\frac{3n+4}{4}+(j-1)}; 1 \leq i \leq \frac{n}{2} - 1,\ i\ is\ odd\ and\ 1 \leq j \leq \frac{n}{2} - 1\}$ are in descending order of distinct even numbers.

E_9: Edges joining distinct odd numbers and the edge labels $f^*\{u_{\frac{3n}{2}+i}\, b_{\frac{3n+4}{4}+(j-1)}; 1 \leq i \leq \frac{n}{2} - 1,\ i\ is\ even\ and\ 1 \leq j \leq \frac{n}{2} - 1\}$ are in descending order of distinct even numbers.

Case 2. $n \equiv 2\ (mod 4)$

Define a function $f : V(T_b(F_n)) \to \{0, 1, 2, \ldots, 3n\}$ such that, vertices are labeled as,

$$\begin{aligned} f(b_i) &= 2i - 1\ \ for\ \ 1 \leq i \leq n,\ \ f(u) = 0 \\ f(u_i) &= 2i,\ \ for\ \ 0 \leq i \leq \frac{3n}{2} \\ f(u_{\frac{3n}{2}+i}) &= (3n - 1) - 2(i - 1)\ \ for\ \ 1 \leq i \leq \frac{n}{2}. \end{aligned}$$

The induced edge labeling function is injective and labeled values are from $E(G)$ to N. There are ten classes of edges, which are distinctly labeled as follows,

E_1: Edges connecting consecutive distinct even numbers and edge labels $f^*\{u_{2i-1}u_{2i}; 1 \leq i \leq \frac{3n-2}{4}\}$ are in increasing sequence of even order.

E_2: $f^*(u_{\frac{3n}{2}} u_{\frac{3n+2}{2}}) = (6n-1)$
E_3: Edges connecting consecutive distinct odd numbers and edge labels,
$f^*\{u_{\frac{3n+2}{2}+i}\, u_{\frac{3n+2}{2}+j}; 1 \leq i \leq \frac{n-2}{4}, i\ is\ odd\ and\ 2 \leq j \leq \frac{n+2}{4}\ j\ is\ even\}$
are in decreasing sequence of even order.
E_4: The edge labels $f^*\{u_0 b_i; 1 \leq i \leq n\}$ are in increasing order of the form $(4K-4m+1)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the edge joining odd numbers and apex vertex 0 are distinct.
E_5: The edge labels $f^*\{u_0 u_i; 1 \leq i \leq \frac{3n}{2}\}$ are in increasing order of the form $4K$, where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex vertex 0 are distinct.
E_6: The edge labels $f^*\{u_0 u_{\frac{3n}{2}+i}; 1 \leq i \leq \frac{n}{2}\}$ are in decreasing order of the form $(9n^2 - 12nm + 6n + 4K - 4m + 1)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the edges joining even numbers and apex vertex 0 are distinct.
E_7: The edge labels $f^*\{u_{2i-1} b_j; 1 \leq i \leq \frac{3n+2}{4}\ and\ 1 \leq j \leq \frac{3n+2}{4}\}$ are in increasing order of the form $(4K+4L-4l+1)$, where $K, L, m, l \in N$ and $K = i^2$, $L = j^2$, $m = i$, and $l = j$.
E_8: The edge labels $f^*\{u_{2i} b_j; 1 \leq i \leq \frac{3n-2}{4}\ and\ 1 \leq j \leq \frac{3n-2}{4}\}$ are in increasing order of the form $(4K+4L-4l+1)$, where $K, L, m, l \in N$ and $K = i^2$, $L = j^2$, $m = i$, and $l = j$.
E_9: Edges joining distinct odd numbers and the edge labels
$f^*\{u_{\frac{3n-2}{2}+i}\, b_{\frac{3(n+2)}{4}+(j-1)}; 3 \leq i \leq \frac{n}{2},\ i\ is\ odd\ and\ 1 \leq j \leq \frac{n-2}{4}\}$ are in descending order of distinct even order.
E_{10}: Edges joining distinct odd numbers and the edge labels
$f^*\{u_{\frac{3n-2}{2}+i}\, b_{\frac{3n+2}{4}+(j-1)}; 2 \leq i \leq \frac{n+2}{2},\ i\ is\ even\ and\ 1 \leq j \leq \frac{n+2}{4}\}$
are in descending order of distinct even numbers.
Case 3. $n \equiv 3\ (mod 4)$
Define a function $f : V(T_b(F_n)) \to \{0, 1, 2, \ldots, 3n\}$ such that, vertices are labeled as,

$$\begin{aligned} f(b_i) &= 2i-1 \quad for \quad 1 \leq i \leq n, \quad f(u) = 0 \\ f(u_i) &= 2i, \quad for \quad 0 \leq i \leq \frac{3n-1}{2} \end{aligned}$$

$$f(u_{\frac{3n-1}{2}+i}) \;=\; 3n-2(i-1) \;\; for \;\; 1 \leq i \leq \frac{n+1}{2}.$$

