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Moduli of continuity of functions in Hölder's class $H^{\alpha,2}_{\omega_k}[0,1)$ by first kind Chebyshev wavelets and its applications in the solution of Lane-Emden differential equations

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Abstract

In this paper, two new moduli of continuity $W\left(\left(f-S_{2^{k-1},0}f\right),\frac{1}{2^k}\right)$, $W\left(\left(f-S_{2^{k-1},M}f\right),\frac{1}{2^k}\right)$ and two estimators $E_{2^{k-1},0}(f)$ and,$E_{2^{k-1},M}(f)$ of a functions f in Hölder's class $H^{\alpha,2}_{\omega_k}[0,1)$ by First kind Chebyshev wavelets have been determined. These moduli of continuity and estimators are new and best possible in wavelet analysis. Applying this technique, Lane -Emden differential equations have been solved by the first of kind Chebyshev wavelet method. These solutions obtained by first kind Chebyshev wavelet method approximately coincided with their exact solutions. This is a significant achievement of this research paper in wavelet analysis.

Keywords: Chebyshev wavelet, Modulus of continuity, Wavelet approximation, Hölder's class, Orthonormal basis, Operational matrix of integration .

2020 AMS subject classifications: 42C40, 65T60, 65L10 , 65L60. [1]

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1 Introduction

Recently, researchers are making attention on wavelets. Wavelets have connections between several branches of mathematical sciences and play an important role in signal processing, engineering and technology.The approximation of functions of a certain class by trigonometric series is a common places of analysis.Approximation of functions belonging to some class by wavelet method has been discussed by many researchers like DeVore[1], Morlet[4] , Meyer[3] and Debnath[2]. Wavelets are new tools to solve differential equations and to estimate the moduli of continuity & the approximation of functions.Wavelets help in the most accurate representation of functions $f \in H^{\alpha,2}_{\omega_k}[0, 1)$ class. Several wavelets are known like that Haar wavelet , Legendre wavelet , Chebyshev wavelet. Haar wavelet is one of simplest in wavelet analysis . Due to its simplicity and better applications, it is used in solution of integral as well as differential equations . Haar wavelet contains a non-smooth character. This is a difficiency of Haar wavelet to estimate the moduli of continuity and the approximation of the smooth function by it. This weak point is almost removed by Chebyshev wavelets and more accurate the moduli of continuity and approximations of functions are obtained. Sripathy[14] discussed the chebyshev wavelet based approximation for solving linear and non-linear differential equations.

Moduli of continuity of functions have been studies by Alexander Babenko[10]. In best of our knowledge, there is no work associated to the modulus of continuity and approximation of a function f in Hölder's class by first kind Chebyshev wavelet method. To make an advanced study in this direction, in this paper, the moduli of continuity and approximation of functions have been determined in Hölder's class $H^{\alpha,2}_{\omega_k}[0, 1)$.

Several linear, as well as non-linear differential equations are solvable by Galerkin, Collocation, and other known methods. These equations can be solved by Chebyshev wavelet technique in very efficient and suitable manners. This motivates us to consider first kind Chebyshev wavelet method for the solution of differential equations. Also, Babolian and Fattahzadeh[5] suggested a method to solve the differential equations by using Chebyshev wavelet operational matrix of integration.

In this paper, the Lane-Emden differential equations has been solved by Chebyshev wavelet technique. The main characteristic of this techique is that it reduces the problem to a system of algebraic equations. The approach is based on converting the given differential equations into integral equations through integration by approximating various signals involved in the equation through truncated orthogonal Chebyshev wavelet series and using the operational matrix P of integration , to eliminate the integral operations.

This paper is organized as follows: Section(1) is introductory in which the importance of moduli of continuity and Chebyshev wavelet of first kind related literature are studied. In section(2), Chebyshev wavelet of first kind ,approximation of function and moduli of continuity of functions in class $H^{\alpha,2}_{\omega_k}[0,1)$ are defined.In section(3), theorem concerning the moduli of continuity of $f-S_{2^{k-1},M}(f)$ has been established and also its detail proof is discussed in section(4).In section(5) corollaries are deduced from theorem of section (3) . In section(6) first kind Chebyshev wavelet operational matrix of integration has been constructed and the product operational matrix has been obtained in section(7). In section(8), Lane-Emden differential equations of index 0, 1 & 2 are solved by Chebyshev wavelet method. Finally, the main conclusions are summarized in section (9).

2 Definitions and Preliminaries

2.1 Chebyshev wavelets of first kind

Wavelets constitute a family of functions constructed from dialation and translation of a single function $\Psi \in L^2(\mathbb{R})$ called mother wavelet .We write

$$\Psi_{b,a}(t) = |a|^{\frac{-1}{2}}\Psi\left(\frac{t-b}{a}\right), \quad a \neq 0. \qquad \text{(Daubechies [6])}$$

If we restrict the values of dialation and translation parameter to $a = a_0^{-k}, b = (2n-1)b_0{a_0}^{-k}, a_0 > 1, b_0 > 0$ respectively,the following family of discrete wavelets are constructed:

$$\Psi_{k,n}(t) = |a_0|^{\frac{k}{2}}\Psi\left(a_0^k t - (2n-1)b_0\right).$$

Now, taking $a_0 = 2, \Psi(t) = \tilde{T}_m(t)$ & $b_0 = 1$ the Chebyshev wavelet $\Psi(k,n,m,t)$ of first kind , generally denoted by $\Psi^{(c_1)}_{n,m}(t)$ over the interval [0,1), is obtained as (Babolian [5])

$$\Psi^{(c_1)}_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\tilde{T}_m(2^k t - 2n + 1), & \dfrac{n-1}{2^k-1} \leq t < \dfrac{n}{2^k-1} \\ 0 & \text{, otherwise} \end{cases} \tag{1}$$

$$\text{where} \qquad \tilde{T}_m(t) = \begin{cases} \dfrac{1}{\sqrt{\pi}}, & m = 0 \\ \sqrt{\dfrac{2}{\pi}}T_m(t) & \text{, otherwise.} \end{cases}$$

where $n = 1,2,...,2^{k-1}$, $m = 0,1,2,...M$ and k is the positive integer. In above definition, T_m are the first kind Chebyshev polynomials of degree m on the interval [-1,1] which are defined by

$$T_m(t) = \cos(m\theta), \quad \theta = \arccos(t) \tag{2}$$

and also satisfy the following recursive formula:

$$T_0(t) = 1 \quad T_1(t) = t, \quad T_{m+1}(t) = 2\,t\,T_m(t) - T_{m-1}(t), \qquad m = 1, 2, 3, ...$$

The set of $\{T_m(t) : m = 0, 1, 2, 3, ...\}$ in the Hilbert space $L^2[-1, 1]$ is a orthogonal set with respect to the weight function $\omega(t) = \frac{1}{\sqrt{1-t^2}}$. Orthogonality of Chebyshev polynomial of first kind on the interval [-1,1] implies that

$$\langle T_m(t), T_n(t)\rangle = \int_{-1}^{1} \frac{T_m(t)\overline{T_n(t)}}{\sqrt{1-t^2}}dt = \begin{cases} \pi, & m = n = 0 \\ 0, & n \neq m. \\ \dfrac{\pi}{2}, & n = m \neq 0. \end{cases}$$

In dealing with Chebyshev wavelets, the weight function $\omega(t)$ for orthogonal Chebyshev polynomials has to dilated and translated to construct orthonomal wavelets. So the first kind Chebyshev wavelets are an orthonormal set with weight function (S. Dhawan[7])

$$\omega_k(t) = \begin{cases} \omega_{1,k}(t), & 0 \leq t < \frac{1}{2^{k-1}}, \\ \omega_{2,k}(t), & \frac{1}{2^{k-1}} \leq t < \frac{2}{2^{k-1}}, \\ \vdots \\ \omega_{2^{k-1},k}(t), & \frac{2^{k-1}-1}{2^{k-1}} \leq t < 1, \end{cases} \tag{3}$$

where $\omega_{n,k}(t) = \omega(2^k t - 2n + 1)$. Furthermore, the set of wavelets $\psi_{n,m}(t)$ makes an orthonormal basis in Hilbert space $L^2_{\omega_k}[0, 1)$, $i.e.$

$$\langle \Psi^{(c_1)}_{n,m}, \Psi^{(c_1)}_{n',m'}\rangle_{\omega_k} = \int_0^1 \Psi^{(c_1)}_{n,m}(t)\Psi^{(c_1)}_{n'm'}(t)\omega_k(t)dt = \delta_{n,n'}\delta_{m,m'}$$

in which δ denotes Kronecker delta function defined by

$$\delta_{n,n'} = \begin{cases} 1, & \text{n=n'} \\ 0, & \text{otherwise} \end{cases}$$

