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Connections between ideals of semisimple EMV-algebras and set-theoretic filters

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Abstract

In this paper, we mainly study connections between ideals of the semisimple EMV-algebra M and filters on some nonempty set Ω. We show that there is a bijection between the set of all closed ideals of M and the set of all filters on Ω. We get that this correspondence also holds between the set of all closed prime ideals of M and the set of all weak ultrafilters on Ω. We prove that the topological space of all closed prime ideals of M and the topological space of all weak ultrafilters on Ω are homeomorphic.

Keywords: Semisimple EMV-algebra; Ideal; Filter; Closure operation; Closed ideal

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1 Introduction

An MV-algebra is an algebra $(M; \oplus, *, 0)$ of type (2, 1, 0, 0) which has the top element 1. The study of MV-algebras is very in-depth and comprehensive, which has important applications in other areas of mathematical research. There are close connections between ideals of a semisimple MV-algebra and filters on some associated nonempty set. Moreover, there exists a bijection between the set of all closed ideals of a semisimple MV-algebra and the set of all filters on some nonempty set. For more details about it, we recommend the monographs Cignoli et al. [2013], Lele et al. [2021].

An EMV-algebra is an algebra $(M; \vee, \wedge, \oplus, 0)$ of type (2, 2, 2, 0), which is a new class of algebraic structures. EMV-algebras cannot guarantee the existence of the top element 1, which are the generalizations of MV-algebras. MV-algebras are termwise equivalent to EMV-algebras with the top element, Dvurečenskij and Zahiri [2019].

We shall mainly study connections between ideals of a semisimple EMV-algebra M and filters on Ω, where $M \subseteq [0,1]^\Omega$ and $[0,1]^\Omega$ is an EMV-clan of fuzzy functions on some nonempty set Ω. This paper is organized as follows. In Section 2, we give some basic notions and theorems on EMV-algebras, which will be used in the paper. In Section 3, we start by introducing the limits of $f \in M$ along a filter F on Ω. We study the connections between ideals of M and filters on Ω. In Section 4, we define a closure operation on M. We exhibit a one-to-one correspondence between the set of all closed ideals of M and the set of all filters on Ω. We show that there is a homeomorphism between the topological space of all closed prime ideals of M and the topological space of all weak ultrafilters on Ω. In addition, there is an example of an ideal that is a non-closed ideal, and some properties of closed ideals are listed.

2 Preliminaries

In this section, we introduce some basic notions and theorems on an EMV-algebra, which will be used in the following sections.

A filter F on a nonempty set Ω is a collection of subsets of Ω satisfying (i) the intersection of two elements in F again belongs to it and (ii) for all $S \in F$, $S \subseteq T \subseteq \Omega$ implies that $T \in F$. By (ii), we have $\Omega \in F$ for any filter F on Ω. A filter F is called proper if $\emptyset \notin F$. It is obvious that if F_1 and F_2 are filters on Ω, $F_1 \cap F_2$ is also a filter of Ω. In fact, for all $S_1, S_2 \in F_1 \cap F_2$, we get $S_1 \cap S_2 \in F_1 \cap F_2$. Moreover, for any $S \in F_1 \cap F_2$ and $S \subseteq T \subseteq \Omega$, which implies $T \in F_1$ and $T \in F_2$. So $T \in F_1 \cap F_2$. We have shown that $F_1 \cap F_2$ is a filter on Ω.

Definition 2.1. *([Cignoli et al., 2013, Definition 1.1.1]) An MV-algebra is an algebra* $(M; \oplus, *, 0, 1)$ *of type* $(2, 1, 0, 0)$ *such that* $(M; \oplus, 0)$ *is a commutative monoid, and for all* $x, y \in M$ *satisfying the following axioms:*
(MV1) $x^{**} = x$;
(MV2) $x \oplus 0^* = 0^*$;
(MV3) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

For all $x, y \in [0, 1]$, the real interval $[0, 1]$ with the operations $x \oplus y = \min\{x + y, 1\}$ and $x^* = 1 - x$ is an MV-algebra. Let $(M; +, 0)$ be a monoid. An element $a \in M$ is called idempotent if it satisfies the equation $a + a = a$. We denote the set of all idempotent elements of *M* by $\mathcal{I}(M)$. We recommend Cignoli et al. [2013] for MV-algebras.