The induced edge labeling function is injective and labeled values are from $E(G)$ to N. There are nine classes of edges, which are distinctly labeled as follows,
E_1: Edges connecting consecutive distinct even numbers and edge labels $f^*\{u_{2i-1}u_{2i}; 1 \leq i \leq \frac{3n-1}{4}\}$ are in increasing sequence of even order.
E_2: Edges connecting consecutive distinct odd numbers and edge labels $f^*\{u_{\frac{3n-1}{2}+i} u_{\frac{3n-1}{2}+j}; 1 \leq i \leq \frac{n-1}{2}$ $where\ i\ is\ odd\ and\ 2 \leq j \leq \frac{n+1}{2}$ $where\ j\ is\ even\}$ are in decreasing sequence of even order.
E_3: The edge labels $f^*\{u_0 b_j; 1 \leq j \leq n\}$ are in increasing order of the form $(4K-4m+1)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the edges joining odd numbers and apex vertex 0 are distinct.
E_4: The edge labels $f^*\{u_0 u_i; 1 \leq i \leq \frac{3n-1}{2}\}$ are in increasing order of the form $4K$, where $K \in N$ and $K = i^2$. Since the edges joining even numbers and apex vertex 0 are distinct.
E_5: The edge labels $f^*\{u_0 u_{\frac{3n-1}{2}+i}; 1 \leq i \leq \frac{n+1}{2}\}$ are in increasing order of the form $(9n^2 - 12nm + 12n + 4K - 8m + 4)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the edges joining even numbers and apex vertex 0 are distinct.
E_6: The edge labels $f^*\{u_{2i-1} b_j; 1 \leq i \leq \frac{3n-1}{4}$ and $1 \leq j \leq \frac{3n-1}{4}\}$ are in increasing order of the form $(4K + 4L - 4l + 1)$, where $K, L, m, l \in N$ and $K = i^2$, $L = j^2$, $m = i$, and $l = j$.
E_7: The edge labels $f^*\{u_{2i} b_j; 1 \leq i \leq \frac{3n-1}{4}$ and $1 \leq j \leq \frac{3n-1}{4}\}$ are in increasing order of the form $(4K + 4L - 4l + 1)$,where $K, L, m, l \in N$ and $K = i^2$, $L = j^2$, $m = i$, and $l = j$.
E_8: Edges joining distinct odd numbers and the edge labels $f^*\{u_{\frac{3n-1}{2}+i} b_{\frac{3n-1}{4}+j}$ $; 1 \leq i \leq \frac{n-1}{2},\ i\ is\ odd\ and\ 1 \leq j \leq \frac{n+1}{4}\}$ are in descending order of distinct even order.
E_9: Edges joining distinct odd numbers and the edge labels $f^*\{u_{\frac{3n-1}{2}+i} b_{\frac{3n-1}{4}+j}; 2 \leq i \leq \frac{n+1}{4}$ $i\ is\ even\ and\ 1 \leq j \leq \frac{n+1}{4}\}$ are in descending order of distinct even numbers. Clearly, the assignment pattern of all the classes of edge are distinct. So, $T_b(F_n)$ admits SDL. □

Example 3.4: The square difference labeling of semitotal-block graph of F_4, F_6 and F_7 are shown in the below figure.