2.2 First kind Chebyshev wavelet expansion and approximation of function

The function $f \in L^2_{\omega_k}[0, 1)$ is expressed in the Chebyshev wavelet series as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(c_1)}_{n,m}(t), \tag{4}$$

where $c_{n,m} = \langle f, \Psi^{(c_1)}_{n,m} \rangle_{\omega_k}$. The $(2^{k-1}, M+1)^{th}$ partial sums of above series (4) is given by

$$S_{2^{k-1},M}(f)(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \Psi^{(c_1)}_{n,m}(t) = C^T \Psi^{(c_1)}(t) \tag{5}$$

in which C and $\Psi^{(c_1)}(t)$ are $2^{k-1}(M+1)$ vectors of the form

$$C^T = [c_{1,0}, c_{1,1}, ...c_{1,M}, c_{2,0}, c_{2,1}..., c_{2,M}, ..., c_{2^{k-1},0}, ..., c_{2^{k-1},M}] \quad \text{and}$$
$$\Psi^{(c_1)} = [\Psi^{(c_1)}_{1,0}, \Psi^{(c_1)}_{1,1}, ..., \Psi^{(c_1)}_{1,M}, \Psi^{(c_1)}_{2,0}, \Psi^{(c_1)}_{2,1}, ..., \Psi^{(c_1)}_{2,z}, ..., \Psi^{(c_1)}_{2^{k-1},0}, ..., \Psi^{(c_1)}_{2^{k-1},M-1}]^T$$

The Chebyshev wavelet approximation $E_{2^{k-1},M}(f)$ of a function $f \in L^2_{\omega_k}[0,1)$ by $(2^{k-1}, (M+1))^{th}$ partial sums $S_{2^{k-1},M}(f)$ of its Chebyshev wavelet series is given by

$$E_{2^{k-1},M}(f) = \min_{S_{2^{k-1},M}(f)} \|f - S_{2^{k-1},M}(f)\|_2$$
$$\text{where,} \quad \|f\|_2 = \left(\int_0^1 |f(t)|^2 \, \omega_k(t) dt \right)^{\frac{1}{2}}$$

If $E_{2^{k-1},M}(f) \to 0$ as $k, M \to \infty$ then $E_{2^{k-1},M}(f)$ is called the best approximation of f of order $(2^{k-1}, M+1)$ (Zygmund[8]).

2.3 Modulus of continuity

The Modulus of continuity of a function $f \in L^2_{\omega_k}[0,1)$ is defined as

$$\begin{aligned} W(f,\delta) &= \sup_{0<h\leq\delta} ||f(\cdot + h) - f(\cdot)||_2 \\ &= \sup_{0<h\leq\delta} \left(\int_0^1 |f(t+h) - f(t)|^2 \omega_k(t) dt \right)^{\frac{1}{2}}. \end{aligned}$$

It is remarkable to note that $W(f,\delta)$ is a non-decresing function of δ and $W(f,\delta) \to 0$ as $\delta \to 0^+$ (Chui[9]).

2.4 Function of Hölder's class

A function f is said to be in Hölder's class $H^{\alpha}_{\omega_k}[0,1)$ of order α, $0 < \alpha \leq 1$, (G.Das[11]) if f satisfies

$$\begin{aligned} \big(f(x+t) - f(x)\big)\omega_k(x) &= O\left(|t|^{\alpha}\right), \quad \forall x, t, x+t \in [0,1). \\ If \ \left(\int_0^1 |f(x+t) - f(x)|^2 \, \omega_k(x) dx \right)^{\frac{1}{2}} &= O\left(|t|^{\alpha}\right), \quad \forall x, t, x+t \in [0,1). \end{aligned}$$

then f is said to be a function of Hölder's class $H^{\alpha,2}_{\omega_k}[0,1)$ of order α, $0<\alpha\leq 1$.
The class $H^{\alpha}_{\omega_k}[0,1)$ is a proper subclass of $H^{\alpha,2}_{\omega_k}[0,1)$.
Let $f\in H^{\alpha}_{\omega_k}[0,1), \alpha=1$. Then

$$\Big(f(x+t)-f(x)\Big)\omega_k(x)=O(|t|)$$

$$\begin{aligned}
\left(\int_0^1 |f(x+t)-f(x)|^2\omega_k(x)dx\right) &= \int_0^1 \frac{O\left(|t|\right)^2}{\omega_k(x)}dx\\
&= O\left(|t|^2\right)\sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\sqrt{1-(2^kx-2n+1)^2}dx\\
&= O\left(|t|^2\right)\sum_{n=1}^{2^{k-1}}\int_0^{\pi}\sin^2\theta\ \frac{d\theta}{2^k}\\
&= O\left(|t|^2\right)\ \frac{\pi}{4}
\end{aligned}$$

$$\left(\int_0^1 |f(x+t)-f(x)|^2\omega_k(x)dx\right)^{\frac{1}{2}} = \left(O\left(|t|\right)^2\right)^{\frac{1}{2}}\frac{\sqrt{\pi}}{2}\quad = O\left(|t|\right)$$

i.e $\quad f\in H^{\alpha,2}_{\omega_k}[0,1),\quad \alpha=1.$

$$Let,\qquad f(t)=\begin{cases} t^2, & 0\leq t<\frac{1}{2}\\ 1, & \frac{1}{2}\leq t<1\end{cases}$$

$$f\left(\frac{1}{2}+\frac{\epsilon}{2}\right)-f\left(\frac{1}{2}-\frac{\epsilon}{2}\right)=1-(\frac{1}{2}-\frac{\epsilon}{2})^2=\frac{3}{4}+\frac{\epsilon}{2}-\frac{\epsilon^2}{4}.$$

$$\left(\frac{f\left(\frac{1}{2}+\frac{\epsilon}{2}\right)-f\left(\frac{1}{2}-\frac{\epsilon}{2}\right)}{\epsilon}\right)\omega_k(x)=\left(\frac{3}{4\epsilon}+\frac{1}{2}-\frac{\epsilon}{4}\right)\omega_k\left(\frac{1}{2}-\frac{\epsilon}{2}\right)\to+\infty \text{ as } \epsilon\to 0^+$$

Therefore, $\quad f\notin H^{\alpha}_{\omega_k}[0,1), \alpha=1.$

$$\begin{aligned}
\int_0^1 \big|f(x+t)-f(x)\big|^2\omega_k(x)dx &= \int_0^{\frac{1}{2}} \big|f(x+t)-f(x)\big|^2\omega_k(x)dx\\
&+\int_{\frac{1}{2}}^1 \big|f(x+t)-f(x)\big|^2\omega_k(x)dx\\
&= \int_0^{\frac{1}{2}} |(x+t)^2-x^2|^2\omega_k(x)dx \quad + \quad 0\\
&\leq \int_0^{\frac{1}{2}} \big|2xt+t^2\big|\frac{2}{\sqrt{3}}dx, \qquad 1\leq\omega(x)\leq\frac{2}{\sqrt{3}}\\
&= \frac{2}{\sqrt{3}}\int_0^{\frac{1}{2}}\left(4x^2t^2+t^4+4xt^3\right)dx
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \left|f(x+t)-f(x)\right|^2 \omega_k(x)dx &= \frac{2}{\sqrt{3}}\left[\frac{4}{3}t^2 \times \frac{1}{8} + t^4 \times \frac{1}{2} + 2t^3 \times \frac{1}{4}\right] \\
&= \frac{2}{\sqrt{3}}\left[\frac{1}{6} \times t^2 + \frac{1}{2} \times t^2 + \frac{1}{2} \times t^2\right] \\
&= \frac{7}{3\sqrt{3}} t^2
\end{aligned}$$

$$\left(\int_0^1 \left|f(x+t)-f(x)\right|^2 \omega_k(x)dx\right)^{\frac{1}{2}} \leq \sqrt{\frac{7}{3\sqrt{3}}}|t|$$

$$\left(\int_0^1 |f(x+t)-f(x)|^2 \omega_k(x)dx\right)^{\frac{1}{2}} = O\left(|t|\right)$$

$$f \in H^{\alpha,2}_{\omega_k}[0,1), \alpha = 1$$

Thus, $H^{\alpha}_{\omega_k}[0,1) \subsetneq H^{\alpha,2}_{\omega_k}[0,1),\ \alpha = 1.$

3 Theorems

In this paper, following theorems have been proved.

