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On properties of fuzzy subspaces of vectorspaces

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Abstract

In this paper, we introduce the notion of normal fuzzy subspace of vector spaces. By using it, we construct new fuzzy subspaces. We also show that, under certain conditions, a fuzzy subspace of a vector space is two-valued and takes 0 and 1.

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1 Introduction

Zadeh in [17] introduced the notion of fuzzy set and started a generalized logic. After that reconsideration of mathematics concepts begun. Also there have been a number of generalizations of this fundamental concept. Fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, group theory, groupoids, real analysis, measure theory etc (see [4], [5], [10] and [12]). In 1977, Katsaras and Liu [7] formulated and studied the concept of a fuzzy subspace of a vector space. Since then, a host of mathematicians are involved in extending the basic concepts and results from the theory of crisp vector spaces to the broader framework of the fuzzy setting. However, not all the results can be fuzzified. In [8], among other concepts and results, the fuzzy coset of a fuzzy subspace is defined and the algebraic nature of fuzzy subspaces under homomorphism is studied. In [9], the fuzzy basis and dimension of a fuzzy subspace are defined and studied. In [1] and [3], the fuzzy subspaces over fuzzy fields are discussed.

In this paper, some properties of fuzzy subspaces of vector spaces are investigated. Specially, some ways are created to construct new fuzzy subspaces from the old. Also the notion of normal fuzzy subspace of vector spaces is introduced. We see some normal fuzzy subspaces can be constructed by a fuzzy subspace. Finally, we show that, when a non-constant normal fuzzy subspace be a maximal in the partial ordered set of normal fuzzy subspaces of a vector space, then this fuzzy subspace is two-valued and takes the values 0 and 1.

2 Preliminaries

An abelian group $(\mathcal{V}, +)$ on a field F is called a *vector space* on F if there exists a map $. : \mathcal{V} \times F \longrightarrow \mathcal{V}$ such that for all $x, y \in \mathcal{V}$ and $a, b \in F$ the following conditions hold:

(i) $1x = x$,

(ii) $(ab)x = a(bx)$,

(iii) $a(x + y) = ax + ay$,

(v) $(a + b)x = ax + by$.

Also a non-empty subset $\mathcal{W}$ of a vector space $\mathcal{V}$ is called a *subspace*, if $\mathcal{W}$ is a vector space on F.

Let X be an ordinary set. By a *fuzzy set* μ in X, we mean a function $\mu : X \longrightarrow [0, 1]$ with the grade of membership $\mu(x)$ for $x \in X$. If $t \in [0, 1)$, then $\mu_t = \{x \in X|\ \mu(x) \geq t\}$ is called a *level subset* of μ.

3 Fuzzy normal subspaces

In what follows, $\mathcal{V}$ is a vector space on a field F, unless otherwise specified.

Definition 3.1. ([7], [11]) A fuzzy set μ of $\mathcal{V}$ is called a *fuzzy subspace* of $\mathcal{V}$, if for all $x, y \in \mathcal{V}$ and $a \in F$ the following conditions hold:

(i) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$,

(ii) $\mu(-x) \geq \mu(x)$,

(iii) $\mu(ax) \geq \mu(x)$.

Clearly, if μ is a fuzzy subspace of $\mathcal{V}$, then $\mu(0) \geq \mu(x)$ for all $x \in \mathcal{V}$. Also, μ is a fuzzy subspace of $\mathcal{V}$ if and only if μ_t is a subspace of $\mathcal{V}$ for all $t \in [0, 1)$.

Lemma 3.2. *If μ is a fuzzy subspace of $\mathcal{V}$, then the set $\mathcal{V}_\mu = \{x \in \mathcal{V} \mid \mu(x) = \mu(0)\}$ is a subspace of $\mathcal{V}$.*

Proof. Let $x, y \in \mathcal{V}_\mu$. Then $\mu(x) = \mu(y) = \mu(0)$. Since μ is a fuzzy subspace, it follows that

$$\mu(x-y) \geq \mu(x) \wedge \mu(y) = \mu(0) \wedge \mu(0) = \mu(0).$$

On the other hand $\mu(x-y) \leq \mu(0)$. Hence we have $\mu(x-y) = \mu(0)$ and so $x - y \in \mathcal{V}_\mu$. Also for any $x \in \mathcal{V}_\mu$ and $a \in F$, we get $\mu(ax) \geq \mu(x) = \mu(0)$. On the other hand $\mu(ax) \leq \mu(0)$. Hence, we obtain $\mu(ax) = \mu(0)$, which shows that $ax \in \mathcal{V}_\mu$. Consequently, the set $\mathcal{V}_\mu$ is a subspace of $\mathcal{V}$. □