EMV-algebras as the generalizations of MV-algebras have many important properties. We recommend Dvurečenskij and Zahiri [2019] for EMV-algebras.

Definition 2.2. *([Dvurečenskij and Zahiri, 2019, Definition 3.1]) An EMV-algebra is an algebra* $(M; \vee, \wedge, \oplus, 0)$ *with type* $(2, 2, 2, 0)$ *satisfying the followings:*
(EMV1) $(M; \vee, \wedge, 0)$ *is a distributive lattice with the least element 0;*
(EMV2) $(M; \oplus, 0)$ *is a commutative ordered monoid with the neutral element 0;*
(EMV3) for all $a, b \in \mathcal{I}(M)$ *with* $a \leq b$ *and for each* $x \in [a, b]$, *the element* $\lambda_{a,b}(x) = \min\{y \in [a, b] \mid x \oplus y = b\}$ *exists in* M, *and* $([a, b]; \oplus, \lambda_{a,b}, a, b)$ *is an MV-algebra;*
(EMV4) for any $x \in M$, *there is* $a \in \mathcal{I}(M)$ *such that* $x \leq a$.

EMV-algebras cannot guarantee the existence of the top element 1. An ideal I of an EMV-algebra M is a nonempty subset satisfying (i) for all $x, y \in I$, $x \oplus y \in I$ and (ii) for each $y \in I$ and $x \in M$, $x \leq y$ can deduce $x \in I$. Let $Ideal(M)$ to denote the set of all ideals of M. An ideal *I* of *M* is proper if $I \neq M$. A proper ideal *I* is called prime if for any $x, y \in M$, $x \wedge y \in I$ implies that $x \in I$ or $y \in I$. We use $\mathcal{P}(M)$ to denote the set of all prime ideals of M. An ideal *I* of *M* is maximal if for all $x \in M \backslash I$, we have $\langle I \cup \{x\} \rangle = M$, where $\langle I \cup \{x\} \rangle = \{z \in M \mid z \leq a \oplus n.x$ for some $a \in I$ and some $n \in \mathbb{N}\}$. The set of all maximal ideals of *M* is denoted by $MaxI(M)$. It is well known that any maximal ideal of M must be prime ([Dvurečenskij and Zahiri, 2019]). An EMV-algebra M is semisimple if and only if $Rad(M) = \{0\}$, where $Rad(M) \triangleq \cap\{I \mid I \in MaxI(M)\}$. The set $Rad(M)$ is called the radical of M.

For two EMV-algebras $(M_1; \vee, \wedge, \oplus, 0)$ and $(M_2; \vee, \wedge, \oplus, 0)$, a mapping $\Phi : M_1 \longrightarrow M_2$ is called an EMV-homomorphism if Φ preserves the operations $\vee, \wedge, \oplus$ and 0, and for each $b \in \mathcal{I}(M_1)$ and for each $x \in [0, b]$, we have $\Phi(\lambda_b(x)) = \lambda_{\Phi(b)}(\Phi(x))$. Every MV-homomorphism is also an EMV-homomorphism, but the converse is not necessarily true ([Dvurečenskij and Zahiri, 2019]). A mapping $s : M \longrightarrow [0, 1]$ is said a state-morphism on M if *s* is an EMV-homomorphism from the EMV-algebra *M* into the EMV-algebra of the real interval $([0,1];\vee,\wedge,\oplus,0)$ with top element, such that there exists an element $x \in M$ with $s(x) = 1$. The set $Ker(s) = \{x \in M \mid s(x) = 0\}$ is called the kernel of the state-morphism s ([Dvurečenskij and Zahiri, 2019]).