Figure a: F_4

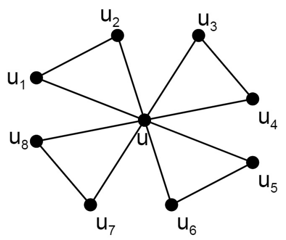


Figure A: $T_b(F_4)$
Case 1: $n \equiv 0(mod4)$

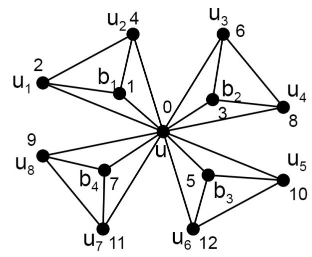


Figure c: F_6

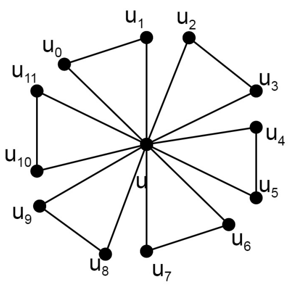


Figure C: $T_b(F_6)$
Case 3: $n \equiv 2(mod4)$

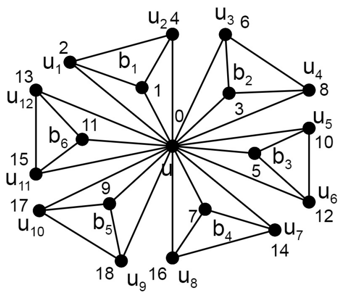


Figure d: F_7

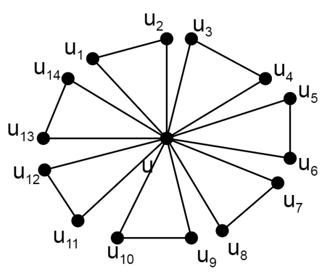
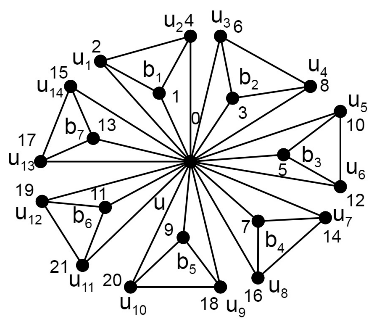


Figure D: $T_b(F_7)$
Case 4: $n \equiv 3(mod4)$



Theorem 3.5. *For* $n \geq 3$, $T_b(P_n \odot K_1)$ *admits SDL.*

Proof. By Theorem A, the vertex set of $T_b(P_n \odot K_1)$ be $V(T_b(P_n \odot K_1)) = 2n+r$ and the edge set be $E(T_b(P_n \odot K_1)) = 6n - 3$. Let$B = b_1, b_2, b_3, ..., b_r$ be the blocks of $(P_n \odot K_1)$ and the total number of blocks (r) are equal to $\mid E(P_n \odot K_1) \mid$ i.e., $\mid B(P_n \odot K_1) \mid = 2n - 1$.
Define a function $f : V(T_b(P_n \odot K_1)) \to \{0, 1, 2, \ldots, (2n + r - 1)\}$ thus, the

vertices are assigned as,

$$\begin{aligned} f(u_i) &= 2i-1 \;\; for \;\; 1 \leq i \leq n \\ f(b_i) &= f(u_n)+2+2(i-1) \;\; for \;\; 1 \leq i \leq (n-1) \\ f(v_j) &= 2j \;\; for \;\; 0 \leq j \leq (n-1), \;\; f(b_n) = (2n+r-1) \\ f(b_{(n+1)+(i-1)}) &= f(v_n)+2+2(i-1) \;\; for \;\; 1 \leq i \leq (n-1). \end{aligned}$$

The induced edge labeling function is injective and values are from $E(G)$ to N. There are eight classes of edges which are distinctly labeled as follows,
E_1: Edges joining (naturally distinct)consecutive odd numbers and the labels $f^*\{u_iu_{i+1}; 1 \leq i \leq (n-1)\}$ are in strictly increasing sequence of even numbers.
E_2: The edge labels $f^*\{u_iv_j; 1 \leq i \leq n \; and \; 0 \leq j \leq (n-1)\}$ are in strictly increasing sequence, having end vertices with the labels in which one of them is even and the other is odd, which receive odd edge labels.
E_3: Edges joining distinct odd numbers and the labels $f^*\{u_ib_i; 1 \leq i \leq (n-1)\}$ are in ascending order of the form $(f(u_n)+2+2(i+1))^2-(4K-4i+1)$, which are distinct odd edge labels.
E_4: $f^*(u_nb_{r-(n-1)}) = 3(4n^2-4n+1)$.
E_5: Edges joining distinct odd and even numbers and the labels $f^*\{v_jb_i; 0 \leq j \leq (n-2) \; and \; 1 \leq j \leq (n-1)\}$ are in ascending order of the form $(f(u_n)+2+2(i+1)))^2-(4K)$, where $K = i^2$ which contains distinct odd edge labels.
E_6: $f^*(v_nb_{r-(n-1)}) = (12n^2-8)$.
E_7: Edges joining distinct odd and even numbers and the labels
$f^*\{u_ib_{(n+1)+(i-1)}; 1 \leq i \leq (n-1)\}$ are in ascending order of the form $(f(v_n)+2)+2(i+1))^2-(4K-4m+1)$, where $K, m \in N$ and $K = i^2 \; and \; m = i$ which contains distinct odd edge labels.
E_8: Edges joining distinct odd and even numbers and the labels
$f^*\{u_{i+1}b_{(n+1)+(i-1)}; 1 \leq i \leq (n-1)\}$ are in ascending order of the form $(f(v_n)+2)+2(i+1))^2-(4K-4m+1)$, where $K, m \in N$ and $K = i^2$ and $m = i$ which contains distinct odd edge labels. Clearly, all the classes of edges are not being same. Hence, $T_b(P_n \odot K_1)$ admits SDL. □

Example 3.5: The square difference labeling of semitotal-block graph of $P_4 \odot K_1$ is shown in the below figure.