Definition 3.3. A fuzzy subspace of $\mathcal{V}$ is said to be *normal* if there exists $x \in \mathcal{V}$ such that $\mu(x) = 1$. Note that if a fuzzy subspace of $\mathcal{V}$ is normal, then $\mu(0) = 1$. Hence μ is a normal fuzzy subspace if and only if $\mu(0) = 1$.

Theorem 3.4. *Let μ be a fuzzy subspace of $\mathcal{V}$ and let $\widetilde{\mu}$ be a fuzzy set in $\mathcal{V}$ defined by $\widetilde{\mu}(x) = \mu(x) + 1 - \mu(0)$ for all $x \in \mathcal{V}$. Then $\widetilde{\mu}$ is a normal fuzzy subspace of $\mathcal{V}$ containing μ.*

Proof. Let $x, y \in \mathcal{V}$ and $a \in F$. Then

$$\widetilde{\mu}(x-y) = \mu(x-y) + 1 - \mu(0) \geq (\mu(x) \wedge \mu(y)) + 1 - \mu(0) =$$

$$(\mu(x) + 1 - \mu(0)) \wedge (\mu(y) + 1 - \mu(0)) = \widetilde{\mu}(x) \wedge \widetilde{\mu}(y).$$

Also we have

$$\widetilde{\mu}(ax) = \mu(ax) + 1 - \mu(0) \geq \mu(x) + 1 - \mu(0) = \widetilde{\mu}(x).$$

Clearly, $\widetilde{\mu}(0) = 1$ and $\mu \subseteq \widetilde{\mu}$. This completes the proof. □

Corollary 3.5. *If μ is a fuzzy subspace of $\mathcal{V}$ satisfying $\widetilde{\mu}(x) = 0$ for some $x \in \mathcal{V}$, then $\mu(x) = 0$.*

Lemma 3.6. *Let $\chi_{\mathcal{W}}$ be the characteristic function of a subset $\mathcal{W} \subseteq \mathcal{V}$. Then $\mathcal{W}$ is a subspace of $\mathcal{V}$ if and only if $\chi_{\mathcal{W}}$ is a fuzzy subspace of $\mathcal{V}$.*

Proof. It is directly followed from discussion after Definition 3.1. □

Theorem 3.7. *For any subspace $\mathcal{W}$ of $\mathcal{V}$, the characteristic function $\chi_{\mathcal{W}}$ is a normal fuzzy subspace of $\mathcal{V}$ and $\mathcal{V}_{\chi_{\mathcal{W}}} = \mathcal{W}$.*

Proof. Straightforward. □

Theorem 3.8. *A fuzzy subspace μ of $\mathcal{V}$ is normal if and only if $\widetilde{\mu} = \mu$.*

Proof. If $\widetilde{\mu} = \mu$, then it is obvious that μ is a normal fuzzy subspace of $\mathcal{V}$. Assume that μ is a normal fuzzy subspace of $\mathcal{V}$ and let $x \in \mathcal{V}$. Then $\widetilde{\mu}(x) = \mu(x) + 1 - \mu(0) = \mu(x)$, and hence $\widetilde{\mu} = \mu$. □

Theorem 3.9. *If μ is a fuzzy subspace of $\mathcal{V}$, then $\widetilde{(\widetilde{\mu})} = \widetilde{\mu}$.*

Proof. Straightforward. □

Theorem 3.10. *Let μ be a fuzzy subspace of $\mathcal{V}$. If there exists a fuzzy subspace ν of $\mathcal{V}$ satisfying $\widetilde{\nu} \subseteq \mu$, then μ is a normal fuzzy subspace of $\mathcal{V}$.*

Proof. Suppose there exists a fuzzy subspace ν of $\mathcal{V}$ such that $\widetilde{\nu} \subseteq \mu$. Then $1 = \widetilde{\nu}(0) \leq \mu(0)$, and therefore $\mu(0) = 1$. □