Theorem 2.1. *([Dvurečenskij and Zahiri, 2019, Theorem 4.2 (ii)]) Let M be an EMV-algebra and s be a state-morphism on M. Then $Ker(s)$ is a maximal ideal of M. In addition, there is a unique maximal ideal I of M such that $s = s_I$, where $s_I : x \longmapsto x/I$ for all $x \in M$.*

Definition 2.3. *([Dvurečenskij and Zahiri, 2019, Definition 4.9]) Let Ω be a nonempty set. A system $T \subseteq [0,1]^{\Omega}$ is called an EMV-clan if it satisfies the following conditions:*
(1) $0 \in T$ such that $0(w) = 0$ for all $w \in \Omega$;
(2) if $a \in T$ is a 0-1-valued function, then $a - f \in T$ for each $f \in T$ with $f(w) \leq a(w)$ for all $w \in \Omega$, and if $f, g \in T$ with $f(w), g(w) \leq a(w)$ for all $w \in \Omega$, then $f \oplus g \in T$, where $(f \oplus g)(w) = \min\{f(w) + g(w), a(w)\}$ for all $w \in \Omega$;
(3) for each $f \in T$, there exists a 0-1-valued function $a \in T$ such that $f(w) \leq a(w)$ for all $w \in \Omega$;
(4) for given $w \in \Omega$, there exists $f \in T$ such that $f(w) = 1$.

From Dvurečenskij and Zahiri [2019, Proposition 4.10], we see that any EMV-clan can be organized into an EMV-algebra. That is, every EMV-clan on some $\Omega \neq \emptyset$ is an EMV-algebra, see Dvurečenskij and Zahiri [2019].

3 Ideals of semisimple EMV-algebras and filters on associated nonempty sets

Let *M* be a semisimple EMV-algebra. By Dvurečenskij and Zahiri [2019, Theorem 4.11], there is an EMV-clan $[0,1]^{\Omega}$ on some $\Omega \neq \emptyset$ such that M is an EMV-subalgebra of $[0,1]^{\Omega}$. In this section, for a semisimple EMV-algebra $M \subseteq [0,1]^{\Omega}$, we shall define the notion of limits along a filter. The connections between ideals of M and filters on Ω are studied. For each *f* ∈ *M* and for all $\varepsilon > 0$, we denote $D(f,\varepsilon) = \{x \in \Omega \mid f(x) < \varepsilon\}$.

Definition 3.1. *Let M be a semisimple EMV-algebra and F be a filter on Ω such that $M \subseteq [0,1]^{\Omega}$. For any $f \in M$ and $t \in [0,1]$, we call that f converges to t along F if for every $\varepsilon > 0$, there is $S \in F$ such that $\mid f(S) - t \mid < \varepsilon$.*

Proposition 3.1. *Let M be a semisimple EMV-algebra and F be a proper filter on Ω such that $M \subseteq [0,1]^{\Omega}$. Then for each $f \in M$, there has at most one limit along F.*

Proof. The proof is similar to Lele et al. [2021, Proposition 2.2].□

For any $f \in M$, the limit of f along a proper filter *F* on Ω does not necessarily exist. But it would be unique if it exists by Proposition 3.1. We denote it by $lim_F f$.

Let I be an ideal of M and F be a filter on Ω. We define

$\mathbf{F}_I = \{S \subseteq \Omega \mid D(f, \varepsilon) \subseteq S$ for some $f \in I$ and $\varepsilon > 0\}$

and

$\mathbf{I}_F = \{f \in M \mid f$ converges to 0 along $F\}$=$\{f \in M \mid D(f, \varepsilon) \in F$ for all $\varepsilon > 0\}$.

Proposition 3.2. *Let M be a semisimple EMV-algebra and F be a filter on Ω such that $M \subseteq [0,1]^\Omega$. For all $f, g \in M$:*
(1) If $lim_F f$ and $lim_F g$ exist, then $lim_F(f \oplus g)$ exists and $lim_F(f \oplus g) = lim_F f \oplus lim_F g$.
(2) If $lim_F f$ exists, then $lim_F \lambda_a(f)$ exists and $lim_F \lambda_a(f) = \lambda_a(lim_F f)$, where a is an idempotent element of M such that $f \in [0, a]$.