Theorem 3.1. *If* $f \in H^{\alpha,2}_{\omega_k}[0,1)$ *and its first kind Chebyshev wavelet expansion be*

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(c_1)}_{n,m}(t)$$

having $(2^{k-1}, M+1)^{th}$ *partial sums*

$$\left(S_{2^{k-1},M}(f)\right)(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\Psi^{(c_1)}_{n,m}(t)$$

Then the Moduli of continuity of $f - \left(S_{2^{k-1},M}(f)\right)$ *satisfies*

i. $$\begin{aligned} W((f - S_{2^{k-1},0}f), \frac{1}{2^k}) &= \underset{0<h\leq\frac{1}{2^k}}{sup} \|(f - S_{2^{k-1},0}f)(\cdot + h) - (f - S_{2^{k-1},0}f)(\cdot)\| \\ &= O\left(\frac{1}{2^{(k-1)\alpha}}\right) \end{aligned}$$

ii. $$\begin{aligned} W((f - S_{2^{k-1},M}f), \frac{1}{2^k}) &= \underset{0<h\leq\frac{1}{2^k}}{sup} \|(f - S_{2^{k-1},M}f)(\cdot + h) - (f - S_{2^{k-1},M}f)(\cdot)\| \\ &= O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right), \quad M \geq 1. \end{aligned}$$

4 Proof

Proof of the theorem (3.1)

(**i**) Error between f(x) and its Chebyshev wavelet expansion in interval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$ is given by

$$
\begin{aligned}
e_n(t) &= c_{n,0}\Psi_{1,0}^{(c_1)}(t) - f\chi_{\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right)}(t)\\
&= \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(x)\Psi_{1,0}^{(c_1)}(x)\omega_k(x)dx\right)\Psi_{1,0}^{(c_1)}(t) - f\chi_{\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right)}(t)\\
&= \frac{2^{\frac{k}{2}}}{\sqrt{\pi}}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(x)\omega_k(x)dx\right)\frac{2^{\frac{k}{2}}}{\sqrt{\pi}} - f\chi_{\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right)}(t)\\
&= \frac{2^k}{\pi}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(x)\omega_k(x)dx\right) - \frac{2^k}{\pi}f\chi_{\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right)}(t)\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\omega_k(x)dx\\
&= \frac{2^k}{\pi}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(f(x)-f(t)\right)\omega_k(x)dx\right).\\
|e_n(t)| &\leq \frac{2^k}{\pi}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |(f(x)-f(t))||\omega_k(x)|dx\right)\\
&= \frac{2^k}{\pi}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |(f(x)-f(t))||\sqrt{\omega_k(x)}||\sqrt{\omega_k(x)}|dx\right)\\
&\leq \frac{2^k}{\pi}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(|(f(x)-f(t))|\right)^2|\omega_k(x)dx\right)^{\frac{1}{2}}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}|\omega_k(x)|dx\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}|\omega_k(x)|dx = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}|\omega(2^kx-2n+1)|dx = \frac{\pi}{2^k}
$$

$$
\begin{aligned}
|e_n(t)| &\leq \frac{2^k}{\pi}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(|(f(x)-f(t))|\right)^2|\omega_k(x)dx\right)^{\frac{1}{2}}\left(\frac{\pi}{2^k}\right)^{\frac{1}{2}}\\
&= \frac{2^k}{\pi}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(|(f(x)-f(t))|\right)^2|\omega(2^kx-2n+1)dx\right)^{\frac{1}{2}}\left(\frac{\pi}{2^k}\right)^{\frac{1}{2}}\\
&= \frac{2^k}{\pi}\left(\int_{-1}^{1}\left(|f\left(\frac{v+2n-1}{2^k}\right)-f(t)|\right)^2|\omega(v)|\frac{dv}{2^k}\right)^{\frac{1}{2}}\left(\frac{\pi}{2^k}\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\begin{aligned}
|e_n(t)| &\le \frac{1}{\sqrt{\pi}}\left(\int_{-1}^{1}\left(\left|f\left(\frac{v+2n-1}{2^k}\right)-f(t)\right|\right)^2|\omega(v)|dv\right)^{\frac{1}{2}}\\
&= \frac{1}{\sqrt{\pi}}\Bigg(\int_{0}^{1}\left|f\left(\frac{v+2n-1}{2^k}\right)-f(t)\right|^2|\omega(v)|dv\\
&+\int_{-1}^{0}\left|f\left(\frac{v+2n-1}{2^k}\right)-f(t)\right|^2|\omega(v)|dv\Bigg)^{\frac{1}{2}}\\
&= \frac{1}{\sqrt{\pi}}\Bigg(\int_{0}^{1}\left|f\left(\frac{v+2n-1}{2^k}\right)-f(t)\right|^2|\omega(v)|dv\\
&+\int_{0}^{1}\left|f\left(\frac{-v+2n-1}{2^k}\right)-f(t)\right|^2|\omega(-v)|dv\Bigg)^{\frac{1}{2}}\\
&= \frac{1}{\sqrt{\pi}}\left(\int_{0}^{1}\left|f\left(\frac{v+2n-1}{2^k}\right)-f(t)\right|^2|\omega(v)|dv\right)^{\frac{1}{2}}\\
&+\frac{1}{\sqrt{\pi}}\left(\int_{0}^{1}\left|f\left(\frac{-v+2n-1}{2^k}\right)-f(t)\right|^2|\omega(v)|dv\right)^{\frac{1}{2}},\\
&\frac{v+2n-1}{2^k}\in\left[\frac{n-\frac{1}{2}}{2^{k-1}},\frac{n}{2^{k-1}}\right) \text{ and } \frac{-v+2n-1}{2^k}\in\left(\frac{n-1}{2^{k-1}},\frac{n-\frac{1}{2}}{2^{k-1}}\right]\\
|e_n(t)| &\le \frac{1}{\sqrt{\pi}}O\left(\frac{1}{2^{(k-1)}}\right)^{\alpha}+\frac{1}{\sqrt{\pi}}O\left(\frac{1}{2^{(k-1)}}\right)^{\alpha}\\
&= \frac{2}{\sqrt{\pi}}O\left(\frac{1}{2^{(k-1)}}\right)^{\alpha} \quad = O\left(\frac{1}{2^{(k-1)\alpha}}\right) \qquad (6)\\
\|e_n\|_2^2 &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}|e_n(t)|^2|\omega_k(t)|dt\\
&= O\left(\frac{1}{2^{(k-1)}}\right)^{2\alpha}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}|\omega_k(t)|dt\\
&= O\left(\frac{1}{2^{2(k-1)\alpha}}\right)\frac{\pi}{2^k}\\
&= \frac{1}{2^k}O\left(\frac{1}{2^{2(k-1)\alpha}}\right)\\
&\le \frac{1}{2^{k-1}}O\left(\frac{1}{2^{2(k-1)\alpha}}\right)\\
\|e_n\|_2^2 &= O\left(\frac{1}{2^{(k-1)(2\alpha+1)}}\right) \qquad (7)
\end{aligned}$$

$$
\begin{aligned}
\left\|f-\left(S_{2^{k-1},0}f\right)\right\|_2^2 &= \int_0^1\left(\sum_{n=1}^{2^{k-1}}e_n(t)\right)^2\omega_k(t)dt\\
&= \int_0^1\left(\sum_{n=1}^{2^{k-1}}(e_n(t))^2\omega_k(t)\right)dt \qquad (8)\\
&+ \sum_{1\leqslant\ n\neq n'\leq\ 2^{k-1}}\sum\int_0^1 e_n(t)e_{n'}(t)\omega_k(t)dt,
\end{aligned}
$$

Due to disjointness of supports of $e_n(t)\&e_{n'}(t)$, equation(8) becomes,

$$
\begin{aligned}
&= \int_0^1\left(\sum_{n=1}^{2^{k-1}}(e_n(t))^2\omega_k(t)\right)dt\\
&= \sum_{n=1}^{2^{k-1}}\int_0^1(e_n(t))^2\omega_k(t)\\
&= \sum_{n=1}^{2^{k-1}}\|e_n(t)\|_2^2\\
&= \sum_{n=1}^{2^{k-1}}O\left(\frac{1}{2^{(k-1)(2\alpha+1)}}\right)\\
&= O\left(\frac{2^{k-1}}{2^{(k-1)(2\alpha+1)}}\right)=O\left(\frac{1}{2^{2(k-1)\alpha}}\right)\\
\left\|f-\left(S_{2^{k-1},0}f\right)\right\|_2 &= O\left(\frac{1}{2^{(k-1)\alpha}}\right) \qquad (9)\\
W\left(\left(f-S_{2^{k-1},0}f\right),\frac{1}{2^k}\right) &= \sup_{0<h\leq\frac{1}{2^k}}\|\left(f-S_{2^{k-1},0}f\right)(\cdot+h)-\left(f-S_{2^{k-1},0}f\right)(\cdot)\|_2\\
&\leq ||\left(f-S_{2^{k-1},0}f\right)||_2+||\left(f-S_{2^{k-1},0}f\right)||_2\\
&= 2||\left(f-S_{2^{k-1},0}f\right)||_2\\
&= 2.O\left(\frac{1}{2^{(k-1)\alpha}}\right)\\
&= O\left(\frac{1}{2^{(k-1)\alpha}}\right)\\
W\left(\left(f-S_{2^{k-1},0}f\right),\frac{1}{2^k}\right) &= O\left(\frac{1}{2^{(k-1)\alpha}}\right). \qquad (10)
\end{aligned}
$$

$(\mathbf{ii})$ Consider

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi_{m,n}^{(c_1)}(t)\\
c_{n,m} &= \langle f, \Psi_{m,n}^{(c_1)}(t)\rangle_{\omega_k}\\
&= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)\Psi_{n,m}^{(c_1)}(t)\omega_k(t)dt\\
&= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)T_m(2^k t-2n+1)\omega_k(t)dt\right)\\
&= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left(f(t)-f\left(\frac{2n-1}{2^k}\right)+f\left(\frac{2n-1}{2^k}\right)\right)g(t)\omega_k(t)dt\\
&= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left(f(t)-f\left(\frac{2n-1}{2^k}\right)\right)g(t)\omega_k(t)dt\right)\\
&+\frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}f\left(\frac{2n-1}{2^k}\right)\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} g(t)\omega_k(t)dt
\end{aligned}
\tag{11}
$$

where $g(t)=T_m(2^k t-2n+1)$

$$
\begin{aligned}
\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} g(t)\omega_k(t)dt &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} T_m(2^k t-2n+1)\omega(2^k t-2n+1)dt\\
&= \int_{\pi}^{0} T_m(\cos\theta)\omega(\cos\theta)\frac{(-\sin\theta)}{2^k}d\theta\\
&= \int_0^{\pi}\frac{\cos m\theta}{\sin\theta}\frac{(\sin\theta)}{2^k}d\theta = \frac{1}{2^k}\left[\frac{\sin m\theta}{m}\right]_0^{\pi} = 0
\end{aligned}
\tag{12}
$$