Corollary 3.11. *Let μ be a fuzzy subspace of $\mathcal{V}$. If there exists a fuzzy subspace ν of $\mathcal{V}$ satisfying $\widetilde{\nu} \subseteq \mu$, then $\widetilde{\mu} = \mu$.*

Proof. It is immediately obtained from Theorem 3.10 and definition of $\widetilde{\mu}$. □

Theorem 3.12. *Let μ be a fuzzy subspace of $\mathcal{V}$ and $f : [0, \mu(0)] \longrightarrow [0, 1]$*

be an increasing map. Define a fuzzy set $\mu^f : \mathcal{V} \longrightarrow [0,1]$ *by* $\mu^f(x) = f(\mu(x))$ *for all* $x \in \mathcal{V}$*. Then* μ^f *is a fuzzy subspace of* $\mathcal{V}$*. In particular, if* $f(t) \geq t$ *for all* $t \in [0, \mu(0)]$ *then* $\mu \subseteq \mu^f$*.*

Proof. Let $x, y \in \mathcal{V}$. Then

$$\mu^f(x-y) = f(\mu(x-y)) \geq f(\mu(x) \wedge \mu(y)) =$$

$$f(\mu(x)) \wedge f(\mu(y)) = \mu^f(x) \wedge \mu^f(y).$$

Also if $a \in F$ and $x \in \mathcal{V}$, then $\mu^f(ax) = f(\mu(ax)) \geq f(\mu(x)) = \mu^f(x)$. Hence μ^f is a fuzzy subspace of $\mathcal{V}$. Assume that $f(t) \geq t$ for all $t \in [0, \mu(0)]$. Then $\mu^f(x) = f(\mu(x)) \geq \mu(x)$ for all $x \in \mathcal{V}$, which means $\mu \subseteq \mu^f$. $\square$

Theorem 3.13. *Let* μ *be a non-constant normal fuzzy subspace of* $\mathcal{V}$*, which is maximal in the partial ordered set of normal fuzzy subspaces of* $\mathcal{V}$ *under fuzzy sets inclusion. Then* μ *is a two-valued fuzzy subspace and takes the values 0 and 1.*

Proof. We know $\mu(0) = 1$. Let $x \in \mathcal{V}$ be such that $\mu(x) \neq 1$. It is enough to show that $\mu(x) = 0$. Assume that there exists $x' \in \mathcal{V}$ such that $0 < \mu(x') < 1$. Define a fuzzy set $\nu : \mathcal{V} \longrightarrow [0,1]$ by $\nu(x) = 1/2(\mu(x) + \mu(x'))$ for all $x \in \mathcal{V}$. Then clearly ν is well-defined. Let $x, y \in \mathcal{V}$. Then

$$\nu(x-y) = 1/2(\mu(x-y) + \mu(x')) \geq 1/2((\mu(x) \wedge \mu(y)) + \mu(x')) =$$

$$(1/2(\mu(x) + \mu(x'))) \wedge (1/2(\mu(y) + \mu(x'))) = \nu(x) \wedge \nu(y).$$

Also if $a \in F$ and $x \in \mathcal{V}$, then

$$\nu(ax) = 1/2(\mu(ax) + \mu(x')) \geq 1/2(\mu(x) + \mu(x')) = \nu(x).$$

Hence ν is a fuzzy subspace of $\mathcal{V}$. Now we have

$$\widetilde{\nu}(x) = \nu(x) + 1 - \nu(0) =$$

$$1/2(\mu(x) + \mu(x')) + 1 - 1/2(\mu(0) + \mu(x')) = 1/2(\mu(x) + 1).$$

So $\widetilde{\nu}(0) = 1/2(\mu(0) + 1) = 1$. Thus $\widetilde{\nu}$ is a normal fuzzy subspace of $\mathcal{V}$. Also $\widetilde{\nu}(0) = 1 > \widetilde{\nu}(x') = 1/2(\mu(x') + 1) > \mu(x')$. We know that $\widetilde{\nu}$ is non-constant. So by $\widetilde{\nu}(x') > \mu(x')$, it follows that μ is not maximal, which is a contradiction. Therefore μ takes only the values 0 and 1. □