Proof. (1) Suppose that $f, g \in M$, $lim_F f$ and $lim_F g$ exist. There exists an idempotent element $a \in \mathcal{I}(M)$ such that $f, g \in [0, a]$. Also, we have $lim_F f, lim_F g \leq a(x)$ for all $x \in \Omega$. In the MV-algebra $([0, a]; \oplus, \lambda_a, 0, a)$, $lim_F f$ and $lim_F g$ also exist. By Lele et al. [2021, Lemma 2.4], we have $lim_F(f \oplus g)$ exists and $lim_F(f \oplus g) = lim_F f \oplus lim_F g$.

(2) Recall that $\lambda_a(f) = \min\{z \in [0, a] \mid z \oplus f = a\}$, where $a \in \mathcal{I}(M)$ with $f \in [0, a]$. Since $([0, a]; \oplus, \lambda_a, 0, a)$ is an MV-algebra, the result follows from Lele et al. [2021, Lemma 2.4].□

Recall that an ultrafilter *U* on Ω is a filter which is maximal, in other words, any filter that contains it is equal to it. An ultrafilter *U* on Ω is equally a collection of subsets of Ω satisfying (i) *U* is proper, (ii) the intersection of two subsets in the collection belongs to it and (iii) for any subset V, $V \in U$ if and only if $\Omega \backslash V \notin U$, see Garner [2020, Definition 2]. From (iii), we see that $\Omega \in U$ for any ultrafilter *U* on Ω. We shall show that the limits along an ultrafilter exist.

Proposition 3.3. *Let M be a semisimple EMV-algebra and U be an ultrafilter on Ω such that $M \subseteq [0,1]^\Omega$. Then, for any $f \in M$, there has a unique limit along U.*

Proof. Suppose that there is no $t \in [0, 1]$ such that $lim_U f = t$. That is, for any $t \in [0, 1]$, there exists $\varepsilon_0 > 0$ such that $f^{-1}(O_t) \notin U$, where $O_t = (t - \varepsilon_0, t + \varepsilon_0)$. In fact, if for all $\varepsilon > 0$, there exists $t_0 \in [0, 1]$ such that $f^{-1}(O_{t_0}) \in U$, where $O_{t_0} = (t_0 - \varepsilon, t_0 + \varepsilon)$. It follows that $lim_U f = t_0$, which is a contradiction. Since [0, 1] is compact, for each open covering $\{O_t \mid t \in [0, 1]\}$ of [0, 1], where $O_t = (t - \varepsilon, t + \varepsilon)$, there exists a finite subset $\{O_{t_1}, O_{t_2}, ..., O_{t_n}\}$ such that $[0, 1] = \bigcup_{i=1}^n O_{t_i}$. Since *U* is an ultrafilter on Ω, we have $\bigcup_{i=1}^n f^{-1}(O_{t_i}) =$

$f^{-1}(\bigcup_{i=1}^{n} O_{t_i}) = f^{-1}([0,1]) = \Omega \in U$. By Garner [2020, Definition 2], there is $j \in \{1, 2, ..., n\}$ such that $f^{-1}(O_{t_j}) \in U$, which is a contradiction. Hence, f has at least one limit along U.

By Proposition 3.1, the uniqueness of the limit is clear.□

Theorem 3.1. *Let M be a semisimple EMV-algebra and U be an ultrafilter on Ω such that $M \subseteq [0,1]^{\Omega}$. Consider the mapping $\Phi_U : M \longrightarrow [0,1]$ given by $\Phi_U(f) = lim_U f$, where $f \in M$. Then Φ_U is an EMV-homomorphism with $Ker(\Phi_U) = \mathbf{I}_U$.*