Therefore equation (11) reduces to,

$$
\begin{aligned}
c_{n,m} &= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left(f(t)-f\left(\frac{2n-1}{2^k}\right)\right)g(t)\omega_k(t)dt\right)\\
&= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left(f(t)-f\left(\frac{2n-1}{2^k}\right)\right)g(t)\sqrt{\omega_k(t)\omega_k(t)}dt\right)\\
|c_{n,m}| &\le \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}}\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left|f(t)-f\left(\frac{2n-1}{2^k}\right)\right|^2|\omega_k(t)|dt\right)^{\frac{1}{2}}\\
&\times\left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left|T_m(2^k t-2n+1)\right|^2|\omega_k(t)|\ dt\right)^{\frac{1}{2}}
\end{aligned}
$$

$$= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left| f(t) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(2^k t - 2n + 1)| dt \right)^{\frac{1}{2}}$$

$$\times \left(\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left| T_m(2^k t - 2n + 1) \right| |\omega(2^k t - 2n + 1)| dt \right)$$

$$= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \left(\int_{-1}^{1} \left| f\left(\frac{v+2n-1}{2^k}\right) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(v)| \frac{dv}{2^k} \right)^{\frac{1}{2}}$$

$$\times \left(\int_0^{\pi} |T_m(\cos\theta)| \, |\omega(\cos\theta)| \frac{\sin\theta}{2^k} d\theta \right)$$

$$= \frac{1}{2^k} \sqrt{\frac{2}{\pi}} \left(\int_{-1}^{1} \left| f\left(\frac{v+2n-1}{2^k}\right) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(v)| dv \right)^{\frac{1}{2}}$$

$$\left(\int_0^{\pi} |\cos m\theta| \left| \frac{1}{\sin\theta} \right| \sin\theta d\theta \right)$$

$$= \frac{1}{2^k} \sqrt{\frac{2}{\pi}} \left(\int_{-1}^{1} \left| f\left(\frac{v+2n-1}{2^k}\right) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(v)| dv \right)^{\frac{1}{2}}$$

$$\left(\int_{-\frac{\pi}{2}}^{\pi/2} \left| \cos(\frac{m\pi}{2} - mt) \right| \, dt \right)$$

$$= \frac{\sqrt{\frac{2}{\pi}}}{2^k} (\int_{-1}^{1} \left| f\left(\frac{v+2n-1}{2^k}\right) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(v)| dv)^{\frac{1}{2}}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \cos\frac{m\pi}{2} \cos mt + \sin\frac{m\pi}{2} \sin mt \right| dt$$

$$\leq \frac{\sqrt{\frac{2}{\pi}}}{2^{k-1}} \left(\int_{-1}^{1} \left| f\left(\frac{v+2n-1}{2^k}\right) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(v)| dv \right)^{\frac{1}{2}}$$

$$\left(\int_0^{\pi/2} (|\cos(mt)| + |\sin(mt)|) \, dt \right)$$

$$\leq \frac{1}{2^{k-1}} \sqrt{\frac{2}{\pi}} \left(\int_{-1}^{1} \left| f\left(\frac{v+2n-1}{2^k}\right) - f\left(\frac{2n-1}{2^k}\right) \right|^2 |\omega(v)| dv \right)^{\frac{1}{2}}$$

$$\times \left[\frac{|\sin(mt)|}{m} + \frac{|\cos(mt)|}{m} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{m.2^k} \sqrt{\frac{2}{\pi}} \left(\left(\frac{1}{2^k}\right)^{\alpha} + \left(\frac{1}{2^k}\right)^{\alpha} \right) = \frac{1}{m.2^{k-1}} \sqrt{\frac{2}{\pi}} \left(\frac{1}{2^{k\alpha}}\right)$$

$$
\begin{aligned}
|c_{n,m}| &\leq \frac{1}{m.2^{k-1}}\sqrt{\frac{2}{\pi}}\left(\frac{1}{2^{k\alpha}}\right). \qquad (13)\\
\text{Let } I(t) &= f(x) - S_{2^{k-1},M}(f)(t)\\
I(t) &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi_{n,m}^{(c_1)}(t) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\Psi_{n,m}^{(c_1)}(t)\\
&= (\sum_{n=1}^{2^{k-1}} + \sum_{\substack{n=1\\2^{k-1}+1}}^{\infty})(\sum_{m=0}^{M} + \sum_{\substack{m=\\M+1}}^{\infty})c_{n,m}\Psi_{n,m}^{c_1}(t) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\Psi_{n,m}^{c_1}(t)\\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}\Psi_{n,m}^{(c_1)}(t), \quad \text{by def}^n \text{ of } \Psi_{n,m}^{(c_1)}\\
(I(t))^2 &= \left(\sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}\Psi_{n,m}^{(c_1)}(t)\right)^2 = \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}^2\left(\Psi_{n,m}^{(c_1)}(t)\right)^2\\
&+ \sum_{n=1}^{2^{k-1}}\sum\sum_{M+1\leq m\neq m'\leq\infty} c_{n,m}c_{n,m'}\Psi_{n,m}^{(c_1)}(t)\Psi_{n,m'}^{(c_1)}(t)\\
&+ \sum\sum_{1\leqslant\ n\neq n'\leq\ 2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}c_{n',m}\Psi_{n,m}^{(c_1)}(t)\Psi_{n',m}^{(c_1)}(t)\\
&+ \sum\sum_{1\leqslant\ n\neq n'\leq\ 2^{k-1}}\sum\sum_{M+1\leq m\neq m'\leq\infty} c_{n,m}c_{n',m'}\Psi_{n,m}^{(c_1)}(t)\Psi_{n',m'}^{(c_1)}(t)\\
||I||_2^2 &= \int_0^1 |f(t) - S_{2^{k-1},M}(f)(t)|^2\omega_n(t)dt\\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} |c_{n,m}|^2\int_0^1 |\Psi_{n,m}^{(c_1)}(t)|^2\omega_n(t)dt\\
&+ \sum_{n=1}^{2^{k-1}}\sum\sum_{M+1\leq m\neq m'\leq\infty} c_{n,m}c_{n,m'}\int_0^1\left(\Psi_{n,m}^{(c_1)}(t)\Psi_{n,m'}^{c}(x)\right)\omega_n(t)dt\\
&+ \sum\sum_{1\leqslant\ n\neq n'\leq\ 2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}c_{n',m}\int_0^1\left(\Psi_{n,m}^{(c_1)}(t)\Psi_{n',m}^{(c_1)}(t)\right)\omega_n(t)dt\\
&+ \sum\sum_{1\leqslant\ n\neq n'\leq\ 2^{k-1}}\sum\sum_{M+1\leq m\neq m'\leq\infty} c_{n,m}c_{n',m'}\int_0^1 \Psi_{n,m}^{(c_1)}(t)\Psi_{n',m'}^{(c_1)}(t)\omega_n(t)dt\\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} |c_{n,m}|^2, \qquad \text{by orthonormality of } \{\Psi_{n,m}^{(c_1)}\}.
\end{aligned}
$$

$$||f - S_{2^{k-1},M}(f)||_2^2 \leq \sum_{n=1}^{2^{k-1}} \sum_{m=M+1}^{\infty} \left[\frac{1}{m.2^{k-1}}\sqrt{\frac{2}{\pi}}\left(\frac{1}{2^{k\alpha}}\right)\right]^2$$

$$= \frac{2}{\pi}.2^{k-1}.\frac{1}{2^{2(k-1)}}.\frac{1}{2^{2k\alpha}} \sum_{m=M+1}^{\infty} \frac{1}{m^2}$$

$$= \frac{2^2}{\pi\ 2^k\ 2^{2k\alpha}} \sum_{m=M+1}^{\infty} \frac{1}{m^2}$$

$$\leq \frac{2^2}{\pi\ 2^k\ 2^{2k\alpha}} \left[\frac{1}{(M+1)^2} + \int_{M+1}^{\infty} \frac{1}{m^2} dm\right]$$

By Cauchy integral test,

$$\leq \frac{2^2}{\pi\ 2^k\ 2^{2k\alpha}} \left[\frac{1}{(M+1)} + \left(\frac{-1}{m}\right)_{M+1}^{\infty}\right]$$

$$= \frac{2^2}{\pi\ 2^k\ 2^{2k\alpha}} \left[\frac{1}{(M+1)} + \frac{1}{M+1}\right]$$

$$\|f - S_{2^{k-1},M}(f)\|_2 \leq 2\sqrt{\frac{2}{\pi}}\ \frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}$$

$$= O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right)$$

$$W\left(\left(f - S_{2^{k-1},M}f\right), \frac{1}{2^k}\right) = \sup_{0<h\leq\frac{1}{2^k}} \|\left(f - S_{2^{k-1},M}f\right)(\cdot + h) - \left(f - S_{2^{k-1},M}f\right)(\cdot)\|_2$$

$$\leq ||\left(f - S_{2^{k-1},0}f\right)||_2 + ||\left(f - S_{2^{k-1},M}f\right)||_2$$

$$= 2||\left(f - S_{2^{k-1},M}f\right)||_2$$

$$= 2.O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right)$$

$$= O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right).$$

Thus, this theorem is completely established.