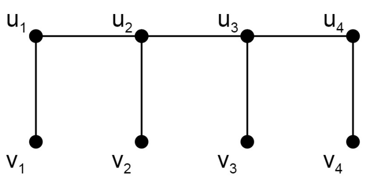


Figure: $P_4 \odot K_1$

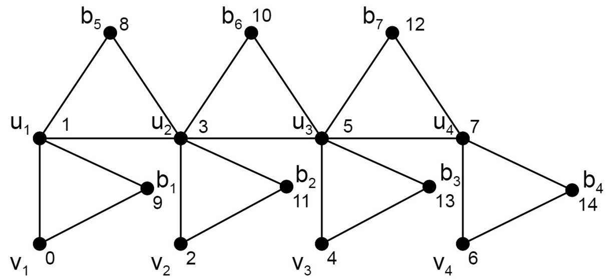


Figure: $T_b\ (P_4 \odot K_1)$

Theorem 3.6. *For* $n \geq 4$, $(T_b(C_n \odot K_1))$ *is SDL.*

Proof. By Theorem A, the vertex set of $T_b(C_n \odot K_1)$ be $V(T_bC_n \odot K_1)) = 3n+1$ and the edge set be $E(T_b(C_n \odot K_1)) = 5n$. Let $B = b_1, b_2, b_3, ..., b_r$ be the blocks of $(C_n \odot K_1)$ and the total number of blocks are equal to $(n+1)$ i.e., $\mid B(C_n \odot K_1) = n+1$.
We consider four cases
Case 1. $n \equiv 0\ (mod 4)$
Define a function $f : V(T_b(C_n \odot K_1)) \to \{0, 1, 2, \ldots, 3n\}$ thus, the vertices are assigned as,

$$\begin{aligned}
f(b_i) &= 4i \ \ for \ \ 1 \leq i \leq \frac{3n}{4}, \ \ f(b_0) = 0 \\
f(u_i) &= 2i-1, \ \ for \ \ 1 \leq i \leq n, \ \ f(v_i) = 2(2i-1) \ \ for \ \ 1 \leq i \leq \frac{3n}{4} \\
f(b_{\frac{3n}{4}+i}) &= f(b_{\frac{3n}{4}}) - 3 - 4(i-1)) \ \ for \ \ 1 \leq i \leq \frac{n}{4} \\
f(v_{\frac{3n}{4}+i}) &= f(v_{\frac{3n}{4}}) + 1 - 4(i-1)) \ \ for \ \ 1 \leq i \leq \frac{n}{4}.
\end{aligned}$$

The induced edge labeling function is injective and values are from $E(G)$ to N. There are nine classes of edges which are distinctly labeled as follows,
E_1: Edges joining consecutive (naturally distinct) odd numbers and common end vertex 0. The labeling of edges $f^*\{b_0u_i; 1 \leq i \leq n\}$ are of the form $(4K-4m+1)$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_2: Edges joining consecutive (naturally distinct) odd numbers and the labeling

of edges $f^*\{u_iu_{i+1}; 1 \leq i \leq (n-1)\}$ are distinct as it form a strictly increasing sequence of even numbers.
E_3: $f^*(u_nu_1) = 4n(n-1)$.
E_4: Edges joining consecutive distinct odd and even numbers and the edge labels $f^*\{u_iv_i; 1 \leq i \leq \frac{3n}{4}\}$ are in increasing sequence of the form $3(4K-4m+1)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the labeling pattern of the edges connecting consecutive odd numbers and consecutive even numbers are distinct.
E_5: Edges joining (naturally distinct) odd numbers, the edge labels $f^*\{u_{\frac{3n}{4}+i} v_{\frac{3n}{4}+i}; 1 \leq i \leq \frac{n}{4}\}$ are of the form $\mid (f(v_{\frac{3n}{4}}) + 1 - 4(m-1))^2 - (4K-4m+1) \mid$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_6: Edges joining consecutive distinct odd and even numbers, the edge labels $f^*\{u_ib_i; 1 \leq i \leq \frac{3n}{4}\}$ are in increasing sequence of the form $(12K + 4m - 1)$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_7: Edges joining (naturally distinct) odd numbers and the edge labels $f^*\{u_{\frac{3n}{4}+i} b_{\frac{3n}{4}+i}; 1 \leq i \leq \frac{n}{4}\}$ are in decreasing sequence of even numbers of the form $\mid (f(b_{\frac{3n}{4}}) - 3 - 4(m-1))^2 - (4K-4m+1) \mid$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_8: Edges joining consecutive (naturally distinct) odd and even numbers and the edge labels $f^*\{v_ib_i; 1 \leq i \leq \frac{3n}{4}\}$ are in increasing sequence of the form $(16m - 4)$, where $m \in N$ and , $m = i$.
E_9: Edges joining (naturally distinct) odd numbers and the edge labels $f^*\{v_{\frac{3n}{4}+i} b_{\frac{3n}{4}+i}; 1 \leq i \leq \frac{n}{4}\}$ are in decreasing sequence of the form $\mid (f(v_{\frac{3n}{4}}) + 1 - 4(m-1))^2 - (f(b_{\frac{3n}{4}}) - 3 - 4(m-1))^2 \mid$, where $K, m \in N$ and $K = i^2$, $m = i$.
Case 2. $n \equiv 1\ (mod 4)$
Define a function $f : V(T_b(C_n \odot K_1)) \to \{0, 1, 2, \ldots, 3n\}$ so, the vertices are labeled as,