Proof. Let $\Phi_U : M \longrightarrow [0,1]$ be a mapping defined by $\Phi_U(f) = lim_U f$, where $f \in M$. By Proposition 3.3, the limit of f along U is unique. So Φ_U is well-defined. For all $f, g \in M$, there is $a \in \mathcal{I}(M)$ such that $f, g \in [0,a]$ and $([0,a]; \oplus, \lambda_a, 0, a)$ is an MV-algebra. Now we consider the restriction of Φ_U on $[0,a]$. From Lele et al. [2021, Proposition 2.6] we see that $\Phi_U \mid_{[0,a]}$ is an MV-homomorphism. Clearly, $\Phi_U(0) = 0$. Also, we have $\Phi_U(f \oplus g) = \Phi_U(f) \oplus \Phi_U(g)$, $\Phi_U(f \vee g) = \Phi_U(f) \vee \Phi_U(g)$ and $\Phi_U(f \wedge g) = \Phi_U(f) \wedge \Phi_U(g)$. That is, Φ_U is an EMV-homomorphism. In addition, $Ker(\Phi_U) = \{f \in M \mid lim_U f = 0\} = \mathbf{I}_U$.□

Theorem 3.2. *Let M be a semisimple EMV-algebra such that $M \subseteq [0,1]^{\Omega}$. We have the followings:*
(1) For each ideal I of M, $\mathbf{F}_I$ is a filter on Ω. Moreover, if I is proper, then $\mathbf{F}_I$ is proper.
(2) For each filter F on Ω, $\mathbf{I}_F$ is an ideal of M. Moreover, if F is proper, then $\mathbf{I}_F$ is proper.

Proof. (1) Let *I* be an ideal of *M*.

(i) For all $\varepsilon > 0$ and $f \in I$, we have $D(f, \varepsilon) = \{x \in \Omega \mid f(x) < \varepsilon\} \subseteq \Omega$. Then $\Omega \in \mathbf{F}_I$.

(ii) Let $S_1 \subseteq S_2 \subseteq \Omega$ and $S_1 \in \mathbf{F}_I$. There exist $f \in I$ and $\varepsilon > 0$ such that $D(f, \varepsilon) \subseteq S_1 \subseteq S_2$. This implies that $S_2 \in \mathbf{F}_I$.

(iii) Suppose that $S_1, S_2 \in \mathbf{F}_I$. There exist $f, g \in I$ and $\varepsilon, \delta > 0$ such that $D(f, \varepsilon) \subseteq S_1$ and $D(g, \delta) \subseteq S_2$. It follows that $D(f, \varepsilon) \cap D(g, \delta) \subseteq S_1 \cap S_2$. In addition, since $D(f \oplus g, \min(\varepsilon, \delta)) \subseteq D(f, \varepsilon) \cap D(g, \delta)$ and $f \oplus g \in I$, we have $D(f, \varepsilon) \cap D(g, \delta) \in \mathbf{F}_I$. By (ii), it now follows that $S_1 \cap S_2 \in \mathbf{F}_I$. So $\mathbf{F}_I$ is a filter on Ω.

Let *I* be a proper ideal. Suppose that $\mathbf{F}_I$ is not proper. Then $\emptyset \in \mathbf{F}_I$. So there exist $f \in I$ and $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ for all $x \in \Omega$. We choose $N \geq 1$ such that $f(x) \geq \varepsilon \geq \frac{1}{N}$. Then $Nf \in I$ and $Nf(x) \geq 1$. It implies that $1 \in I$ and $I = M$, which is a contradiction. Therefore, $\mathbf{F}_I$ is proper.

(2) Let *F* be a filter on Ω.

(i) Since $0 \in \mathbf{I}_F$, we have $\mathbf{I}_F \neq \emptyset$.

(ii) For all $f, g \in \mathbf{I}_F$, by Proposition 3.2, we have $lim_F(f \oplus g) = lim_F f \oplus lim_F g = 0$. So $f \oplus g \in \mathbf{I}_F$.

(iii) Suppose that $f \in M$, $g \in \mathbf{I}_F$ and $f \leq g$. We have $lim_F f \leq lim_F g = 0$. Then $f \in \mathbf{I}_F$. Therefore, $\mathbf{I}_F$ is an ideal of *M*.