5 Corollary

Corolary 5.1. *If $f \in H^{\alpha,2}_{\omega_k}[0,1)$ and its first kind Chebyshev wavelet expansion be*

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(c_1)}_{n,m}(t)$$

having $(2^{k-1}, M+1)^{th}$ partial sums

$$\left(S_{2^{k-1},M}(f)\right)(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\Psi^{(c_1)}_{n,m}(t)$$

Then the first kind Chebyshev wavelet approximation $E_{2^{k-1},M}(f)$ of f is given by

$$\begin{aligned}(i)\, E_{2^{k-1},0}(f) = min||f - (S_{2^{k-1},0}f)||_2 = &\quad min||f - \sum_{n=1}^{2^{k-1}} c_{n,0}\Psi^{(c_1)}_{n,0}(t)||_2 \\ &= O\left(\frac{1}{2^{(k-1)\alpha}}\right) \\ (ii)\, E_{2^{k-1},M}(f) = min||f - (S_{2^{k-1},M}f)||_2 = &\quad min||f - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\Psi^{(c_1)}_{n,m}(t)||_2 \\ &= O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right)\end{aligned}$$

The proof of corollary (5.1) can be developed parallel to the proof of theorem (3.1) independently.

Remark
If $f \in H^{\alpha,2}_{\omega_k}[0,1)$, then the moduli of continuity $W\left(\left(f - S_{2^{k-1},M}f\right), \frac{1}{2^k}\right) = O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right)$ and approximation $E_{2^{k-1},M}(f) = O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right)$ tends to 0 as k $\to \infty$, M$\to \infty$. Hence $W\left(\left(f - S_{2^{k-1},M}f\right), \frac{1}{2^k}\right)$ and $E_{2^{k-1},M}(f)$ are best possible modulus of continuity and approximation of functions respectively in wavelet analysis.It is also observed that

$$W\left(\left(f - S_{2^{k-1},M}f\right), \frac{1}{2^k}\right) \leq E_{2^{k-1},M}(f).$$

Hence modulus of continuity is sharper than the approximation of function in $H^{\alpha,2}_{\omega_k}[0,1)$ by first kind Chebyshev wavelet method.

6 First kind Chebyshev wavelet operational matrix of integration

In this section, the operational matix P of integration of order $2^{k-1}(M+1) \times 2^{k-1}(M+1)$has been derived. This plays a great role in dealing with the problem of Lane-Emden differential equations (Babolian [5]) .
First the matrix $P_{8\times8}$ has been obtained for $M = 3$ and $k = 2$. There are eight basis wavelet functions given below:

$$\Psi_{1,0}(t) = \begin{cases} \frac{2}{\sqrt{\pi}}, & 0 \le t < \frac{1}{2}, \\ 0, & \frac{1}{2} \le t < 1, \end{cases}$$

$$\Psi_{1,1}(t) = \begin{cases} 2\sqrt{\frac{2}{\pi}}(4t-1), & 0 \le t < \frac{1}{2}, \\ 0, & \frac{1}{2} \le t < 1, \end{cases}$$

$$\Psi_{1,2}(t) = \begin{cases} 2\sqrt{\frac{2}{\pi}}(32\,t^2-16t+1), & 0 \le t < \frac{1}{2}, \\ 0, & \frac{1}{2} \le t < 1, \end{cases}$$

$$\Psi_{1,3}(t) = \begin{cases} 2\sqrt{\frac{2}{\pi}}(256\,t^3-192\,t^2+36t-1), & 0 \le t < \frac{1}{2}, \\ 0, & \frac{1}{2} \le t < 1, \end{cases}$$

$$\Psi_{2,0}(t) = \begin{cases} 0, & 0 \le t < \frac{1}{2}, \\ \frac{2}{\sqrt{\pi}}, & \frac{1}{2} \le t < 1, \end{cases}$$

$$\Psi_{2,1}(t) = \begin{cases} 0, & 0 \le t < \frac{1}{2}, \\ 2\sqrt{\frac{2}{\pi}}(4t-3), & \frac{1}{2} \le t < 1, \end{cases}$$

$$\Psi_{2,2}(t) = \begin{cases} 0, & 0 \le t < \frac{1}{2}, \\ 2\sqrt{\frac{2}{\pi}}(32\,t^2-48t+17), & \frac{1}{2} \le t < 1, \end{cases}$$

$$\Psi_{2,3}(t) = \begin{cases} 0, & 0 \le t < \frac{1}{2}, \\ 2\sqrt{\frac{2}{\pi}}(256\,t^3-576\,t^2+420t-99), & \frac{1}{2} \le t < 1. \end{cases}$$

Let,

$$\Psi(t) = [\Psi_{10}, \Psi_{11}, \Psi_{12}, \Psi_{13}, \Psi_{20}, \Psi_{21}, \Psi_{22}, \Psi_{23}]^T \tag{14}$$

By integrating the basis functions from 0 to t and expressing them in term of Chebyshev wavelet series ,

$$\int_0^t \Psi_{1,0}(x) = \begin{cases} \frac{2}{\sqrt{\pi}}t, & 0 \le t < \frac{1}{2}, \\ \frac{1}{\sqrt{\pi}}, & \frac{1}{2} \le t < 1, \end{cases}$$
$$= \left[\frac{1}{4}, \frac{1}{4\sqrt{2}}, 0, 0, \frac{1}{2}, 0, 0, 0\right] \Psi(t)$$

$$\int_0^t \Psi_{1,1}(x) = \begin{cases} 2\sqrt{\frac{2}{\pi}}(4t-1), & 0 \le t < \frac{1}{2}, \\ 0, & \frac{1}{2} \le t < 1, \end{cases}$$
$$= \left[-\frac{1}{8\sqrt{2}}, 0, \frac{1}{16}, 0, 0, 0, 0, 0\right] \Psi(t)$$

$$\int_0^t \Psi_{1,2}(x) = \begin{cases} 2\sqrt{\frac{2}{\pi}}(32\,t^2 - 16t + 1), & 0 \le t < \frac{1}{2}, \\ 0, & \frac{1}{2} \le t < 1, \end{cases}$$
$$= \left[-\frac{1}{6\sqrt{2}}, -\frac{1}{8}, 0, \frac{1}{24}, -\frac{1}{3\sqrt{2}}, 0, 0, 0\right] \Psi(t)$$

$$\int_0^t \Psi_{1,3}(x) = \begin{cases} 2\sqrt{\frac{2}{\pi}}(256\,t^3 - 192\,t^2 + 36t - 1), & 0 \le t < \frac{1}{2}, \\ 0, & \frac{1}{2} \le t < 1, \end{cases}$$
$$= \left[\frac{1}{16\sqrt{2}}, 0, -\frac{1}{16}, 0, 0, 0, 0, 0\right] \Psi(t)$$

$$\int_0^t \Psi_{2,0}(x) = \begin{cases} 0, & 0 \le t < \frac{1}{2}, \\ \frac{2}{\sqrt{\pi}}, & \frac{1}{2} \le t < 1, \end{cases}$$
$$= \left[0, 0, 0, 0, \frac{1}{4}, \frac{1}{4\sqrt{2}}, 0, 0\right] \Psi(t)$$

$$\int_0^t \Psi_{2,1}(x) = \begin{cases} 0, & 0 \le t < \frac{1}{2}, \\ 2\sqrt{\frac{2}{\pi}}(4t-3), & \frac{1}{2} \le t < 1, \end{cases}$$
$$= \left[0, 0, 0, 0, -\frac{1}{8\sqrt{2}}, 0, \frac{1}{16}, 0\right] \Psi(t)$$

$$\int_0^t \Psi_{2,2}(x) = \begin{cases} 0, & 0 \le t < \frac{1}{2}, \\ 2\sqrt{\frac{2}{\pi}}(32\,t^2 - 48t + 17), & \frac{1}{2} \le t < 1, \end{cases}$$
$$= \left[0, 0, 0, 0, -\frac{1}{6\sqrt{2}}, -\frac{1}{8}, 0, \frac{1}{24}\right] \Psi(t)$$

$$\int_0^t \Psi_{2,3}(x) = \begin{cases} 0, & 0 \le t < \frac{1}{2}, \\ 2\sqrt{\frac{2}{\pi}}(256\,t^3 - 576\,t^2 + 420t - 99), & \frac{1}{2} \le t < 1. \end{cases}$$
$$= \left[0, 0, 0, 0, \frac{1}{16\sqrt{2}}, 0, -\frac{1}{16}, 0\right] \Psi(t)$$