$$\begin{aligned} f(b_i) &= 4i \;\; for \;\; 1 \leq i \leq \frac{3(n-1)}{4}, \;\; f(b_0) = 0 \\ f(u_i) &= 2i - 1 \;\; for \;\; 1 \leq i \leq n, \;\; f(v_i) = 2(2i-1) \;\; for \;\; 1 \leq i \leq \frac{3n+1}{4} \end{aligned}$$

$$f(b_{\frac{3(n-1)}{4}+i}) = f(b_{\frac{3(n-1)}{4}} + 3 - 4(i-1) \;\; for \;\; 1 \leq i \leq \frac{n+3}{4}$$
$$f(v_{\frac{3n+1}{4}+i}) = f(v_{\frac{3n}{4}}) - 1 - 4(i-1) \;\; for \;\; 1 \leq i \leq \frac{n-1}{4}.$$

The induced edge labeling function is injective and values are from $E(G)$ to N. There are ten classes of edges which are distinctly labeled as follows,
E_1: Edges joining consecutive (naturally distinct) odd numbers and common end vertex. The labeling of edges $f^*\{b_0u_i; 1 \leq i \leq n\}$ are of the form $(4K-4m+1)$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_2: Edges joining consecutive (naturally distinct) odd numbers and the labeling of edges $f^*\{u_iu_{i+1}; 1 \leq i \leq (n-1)\}$ are distinct as it form a strictly increasing sequence of even numbers.
E_3: $f^*(u_nu_1) = 4n(n-1)$.
E_4: Edges joining consecutive distinct odd and even numbers and the edge labels $f^*\{u_iv_i; 1 \leq i \leq \frac{3n+1}{4}\}$ are in increasing sequence of the form $3(4K-4m+1)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the labeling pattern of the edges connecting consecutive odd numbers and consecutive even numbers are distinct.
E_5: Edges joining (naturally distinct) odd numbers and the edge labels,
$f^*\{u_{\frac{3n+1}{4}+i}v_{\frac{3n+1}{4}+i}; 1 \leq i \leq \frac{n}{4}\}$ are of the form $(f(v_{\frac{3n+1}{4}}) - 1 - 4(m-1))^2 - (4K-4m+1)$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_6: Edges joining consecutive distinct odd and even numbers, the edge labels $f^*\{u_ib_i; 1 \leq i \leq \frac{3(n-1)}{4}\}$ are in increasing sequence of the form $(12K+4m-1)$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_7: Edges joining (naturally distinct) odd numbers and the edge labels,
$f^*\{u_{\frac{3(n-1)}{4}+i}b_{\frac{3(n-1)}{4}+i}; 1 \leq i \leq \frac{n+3}{4}\}$ are in decreasing sequence of the form $(f(b_{\frac{3(n-1)}{4}}) + 3 - 4(m-1))^2 - (4K-4m+1)$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_8: Edges joining consecutive distinct odd and even numbers, the edge labels $f^*\{v_ib_i; 1 \leq i \leq \frac{3(n-1)}{4}\}$ are in increasing sequence of the form $(16m-4)$, where $m \in N$ and , $m = i$.
E_9: $f^*(v_{\frac{3n+1}{4}}b_{\frac{3(n+1)}{4}}) = (6n-1)$.
E_{10}: Edges joining distinct odd numbers and the the edge labels

$f^*\{v_{\frac{3n+1}{4}+i} b_{\frac{3(n+1)}{4}+1}; 1 \leq i \leq \frac{n-1}{4}\}$ are in decreasing order of the form $\mid (f(v_{\frac{3n+1}{4}}) - 1 - 4(m-1))^2 - (f(b_{\frac{3n+1}{4}} - 4 - 4(m-1))^2 \mid$, where $m \in N$ and $m = i$.