Let *F* be a proper filter. If $\mathbf{I}_F$ is not proper, then $\mathbf{I}_F = M$. For all $f \in \mathbf{I}_F = M$, for all $\varepsilon > 0$, we have $D(f, \varepsilon) \in F$. There exists $a \in \mathcal{I}(M)$ such that $f \leq a$ and $a \in M = \mathbf{I}_F$. So for any $x \in \Omega$, there is $g(x) > 0$ such that $a(x) \geq g(x)$, where $g \in [0, a]$. It follows that $\emptyset = D(a, g(x)) \in F$, which is a contradiction. Hence, $\mathbf{I}_F$ is proper.□

Proposition 3.4. *Let M be a semisimple EMV-algebra such that $M \subseteq [0,1]^\Omega$. Then we have the followings:*
(1) For each ideal I of M, $I \subseteq \mathbf{I}_{\mathbf{F}_I}$.
(2) For each filter F on Ω, $\mathbf{F}_{\mathbf{I}_F} \subseteq F$.
(3) For each filter F on Ω, $\mathbf{F}_{\mathbf{I}_F} = F$ if $\{0,1\}^\Omega \subseteq M$.

Proof. The proof is similar to Lele et al. [2021, Proposition 2.8].□

Proposition 3.5. *Let M be a semisimple EMV-algebra such that $M \subseteq [0,1]^\Omega$. We have the followings:*
(1) If $\{0,1\}^\Omega \subseteq M$, then for each maximal ideal K of M, $\mathbf{F}_K$ is an ultrafilter on Ω.
(2) $\mathbf{I}_U$ is a maximal ideal of M if U is an ultrafilter on Ω.
(3) If $\{0,1\}^\Omega \subseteq M$, the converse of (2) is true.

Proof. (1) Let K be a maximal ideal of M and $S \subseteq \Omega$. Suppose $S \notin \mathbf{F}_K$. We will show that $\Omega \backslash S \in \mathbf{F}_K$.

We define $f \in M$ by

$$f(x) = \begin{cases} 0 & x \in S, \\ 1 & x \notin S. \end{cases}$$

Then we have $D(f, 0.5) = S \notin \mathbf{F}_K$. It follows that $f \notin K$. Let $b \in \mathcal{I}(M)$ such that $f \in [0, b]$. It follows from $f \notin K$ that $f \notin K_b$, where $K_b = K \cap [0, b]$. Since K is a maximal ideal of M, by Dvurečenskij and Zahiri [2019, Proposition 3.22], K_b is a maximal ideal of the MV-algebra $([0,b]; \oplus, \lambda_b, 0, b)$. By the maximality of K_b, there exists $n \geq 1$ such that $\lambda_b(nf) \in K_b$. Then $\lambda_b(nf) \in K$. Notice that $nf = f$, which follows that $\lambda_b(f) = \lambda_b(nf) \in K$. In addition, we also have $\Omega \backslash S = \Omega \backslash D(f, 0.5) = D(\lambda_b(f), 0.5) \in \mathbf{F}_K$. Hence, by Freiwald [2014, Chapter IX, Theorem 3.5], $\mathbf{F}_K$ is an ultrafilter on Ω.

(2) Let *U* be an ultrafilter on Ω. From Theorem 3.1, there is an EMV-homomorphism $\Phi_U : M \longrightarrow [0,1]$ defined by $\Phi_U(f) = lim_U f$. Since $M \subseteq [0,1]^\Omega$ is semisimple, for given $w \in \Omega$, there is $f \in M$ such that $f(w) = 1$. So for

$\{w\} \subseteq \Omega \in U$ and all $\varepsilon > 0$, we have $f(\{w\}) \subseteq (1-\varepsilon, 1+\varepsilon)$, which implies that there exists $f \in M$ such that $\Phi_U(f) = lim_U f = 1$. Hence, Φ_U is a state-morphism on M. By Theorem 2.1, $Ker(\Phi_U) = \mathbf{I}_U$ is a maximal ideal of *M*.