Thus, $\displaystyle\int_0^t \Psi(x)dx = P_{8\times 8}\Psi(t)$ (M. A. Fariborzi[13]) (15)

$$\text{where } P_{8\times 8} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4\sqrt{2}} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{8\sqrt{2}} & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6\sqrt{2}} & -\frac{1}{8} & 0 & \frac{1}{24} & -\frac{1}{3\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{16\sqrt{2}} & 0 & -\frac{1}{16} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{8\sqrt{2}} & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{6\sqrt{2}} & -\frac{1}{8} & 0 & \frac{1}{24} \\ 0 & 0 & 0 & 0 & \frac{1}{16\sqrt{2}} & 0 & -\frac{1}{16} & 0 \end{pmatrix} \qquad (16)$$

7 The Product Operation matrix (POM)

In this section , the product operation matrix has been obtained. This has important role in solving differential equations (Babolian [5]).
By Equation (14),

$$\Psi\,\Psi^T =$$

$$\begin{pmatrix} \Psi_{1,0}\Psi_{1,0} & \Psi_{1,0}\Psi_{1,1} & \Psi_{1,0}\Psi_{1,2} & \Psi_{1,0}\Psi_{1,3} & \Psi_{1,0}\Psi_{2,0} & \Psi_{1,0}\Psi_{2,1} & \Psi_{1,0}\Psi_{2,2} & \Psi_{1,0}\Psi_{2,3} \\ \Psi_{1,1}\Psi_{1,0} & \Psi_{1,1}\Psi_{1,1} & \Psi_{1,1}\Psi_{1,2} & \Psi_{1,1}\Psi_{1,3} & \Psi_{1,1}\Psi_{2,0} & \Psi_{1,1}\Psi_{2,1} & \Psi_{1,1}\Psi_{2,2} & \Psi_{1,1}\Psi_{2,3} \\ \Psi_{1,2}\Psi_{1,0} & \Psi_{1,2}\Psi_{1,1} & \Psi_{1,2}\Psi_{1,2} & \Psi_{1,2}\Psi_{1,3} & \Psi_{1,2}\Psi_{2,0} & \Psi_{1,2}\Psi_{2,1} & \Psi_{1,2}\Psi_{2,2} & \Psi_{1,2}\Psi_{2,3} \\ \Psi_{1,3}\Psi_{1,0} & \Psi_{1,3}\Psi_{1,1} & \Psi_{1,3}\Psi_{1,2} & \Psi_{1,3}\Psi_{1,3} & \Psi_{1,3}\Psi_{2,0} & \Psi_{2,1}\Psi_{2,1} & \Psi_{1,3}\Psi_{2,2} & \Psi_{1,3}\Psi_{2,3} \\ \Psi_{2,0}\Psi_{1,0} & \Psi_{2,0}\Psi_{1,1} & \Psi_{2,0}\Psi_{1,2} & \Psi_{2,0}\Psi_{1,3} & \Psi_{2,0}\Psi_{2,0} & \Psi_{2,0}\Psi_{2,1} & \Psi_{2,0}\Psi_{2,2} & \Psi_{2,0}\Psi_{2,3} \\ \Psi_{2,1}\Psi_{1,0} & \Psi_{2,1}\Psi_{1,1} & \Psi_{2,1}\Psi_{1,2} & \Psi_{2,1}\Psi_{1,3} & \Psi_{2,1}\Psi_{2,0} & \Psi_{2,1}\Psi_{2,1} & \Psi_{2,1}\Psi_{2,2} & \Psi_{2,1}\Psi_{2,3} \\ \Psi_{2,2}\Psi_{1,0} & \Psi_{2,2}\Psi_{1,1} & \Psi_{2,2}\Psi_{1,2} & \Psi_{2,2}\Psi_{1,3} & \Psi_{2,2}\Psi_{2,0} & \Psi_{2,2}\Psi_{2,1} & \Psi_{2,2}\Psi_{2,2} & \Psi_{2,2}\Psi_{2,3} \\ \Psi_{2,3}\Psi_{1,0} & \Psi_{2,3}\Psi_{1,1} & \Psi_{2,3}\Psi_{1,2} & \Psi_{2,3}\Psi_{1,3} & \Psi_{2,3}\Psi_{2,0} & \Psi_{2,3}\Psi_{2,1} & \Psi_{2,3}\Psi_{2,2} & \Psi_{2,3}\Psi_{2,3} \end{pmatrix}$$

Substituting the values of each entry of above symmetric matrix , and after simplification expressing them in term of Chebyshev wavelet series ,

$$\Psi\,\Psi^T = \begin{pmatrix} A & O_{4\times 4} \\ O_{4\times 4} & B \end{pmatrix}$$

where $O_{4\times 4}$ is a 4×4 zero matrix and

$$A = \begin{pmatrix} \frac{2}{\sqrt{\pi}}\Psi_{1,0} & \frac{2}{\sqrt{\pi}}\Psi_{1,1} & \frac{2}{\sqrt{\pi}}\Psi_{1,2} & \frac{2}{\sqrt{\pi}}\Psi_{1,3} \\ \frac{2}{\sqrt{\pi}}\Psi_{1,1} & \frac{2}{\sqrt{\pi}}\Psi_{1,0} + \sqrt{\frac{2}{\pi}}\Psi_{1,2} & \sqrt{\frac{2}{\pi}}\Psi_{1,1} + \sqrt{\frac{2}{\pi}}\Psi_{1,3} & \sqrt{\frac{2}{\pi}}\Psi_{1,2} \\ \frac{2}{\sqrt{\pi}}\Psi_{1,2} & \sqrt{\frac{2}{\pi}}\Psi_{1,1} + \sqrt{\frac{2}{\pi}}\Psi_{1,3} & \frac{2}{\sqrt{\pi}}\Psi_{1,0} & \sqrt{\frac{2}{\pi}}\Psi_{1,1} \\ \frac{2}{\sqrt{\pi}}\Psi_{1,3} & \sqrt{\frac{2}{\pi}}\Psi_{1,2} & \sqrt{\frac{2}{\pi}}\Psi_{1,1} & \frac{2}{\sqrt{\pi}}\Psi_{1,0} \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{2}{\sqrt{\pi}}\Psi_{2,0} & \frac{2}{\sqrt{\pi}}\Psi_{2,1} & \frac{2}{\sqrt{\pi}}\Psi_{2,2} & \frac{2}{\sqrt{\pi}}\Psi_{2,3} \\ \frac{2}{\sqrt{\pi}}\Psi_{2,1} & \frac{2}{\sqrt{\pi}}\Psi_{1,0} + \sqrt{\frac{2}{\pi}}\Psi_{1,2} & \sqrt{\frac{2}{\pi}}\Psi_{2,1} + \sqrt{\frac{2}{\pi}}\Psi_{2,3} & \sqrt{\frac{2}{\pi}}\Psi_{2,2} \\ \frac{2}{\sqrt{\pi}}\Psi_{2,2} & \sqrt{\frac{2}{\pi}}\Psi_{2,1} + \sqrt{\frac{2}{\pi}}\Psi_{2,3} & \frac{2}{\sqrt{\pi}}\Psi_{2,0} & \sqrt{\frac{2}{\pi}}\Psi_{2,1} \\ \frac{2}{\sqrt{\pi}}\Psi_{2,3} & \sqrt{\frac{2}{\pi}}\Psi_{2,2} & \sqrt{\frac{2}{\pi}}\Psi_{2,1} & \frac{2}{\sqrt{\pi}}\Psi_{2,0} \end{pmatrix}$$

8 Solution of the Lane -Emden differential equation by first kind Chebyshev wavelet

Many problems in the literature of mathematical physics can be distinctively formulated as equations of Lane-Emden type as follows:

$$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} + L(y) = 0 \qquad \text{(A-M \quad Wazwaz[12])}$$

where $L(y)$ is some given function of y.

A difficult element in the analysis of this type of equations is the singularity behaviour that occurs at $t = 0$. Most algorithms currently in use for handling the Lane-Emden type problems are based on either series solution or perturbation techniques. In recent years, a lot of attention has been devoted to the study of wavelet theory. The first kind Chebyshev wavelet method accurately computes the solution of differential equations and it is of great interest to solve this type of problem.

The most popular form of $L(y)$ attracted by the scientific community is

$$L(y) = y^n$$

where n is constant parameter.

The standard Lane-Emden equation of index n is of the form

$$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} + y^n = 0, \qquad \text{with} \qquad y(0) = 1,\ y'(0) = 0 \tag{17}$$

It is a basic equation in the theory of stellar structure. The equation describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules .

It was physically shown that interesting values of n lie in the interval $[0, 5]$. In addition, exact solutions exist only for $n = 0, 1, 5$. Notice that equation (17) is linear for $n = 0, 1$ and non-linear otherwise.