Case 3. $n \equiv 2\ (mod 4)$

Define a function $f : V(T_b(C_n \odot K_1) \to \{0, 1, 2, \ldots, 3n\}$ such that, vertices are labeled as,

$$\begin{aligned}
f(b_i) &= 4i \ \ for \ \ 1 \leq i \leq \frac{3n-2}{4}, \ \ f(b_0) = 0 \\
f(u_i) &= 2i - 1, \ \ for \ \ 1 \leq i \leq n \\
f(v_i) &= 2(2i-1) \ \ for \ \ 1 \leq i \leq \frac{3n+2}{4} \\
f(b_{\frac{3n-2}{4}+i}) &= f(b_{\frac{3n-2}{4}}) + 1 - 4(i-1) \ \ for \ \ 1 \leq i \leq \frac{n+2}{4} \\
f(v_{\frac{3n+2}{4}+i}) &= f(v_{\frac{3n+2}{4}}) - 3 - 4(i-1) \ \ for \ \ 1 \leq i \leq \frac{n-2}{4}.
\end{aligned}$$

The induced edge labeling function is injective and values are from $E(G)$ to N. There are nine classes of edges which are distinctly labeled as follows,

E_1: Edges joining consecutive (naturally distinct) odd numbers and the labeling of edges $f^*\{b_0u_i; 1 \leq i \leq n\}$ are of the form $(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2$, $m = i$.

E_2: Edges joining consecutive (naturally distinct) odd numbers and the labeling of edges $f^*\{u_iu_{i+1}; 1 \leq i \leq (n-1)\}$ are distinct as it form a strictly increasing sequence of even numbers.

$E_3 : f^*(u_nu_1) = 4n(n-1)$.

E_4: Edges joining consecutive distinct odd and even numbers and the edge labels $f^*\{u_iv_i; 1 \leq i \leq \frac{3n+2}{4}\}$ are in increasing sequence of the form $3(4K-4m+1)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the labeling pattern of the edges connecting consecutive odd numbers and consecutive even numbers are distinct.

E_5: Edges joining (naturally distinct) odd numbers and the edge labels, $f^*\{u_{\frac{3n+2}{4}+i} v_{\frac{3n+2}{4}+i}; 1 \leq i \leq \frac{n-2}{4}\}$ are of the form $\mid (f(v_{\frac{3n+2}{4}}) - 3 - 4(m-1))^2 - (4K - 4m + 1) \mid$, where $K, m \in N$ and $K = i^2$, $m = i$.

E_6: Edges joining consecutive distinct odd and even numbers, the edge labels $f^*\{u_ib_i; 1 \leq i \leq \frac{3n-2}{4}\}$ are in increasing sequence of the form $(12K+4m-1)$,

where $K, m \in N$ and $K = i^2$, $m = i$.
E_7: Edges joining (naturally distinct) odd numbers and the edge labels, $f^*\{u_{\frac{3n+2}{4}+i} b_{\frac{3n+2}{4}+i}; 1 \leq i \leq \frac{n+2}{4}\}$ are in decreasing sequence of the form $\mid (f(b_{\frac{3n+2}{4}}) + 3 - 4(m-1))^2 - (4K - 4m + 1 \mid)$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_8: Edges joining consecutive distinct odd and even numbers, the edge labels $f^*\{v_i b_i; 1 \leq i \leq \frac{3n-2)}{4}\}$ are in increasing sequence of the form $(16m - 4)$, where $m \in N$ and , $m = i$.
E_9: Edges joining naturally distinct odd numbers except first end vertex and the edge labels $f^*\{v_{\frac{3n-2}{4}+i} b_{\frac{3n-2}{4}+i}; 1 \leq i \leq \frac{n+2}{4}\}$ are in decreasing sequence of the form $\mid (f(v_{\frac{3n-2}{4}}) + 4 - 3(m-1))^2 - (f(b_{\frac{3(n-1)}{4}}) + 1 - 4(m-1))^2 \mid$, where $m \in N$ and $m = i$.
Case 4. $n \equiv 3 \ (mod 4)$
Define a function $f : V(T_b(C_n \odot K_1)) \to \{0, 1, 2, \ldots, 3n\}$ such that, vertices are labeled as,

$$\begin{aligned}
f(b_i) &= 4i \ \ for \ \ 1 \leq i \leq \frac{3n-1}{4}, \ \ f(b_0) = 0 \\
f(u_i) &= 2i - 1, \ \ for \ \ 1 \leq i \leq n \\
f(v_i) &= 2(2i-1) \ \ for \ \ 1 \leq i \leq \frac{3n-1}{4} \\
f(b_{\frac{3n-1}{4}+i}) &= f(b_{\frac{3n-1}{4}}) - 1 - 4(i-1) \ \ for \ \ 1 \leq i \leq \frac{n+1}{4} \\
f(v_{\frac{3n-1}{4}+i}) &= f(v_{\frac{3n-1}{4}}) + 3 - 4(i-1) \ \ for \ \ 1 \leq i \leq \frac{n+1}{4}.
\end{aligned}$$