(3) If $\mathbf{I}_U$ be a maximal ideal of M. Then $\mathbf{F}_{\mathbf{I}_U}$ is an ultrafilter on Ω by (1). By Proposition 3.4 (3), $U = \mathbf{F}_{\mathbf{I}_U}$ is an ultrafilter.□

Proposition 3.6. *Let M be a semisimple EMV-algebra and F be a filter on Ω such that $\{0,1\}^\Omega \subseteq M \subseteq [0,1]^\Omega$. Then for any $f \in M$, F is an ultrafilter if and only if f has a unique limit along F.*

Proof. $\Rightarrow$: If *F* is an ultrafilter. By Proposition 3.3 we see that f has a unique limit along *F*.

$\Leftarrow$: Suppose that f has a unique limit along F, where $f \in M$. Consider the mapping $\Phi_F : M \longrightarrow [0,1]$ defined by $\Phi_F(f) = lim_F f$. We have that Φ_F is well-defined. By the proof of Proposition 3.5, Φ_F is a sate-morphism on M. So $Ker(\Phi_F) = \mathbf{I}_F$ is a maximal ideal of *M* by Theorem 2.1. Therefore, *F* is an ultrafilter on Ω by Proposition 3.5 (3).□

4 Closed ideals of semisimple EMV-algebras

In this section, we introduce the notions of closure operations and c-closed ideals on EMV-algebras. We get a bijection between the set of all closed ideals of M and the set of all filters on Ω. We exhibit a homeomorphism between the topological space of all closed prime ideals of M and the topological space of all weak ultrafilters on Ω.

Definition 4.1. *A closure operation on an EMV-algebra M is a mapping $c : Ideal(M) \longrightarrow Ideal(M)$ satisfying the following conditions: for all $I, J \in Ideal(M)$,*
(C1) $I \subseteq I^c$;
(C2) if $I \subseteq J$, then $I^c \subseteq J^c$;
(C3) $I^{cc} = I^c$; where I^c=$c(I)$.

Proposition 4.1. *Let M be a semisimple EMV-algebra and $M \subseteq [0,1]^\Omega$. For each ideal I of M, we denote $I^c = \mathbf{I}_{\mathbf{F}_I}$. Then c is a closure operation on M.*

Proof. The proof is similar to Lele et al. [2021, Proposition 3.1].□

An ideal I of M is called c-closed if $I^c = I$. We frequently prefer to call an ideal is closed instead of c-closed. The set of all closed ideals of M is denoted by $\mathcal{C}(M)$. In the subsequent sections, we shall mainly study closed ideals of M, where the closure operation is given by Proposition 4.1. Now we show that any maximal ideal must be contained in $\mathcal{C}(M)$.

Proposition 4.2. *Let M be a semisimple EMV-algebra and $M \subseteq [0,1]^{\Omega}$. Every maximal ideal of M is a closed ideal.*

Proof. Let I be a maximal ideal of M. $\mathbf{I}_{\mathbf{F}_I}$ is a proper ideal by Theorem 3.2. By Proposition 3.4 (1), we have $I \subseteq \mathbf{I}_{\mathbf{F}_I}$. Suppose $I \subsetneqq \mathbf{I}_{\mathbf{F}_I}$. For any $f \in \mathbf{I}_{\mathbf{F}_I} \backslash I$, by the maximality of I, we have $M = \langle I \cup \{f\} \rangle \subseteq \mathbf{I}_{\mathbf{F}_I}$, which is a contradiction. So $I = \mathbf{I}_{\mathbf{F}_I}$. We have shown that I is closed.□

Theorem 4.1. *Let M be a semisimple EMV-algebra such that $\{0,1\}^{\Omega} \subseteq M \subseteq [0,1]^{\Omega}$. Then there is a bijection between the set of all closed ideals of M and the set of all filters on Ω.*