Example (1). For $n = 0$, the Lane-Emden differential equation (17) reduces to

$$t\,y'' + 2\,y' + t = 0 \qquad \text{with} \quad y(0) = 1, y'(0) = 0 \tag{18}$$

The exact solution of the equation (18) is $y(t) = 1 - \frac{t^2}{6}$.
The equation (18) has been solved using first kind Chebyshev wavelet. First it is assumed that $y''(t)$ can be expanded in terms of Chebyshev wavelet as

$$y''(t) = C^T\Psi(t) \tag{19}$$

where $\Psi(t)$ is given by equation (14). By integrationg equation (19) twice with respect to t from 0 to t and using conditions of equation (18) & (15) following equations are obtained ,

$$y'(t) = C^T\, P\, \Psi(t) \tag{20}$$

$$\text{and,}\quad y(t) = C^T\, P^2\, \Psi(t) + d^T\, \Psi(t) \tag{21}$$

$$\text{where,}\quad d^T = [\frac{\sqrt{\pi}}{2}\ ,0\,,\ 0\ ,0\ ,\frac{\sqrt{\pi}}{2}\ ,0\ ,0,\ 0] \tag{22}$$

Using equation $(14), t$ is expressed as

$$t = e^T\Psi(x) \tag{23}$$

$$\text{where,}\quad e^T = [\frac{\sqrt{\pi}}{8},\ \frac{\sqrt{\frac{\pi}{2}}}{8},\ 0,\ 0,\ \frac{3\sqrt{\pi}}{8},\ \frac{\sqrt{\frac{\pi}{2}}}{8},\ 0,\ 0] \tag{24}$$

Substituting values from Equation (19) to (24) in Equation (18),

$$e^T\Psi(t)\Psi^T(t)C + 2C^TP\Psi(t) + e^T\Psi(t) = 0 \tag{25}$$

The following property of the product of two first kind Chebyshev wavelet vectors will also be used:

$$e^T\Psi(t)\Psi^T(t) \simeq \Psi^T(t)\tilde{e} \tag{26}$$

where $\tilde{e}$ is 8×8 matrix. Let us establish equation (26) & find $\tilde{e}$,

$$e^T\Psi\Psi^T = \frac{1}{4}\Big[\Psi_{1,0}+\frac{1}{\sqrt{2}}\Psi_{1,1},\ \frac{1}{\sqrt{2}}\Psi_{1,0}+\Psi_{1,1}+\frac{1}{2}\Psi_{1,2},\ \Psi_{1,2}+\frac{1}{2}\Psi_{1,1}+\frac{1}{2}\Psi_{1,3},\ \Psi_{1,3} +\frac{\Psi_{1,2}}{2},\ 3\Psi_{2,0}+\frac{\Psi_{2,1}}{\sqrt{2}},\ 3\Psi_{2,1}+\frac{1}{\sqrt{2}}\Psi_{2,0}+\frac{\Psi_{2,2}}{2},\ \frac{\Psi_{2,1}}{2}+3\Psi_{2,2}+\frac{1}{2}\Psi_{2,3},\ 3\Psi_{2,3}+\frac{1}{2}\Psi_{2,2}\Big]$$

$$e^T\Psi\Psi^T = \begin{pmatrix}\Psi_{1,0}\\ \Psi_{1,1}\\ \Psi_{1,2}\\ \Psi_{1,3}\\ \Psi_{2,0}\\ \Psi_{2,1}\\ \Psi_{2,2}\\ \Psi_{2,3}\end{pmatrix} \cdot \frac{1}{4}\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0\\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 3 & \frac{1}{\sqrt{2}} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 3 & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 3 & \frac{1}{2}\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 3 \end{pmatrix}$$

where

$$\tilde{e} = \frac{1}{4}\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 3 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 3 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 3 \end{pmatrix}_{8\times 8} \tag{27}$$

Using Equation (26), Equation (25) becomes

$$\begin{aligned} (\Psi^T(t)\tilde{e})C + \Psi^T(t)(2P^TC) + \Psi^T(t)e = 0 \\ \Psi^T(t)\ \left[\tilde{e}\,C + (2\,P^T\,C) + e\right] = 0 \\ \tilde{e}\,C + (2\,P^T\,C) + e = 0 \end{aligned} \tag{28}$$

Equation (28) is a set of algebraic equations which is solved for C in following form

$$C = \left(\tilde{e} + 2\,P^T\right)^{-1}(-e)$$

$$C = \begin{pmatrix} -0.295408975150919 \\ 0.000000000000000 \\ -0.000000000000000 \\ -0.000000000000000 \\ -0.295408975150919 \\ 0.000000000000000 \\ -0.000000000000000 \\ 0.000000000000000 \end{pmatrix} \tag{29}$$

By substituting the Chebyshev wavelet coefficients C from equation (29) into equation (21), the explicit form of the approximate solution of equation (18) is,

$$y(t) = \big[0.872379629742559, -0.013055355597036, -0.003263838899259, 0,$$
$$0.798527385954829, -0.039166066791109, -0.003263838899259, 0\big]\,\Psi(t)$$

Using the simplified value of Ψ(t) from equation (14), $y(t)$ becomes

$$y(t)= \begin{cases} 1 - 0.166666666666660t^2, & 0 \le t < \frac{1}{2}, \\ 1 - 6.357719098970234.10^{-15}\,t - 0.166666666666661\,t^2, & \frac{1}{2} \le t < 1. \end{cases}$$

The exact solution and the solution obtained by Chebyshev wavelet method of differential equation (18) at different values of t are given in Table (1):

Variable (t)	Exact solution	Chebyshev solution	Absolute error
0	1.000000000000000	1.000000000000000	0
0.1	0.998333333333333	0.998333333333333	0
0.2	0.993333333333333	0.993333333333334	0.000000000000001
0.3	0.985000000000000	0.985000000000001	0.000000000000001
0.4	0.973333333333333	0.973333333333334	0.000000000000001
0.5	0.958333333333333	0.958333333333333	0
0.6	0.940000000000000	0.939999999999999	0.000000000000001
0.7	0.918333333333333	0.918333333333333	0
0.8	0.893333333333333	0.893333333333333	0
0.9	0.865000000000000	0.865000000000000	0

Table(1): Comparison table for the exact and Chebyshev wavelet solution of Lane-Emden equation of index 0.

This table shown that the solution of equation (18) obtained by first kind Chebyshev wavelet method nearly coincides with its exact solution.The graphs of exact and first kind Chebyshev wavelet solutions of Lane -Emden differential equation for $n = 0$ are drawn in figure (1).

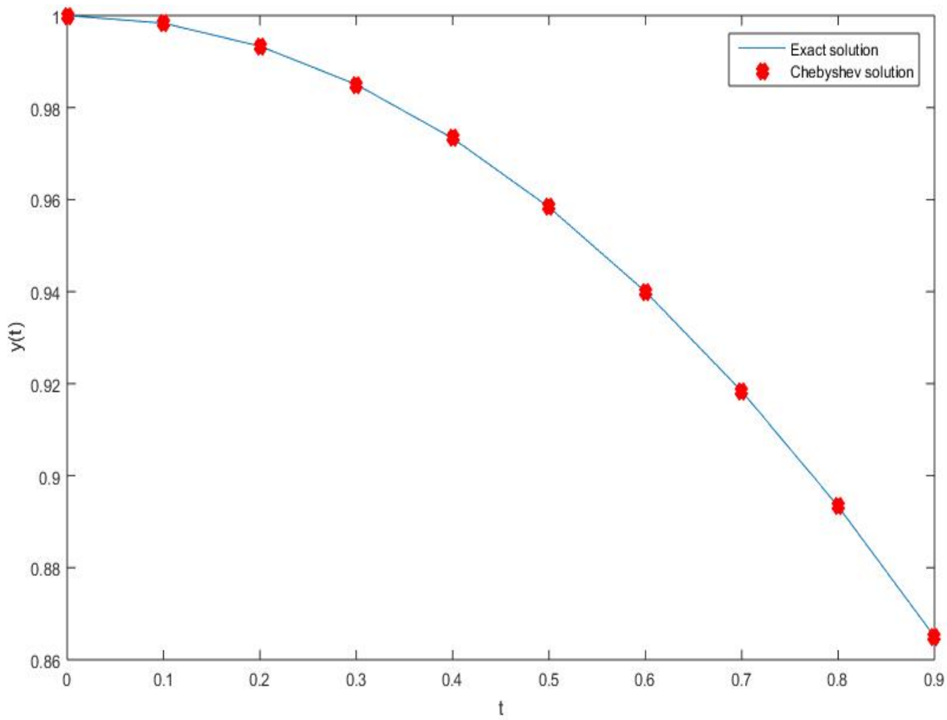

Figure 1:The graphs of first kind Chebyshev wavelet and exact solutions

Example (2). For $n = 1$, the Lane-Emden differential equation (17) reduces to

$$t\,y'' + 2\,y' + t\,y = 0 \qquad \text{with} \quad y(0) = 1, y'(0) = 0 \tag{30}$$

The exact solution of the equation (30) is $y(t) = \begin{cases} \frac{\sin(t)}{t}, & t \neq 0 \\ 1 & t = 0. \end{cases}$

Substituting values from Equation $(19) - (24)$ in equation (30) ,

$$e^T \Psi(t) \Psi^T(t) C + 2 C^T P \Psi(t) + e^T \Psi(t) (C^T\, P^2 + d^T) \Psi(t) = 0$$
$$C = \left[\tilde{e} + 2\,P^T + \tilde{e}\,(P^2)^T\right]^{-1} (-\tilde{e}\,d)$$

$$C = [-0.2871902685, 0.0077317973, 0.0019076755, -0.0000134466, \\ -0.2449764764, 0.0218383818, 0.0015733213, -0.0000401720]^T \tag{31}$$