Theorem 3.14. *Let μ be a fuzzy subspace of $\mathcal{V}$ and let $\overline{\mu}$ be a fuzzy set in $\mathcal{V}$ defined by $\overline{\mu}(x) = \mu(x)/\mu(0)$ for all $x \in \mathcal{V}$. Then $\overline{\mu}$ is a normal fuzzy subspace of $\mathcal{V}$ containing μ.*

Proof. For any $x, y \in \mathcal{V}$, we have

$$\mu(x - y) = \mu(x - y)/\mu(0) \geq (1/\mu(0))(\mu(x) \wedge \mu(y)) =$$

$$(\mu(x)/\mu(0)) \wedge (\mu(y)/\mu(0)) = \overline{\mu}(x) \wedge \overline{\mu}(y).$$

Also if $a \in F$ and $x \in V$ we get

$$\overline{\mu}(ax) = (\mu(ax)/\mu(0)) \geq (\mu(x)/\mu(0)) = \overline{\mu}(x).$$

Hence $\overline{\mu}$ is a fuzzy subspace of $\mathcal{V}$. Clearly $\overline{\mu}(0) = 1$ and $\mu \subseteq \overline{\mu}$. □

Corollary 3.15. *If μ is a fuzzy subspace of $\mathcal{V}$ satisfying $\overline{\mu}(x) = 0$ for some $x \in \mathcal{V}$, then $\mu(x) = 0$.*

Proof. Obvious. □

Theorem 3.16. *Let μ be a non-constant fuzzy subspace of $\mathcal{V}$ such that $\widetilde{\mu}$ is a maximal in the partial ordered set of normal fuzzy subspace of $\mathcal{V}$ under fuzzy sets inclusion. Then*

(1) *μ is normal.*

(2) *μ takes only the values 0 and 1.*

(3) $\chi_{\mathcal{V}_\mu} = \mu$.

(4) *$\mathcal{V}_\mu$ is a maximal subspace of $\mathcal{V}$.*

Proof. Since μ is non-constant, so $\widetilde{\mu}$ is non-constant maximal. Also $\widetilde{\mu}$ is normal, which implies $\widetilde{\mu}$ takes only values 0 and 1 by Theorem 3.13. . If $\widetilde{\mu}(x) = 1$, then $\mu(x) = \mu(0)$ and if $\widetilde{\mu}(x) = 0$, then $\mu(x) = \mu(0) - 1$. By Corollary 3.5, we have $\mu(x) = 0$ which implies $\mu(0) = 1$. Therefore μ is normal, and also $\widetilde{\mu} = \mu$ by Theorem 3.8, which proves (1) and (2).

(3) Clearly $\chi_{\mathcal{V}_\mu} \subseteq \mu$ and $\chi_{\mathcal{V}_\mu}$ takes only the values 0 and 1. Let $x \in \mathcal{V}$ and $\mu(x) = 0$, then $\mu \subseteq \chi_{\mathcal{V}_\mu}$. If $\mu(x) = 1$ then $x \in \mathcal{V}_\mu$ and so $\chi_{\mathcal{V}_\mu}(x) = 1$. In any case $\mu \subseteq \chi_{\mathcal{V}_\mu}$.

(4) Since μ is non-constant, $\mathcal{V}_\mu$ is a proper subspace of $\mathcal{V}$. Let $\mathcal{W}$ be a subspace of $\mathcal{V}$ such that $\mathcal{V}_\mu \subseteq \mathcal{W}$. Then we obtain $\mu = \chi_{\mathcal{V}_\mu} \subseteq \chi_{\mathcal{W}}$. Since μ and $\chi_{\mathcal{W}}$ are normal and $\mu = \widetilde{\mu}$ is maximal in the partial ordered set of normal fuzzy subspaces under fuzzy sets inclusion, we have $\mu = \chi_{\mathcal{W}}$ or $\chi_{\mathcal{W}}(x) = 1$ for all $x \in \mathcal{V}$, so $\mathcal{W} = \mathcal{V}$. If $\mu = \chi_{\mathcal{W}}$ then $\mathcal{V}_\mu = \mathcal{V}_{\chi_{\mathcal{W}}} = \mathcal{W}$ by Theorem 3.7 . Therefore $\mathcal{V}_\mu$ is a maximal subspace of $\mathcal{V}$. □

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