The induced edge labeling function is injective and values are from $E(G)$ to N. There are nine classes of edges which are distinctly labeled as follows,
E_1: Edges joining consecutive (naturally distinct) odd numbers and the labeling of edges $f^*\{b_0 u_i; 1 \leq i \leq n\}$ are of the form $(4K - 4m + 1)$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_2: Edges joining consecutive (naturally distinct) odd numbers and the labeling of edges $f^*\{u_i u_{i+1}; 1 \leq i \leq (n-1)\}$ are distinct as it form a strictly increasing sequence of even numbers.
E_3 : $f^*(u_n u_1) = 4n(n-1)$.

E_4: Edges joining consecutive distinct odd and even numbers and the edge labels $f^*\{u_iv_i; 1 \le i \le \frac{3n-1}{4}\}$ are in increasing sequence of the form $3(4K-4m+1)$, where $K, m \in N$ and $K = i^2$, $m = i$. Since the labeling pattern of the edges connecting consecutive odd numbers and consecutive even numbers are distinct.
E_5: Edges joining naturally distinct odd numbers, the edge labels such as $f^*\{u_{\frac{3n-1}{4}+i} v_{\frac{3n-1}{4}+i}; 1 \le i \le \frac{n+1}{4}\}$ are of the form $(f(v_{\frac{3n-1}{4}}) + 1 - 4(m-1))^2 - (4K - 4m + 1)$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_6: Edges joining consecutive distinct odd and even numbers, the edge labels $f^*\{u_ib_i; 1 \le i \le \frac{3n-1}{4}\}$ are in increasing sequence of the form $(12K+4m-1)$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_7: Edges joining (naturally distinct) odd numbers, the edge labels $f^*\{u_{\frac{3n-1}{4}+i} b_{\frac{3n-1}{4}+i}; 1 \le i \le \frac{n+1}{4}\}$ are in decreasing sequence of the form $\mid (f(b_{\frac{3n-1)}{4}}) - 1 - 4(m-1))^2 - (4K - 4m + 1) \mid$, where $K, m \in N$ and $K = i^2$, $m = i$.
E_8: Edges joining consecutive distinct even numbers and the edge labels $f^*\{v_ib_i; 1 \le i \le \frac{3n-1}{4}\}$ are in increasing sequence of the form $(16m - 4)$, where $m \in N$ and , $m = i$.
E_9: Edges joining (naturally distinct) odd numbers and the edge labels $f^*\{v_{\frac{3n-1}{4}+i} b_{\frac{3n-1}{4}+i}; 1 \le i \le \frac{n+1}{4}\}$ are in decreasing sequence of the form $\mid (f(v_{\frac{3n-1}{4}}) + 3 - 4(m-1))^2 - (f(b_{\frac{3(n-1)}{4}}) - 1 - 4(m-1))^2 \mid$, where $m \in N$ and $m = i$. Clearly, the labeling of all the classes of edges are not being same and the function f is SDL and the graph $T_b(C_n \odot K_1)$ admits SDG. □
Example 3.6: The square difference labeling of semitotal-block graph of $C_4 \odot K_1$, $C_5 \odot K_1$, $C_6 \odot K_1$, and $C_7 \odot K_1$ are shown in the below figure

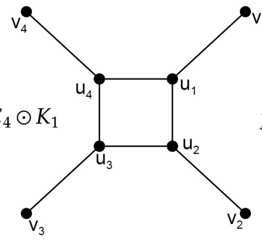
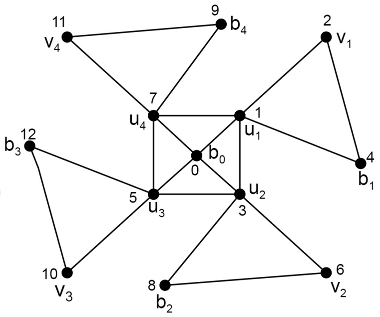