By substituting the Chebyshev wavelet coefficients C from equation (31) into equation (21), the explicit form of the approximate solution of equation (30) is,

$$y(t) = \big[0.872505218190184, -0.012913303990151, -0.003192905781766, \\ 0.000020169906045, 0.801594934932526, -0.036835535048721, \\ -0.002723022009203,\ 0.000056975401025\ \big]\ \Psi(t)$$

or, $y(t)= \begin{cases} 0.9999960361853 + 2.5435469133605.10^{-4}\,t - 0.1692243050960\,t^2 + \\ 0.0082397473925\,t^3, & 0 \leq t < \frac{1}{2} \\ 0.9979746687982 + 0.0116372814448\,t - 0.1914197434042\,t^2 + \\ 0.0232754139255\,t^3 & \frac{1}{2} \leq t < 1. \end{cases}$

Table (2): Comparisons between the exact solution and numerical solutions for various values of t

Variable (t)	Exact solution	Chebyshev solution	Absolute error
0	1	0.999996036185	$0.0396381467104\times10^{-4}$
0.1	0.99833416646	0.998337468350	$0.0330188261299\times10^{-4}$
0.2	0.99334665397	0.993343852898	$0.0280107641192\times10^{-4}$
0.3	0.98506735553	0.985064628313	$0.0272722411598\times10^{-4}$
0.4	0.97354585577	0.973549233079	$0.0337730799104\times10^{-4}$
0.5	0.95885107720	0.958847800410	$0.0327679807199\times10^{-4}$
0.6	0.94107078899	0.941073419447	$0.0263045585602\times10^{-4}$
0.7	0.92031098176	0.920308558518	$0.0242325006294\times10^{-4}$
0.8	0.89669511362	0.896692870105	$0.0224351906097\times10^{-4}$
0.9	0.87036323291	0.870366006692	$0.0277377353607\times10^{-4}$

This table shown that the solution of equation (30) obtained by first kind Chebyshev wavelet method nearly coincides with its exact solution.The graphs of exact and first kind Chebyshev wavelet solutions of Lane -Emden differential equation for $n = 1$ are drawn in figure (2).

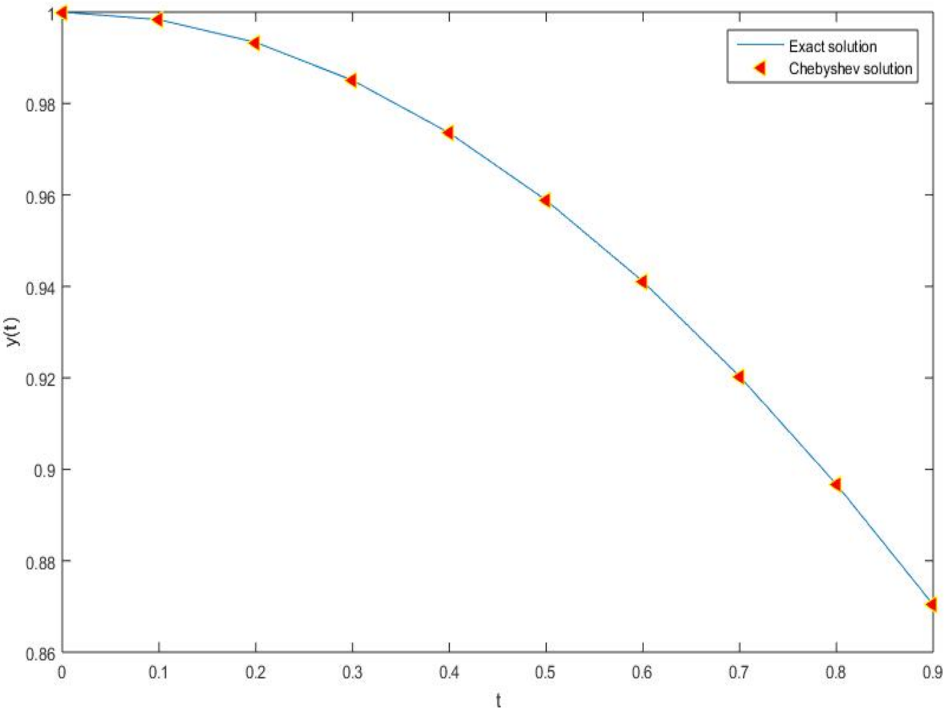

Figure 2: The graphs of first kind Chebyshev wavelet and exact solutions

Example (3). For $n = 2$, The Lane-Emden differential equation is

$$t\,y'' + 2\,y' + t\,y^2 = 0 \qquad \text{with} \quad y(0) = 1, y'(0) = 0 \tag{32}$$

The Taylor series expansion of function $L(y)$ about a point $y = 1$ is given as,

$$L(y) = \sum_{m=0}^{\infty} \frac{(y-1)^m}{m!} \frac{d^m L(y)}{dy^m}\bigg|_{y=1}$$

$$y^2 = 1 + 2\,(y-1) + (y-1)^2$$

From equation (21) & (22),

$$= d^T\Psi(t) + 2\,C^T\,P^2\,\Psi(t) + C^T\,P^4\Psi(t)$$

Substituting the Taylor series exapansion of y^2 and values from equation (19) to (24) in equation (32) ,

$$e^T\Psi(t)\Psi^T(t)C + 2C^T P\Psi(t) + e^T\Psi(t)\Psi^T(t)\,\left[d + 2\,(P^T)^2\,C + (P^T)^4\,C\right] = 0$$

$$C = \left[\tilde{e} + 2P^T + 2\,\tilde{e}\,(P^T)^2 + \tilde{e}\,(P^T)^4\right]^{-1}(-\tilde{e}\,d) \tag{33}$$

$$\begin{aligned} C = [&-0.279001698770205, 0.015429503002548, 0.003798307476540,\\ &-0.000031441187989, -0.195293820350412, 0.043103314900479,\\ &\ 0.003011315345934, -0.000095144707040]^T \end{aligned} \tag{34}$$

By substituting the Chebyshev wavelet coefficients C from equation (34) into equation (21), the explicit form of the approximate solution of equation (32) is,

$$\begin{aligned} y(t) = \big[&0.872630597117721, -0.012771501223632, -0.003122128096052,\\ &0.000040262875496, 0.804642516403461, -0.034524067315646,\\ &-0.002189080129024, 0.000112495988561\big]\,\Psi(t) \end{aligned}$$

$$\text{or, } y(t) = \begin{cases} 0.9999921077841 + 5.0666971784118.10^{-4}\,t - 0.1717663081650\,t^2 \\ +0.016448064886675\,t^3, \qquad 0 \le t < \frac{1}{2} \\ \\ 0.996061399263954 + 0.022704430920508\,t - 0.215186679101439\,t^2 \\ +0.045956511961640\,t^3 \qquad \frac{1}{2} \le t < 1. \end{cases}$$

9 Conclusion

(i) In theorem (3.1) , the moduli of continuity has been estimated as following:

$$W\left(\left(f - S_{2^{k-1},0}f\right), \frac{1}{2^k}\right) = O\left(\frac{1}{2^{(k-1)\alpha}}\right) \to 0 \quad \text{as} \quad k \to \infty,$$
$$W\left(\left(f - S_{2^{k-1},M}f\right), \frac{1}{2^k}\right) = O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right) \to 0 \ \text{ as } \ k \to \infty, M \to \infty.$$

(ii) In corollary (5.1) ,

$$E_{2^{k-1},0}(f) = O\left(\frac{1}{2^{(k-1)\alpha}}\right) \to 0 \quad \text{as} \quad k \to \infty,$$
$$E_{2^{k-1},M}(f) = O\left(\frac{1}{2^{k(\alpha+1/2)}\sqrt{M+1}}\right) \to 0 \quad \text{as} \quad k \to \infty, M \to \infty.$$

Thus $W\left(\left(f - S_{2^{k-1},0}f\right), \frac{1}{2^k}\right), W\left(\left(f - S_{2^{k-1},M}f\right), \frac{1}{2^k}\right), E_{2^{k-1},0}(f)$ & $E_{2^{k-1},M}(f)$ are best possible estimators in wavelet analysis.

(iii) From theorem (3.1) and corollary (5.1) , it is observed that

$$W\left(\left(f - S_{2^{k-1},0}f\right), \frac{1}{2^k}\right) \leq 2\, E_{2^{k-1},0}(f)$$
$$W\left(\left(f - S_{2^{k-1},M}f\right), \frac{1}{2^k}\right) \leq 2\, E_{2^{k-1},M}(f)$$

Hence moduli of continuity $W\left(\left(f - S_{2^{k-1},0}f\right), \frac{1}{2^k}\right),\ W\left(\left(f - S_{2^{k-1},M}f\right), \frac{1}{2^k}\right)$ are better and sharper than approximations $E_{2^{k-1},0}(f)$, $E_{2^{k-1},M}(f)$ respectively.

(iv) Solution of Lane-Emden differential equations by first kind Chebyshev wavelet is approximately same as exact solution of differential equations. This is the significant achievement of this research paper . Calculations performed in section(8) demonstrate that the accuracy of Chebyshev wavelet method is quite high. In this method , there is no complex integral or methodology. Application of this proposed method is very simple & gives the explicite form of approximate solutions to the Lane - Emden differential equations. These are the main advantages of the method. This method is also very convenient for solving the boundary value problems. Hence, this proposed method is very reliable, simple, fast & computationally efficients method.

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