Figure a : $C_4 \odot K_1$

Figure A : $T_b(C_4 \odot K_1)$
Case 1 : $n \equiv 0(mod4)$



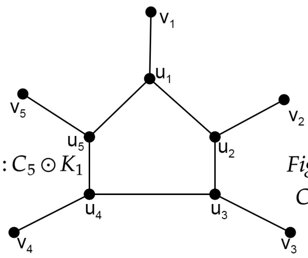
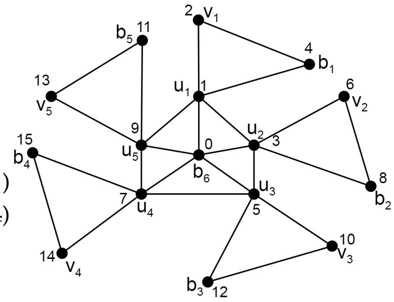


Figure b : $C_5 \odot K_1$

FigureB : $T_b(C_5 \odot K_1)$
Case 2 : $n \equiv 1(mod4)$



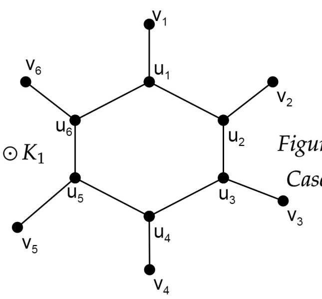
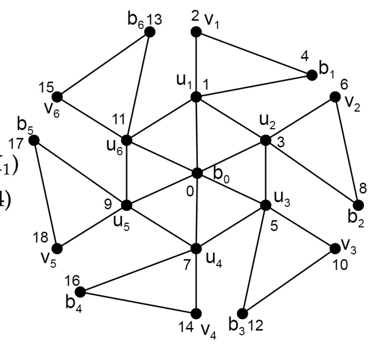


Figure c : $C_4 \odot K_1$

Figure C : $T_b(C_6 \odot K_1)$
Case 3 : $n \equiv 2(mod4)$



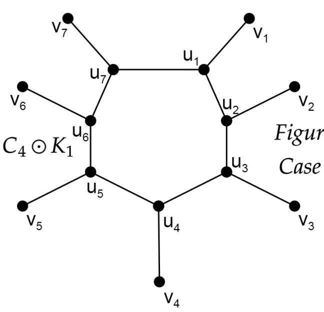
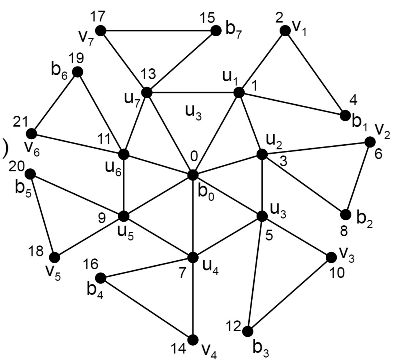


Figure d : $C_4 \odot K_1$

Figure D : $T_b(C_7 \odot K_1)$
Case 4 : $n \equiv 3(mod4)$



4 Conclusion

In this article, we have studied on vertex labeling functions called square sum and square difference labeling and obtained the results for semitotal-block graph for some class of graphs which admits both SSL and SDL. This work contributes sevaral new results to the theory of graph labeling.

Acknowledgements

Authors are thankful to Karnatak University, Dharwad, Karnataka, India for the support through University Research Studentship (URS), No. KU/Sch/URS/2021/623, dated: 26/10/2021.

References

J. Babujee and S. Babitha. On square sum labeling in graphs. *Int.Rev.Fuzzy Math.*, 7(2):81–87, 2012.

J. Bondy and U. Murthy. *Graph Theory With Applications*. Elsevier Science, 1976.

D. Burton. *Elementary Number Theory*. Tata Magraw Hill, 2006.

F.Harary. *Graph Theory*. Addison-Wesleyl, 1969.

J. A. Gallian. A dynamic survey of graph labeling. *Electronic Journal of combinatorics*, (DynamicSurveys), 2020.

G. Ghodasara and M. Patel. Innovative results on square sum labeling of graphs. *J. Graph Label.*, 4(1):15–24, 2018.

G. Ghodasara and M. J. Patel. Some bistar related square sum graphs. *International Journal of Mathematics Trends and Technology*, 47(3):172–177, 2017.

V. Govindan and S. Dhivya. Square sum and square difference labeling of graphs. *J.Appl.Sci. Comput.*, 4(2):1687–1692, 2019.

V. R. Kulli. The semitotal-block graph and the total-block graph of a graph. 1976.

R. Sebastian and K. Germina. Square sum labeling of class of planar graphs. *Proyecciones (Antofagasta)*, 34(1):55–68, 2015.

J. Shiama. Square difference labeling for some graphs. *International journal of computer applications*, 44(4):30–33, 2012.