Proof. Let $F(\Omega)$ to denote the set of all filters on Ω. Define two mappings:
$\Theta : \mathcal{C}(M) \longrightarrow F(\Omega)$ by $\Theta(I) = \mathbf{F}_I$ and $\Upsilon : F(\Omega) \longrightarrow \mathcal{C}(M)$ by $\Upsilon(F) = \mathbf{I}_F$. By Theorem 3.2 and Proposition 3.4(3), Θ and Υ are well-defined. For any $I \in \mathcal{C}(M)$ and $F \in F(\Omega)$, we get $\Theta\Upsilon(F) = \Theta(\mathbf{I}_F) = \mathbf{F}_{\mathbf{I}_F} = F$ and $\Upsilon\Theta(I) = \Upsilon(\mathbf{F}_I) = \mathbf{I}_{\mathbf{F}_I} = I$. So $\Theta\Upsilon$ and $\Upsilon\Theta$ are identical mappings. Hence, Θ is a bijection.□

Remark 4.1. *From Theorem 4.1, we get a one-to-one correspondence between the set of all closed ideals of M and the set of all filters on Ω. We shall study the restriction of this correspondence. We define $\mathcal{C}_M(M) = \{I \in \mathcal{C}(M) \mid I \in MaxI(M)\}$ and $F_U(\Omega) = \{F \mid F$ is an ultrafilter on $\Omega\}$. Suppose that $\{0,1\}^{\Omega} \subseteq M \subseteq [0,1]^{\Omega}$. It is easy to verify that there is also a bijection between $\mathcal{C}_M(M)$ and $F_U(\Omega)$.*

In fact, define two mappings $\Psi : F_U(\Omega) \longrightarrow \mathcal{C}_M(M)$ given by $\Psi(U) = \mathbf{I}_U$ and $\Psi' : \mathcal{C}_M(M) \longrightarrow F_U(\Omega)$ given by $\Psi'(I) = \mathbf{F}_I$. From Proposition 3.4 (3) and Proposition 3.5 we see that Ψ and Ψ' are well-defined. Similar to Theorem 4.1, we can prove that Ψ is a bijection.

Next, we will study a special class of filters on Ω, which corresponds to closed prime ideals of M. A filter F on Ω is called a weak ultrafilter if $\mathbf{I}_F$ is a prime ideal of M. We denote the set of all weak ultrafilters on Ω by $W(\Omega)$.

Proposition 4.3. *Let M be a semsimple EMV-algebra and $M \subseteq [0,1]^{\Omega}$. Every ultrafilter on Ω is a weak ultrafilter.*

Proof. Let F be an ultrafilter on Ω. Then $\mathbf{I}_F$ is a maximal ideal of M by Proposition 3.5 (2). So $\mathbf{I}_F$ is prime ([Dvurečenskij and Zahiri, 2019]). Hence, F is a weak ultrafilter.□

Proposition 4.4. *Let M be a semisimple EMV-algebra and $M \subseteq [0,1]^{\Omega}$. If I is a prime ideal of M, $\mathbf{F}_I$ is a weak ultrafilter on Ω.*

Proof. Let I be a prime ideal of M. Then $\mathbf{F}_I$ is proper. It follows that $\mathbf{I}_{\mathbf{F}_I}$ is a proper ideal by Theorem 3.2. Suppose that $f \wedge g \in \mathbf{I}_{\mathbf{F}_I}$ for $f, g \in M$. We get $D(f \wedge g, \varepsilon) \in \mathbf{F}_I$ for all $\varepsilon > 0$. Since $D(f, \varepsilon), D(g, \varepsilon) \subseteq D(f \wedge g, \varepsilon) \in \mathbf{F}_I$, we have that at least one of $D(f, \varepsilon)$ and $D(g, \varepsilon)$ is nonempty. That is, $f \in \mathbf{I}_{\mathbf{F}_I}$ or $g \in \mathbf{I}_{\mathbf{F}_I}$. In fact, suppose that $D(f, \varepsilon)$ and $D(g, \varepsilon)$ are empty sets. It follows that $\emptyset = D(f \wedge g, \varepsilon) \in \mathbf{F}_I$, which is a contradiction. We have shown that $\mathbf{F}_I$ is a weak ultrafilter on Ω.□

Theorem 4.2. *Let M be a semisimple EMV-algebra such that* $\{0,1\}^\Omega \subseteq M \subseteq [0,1]^\Omega$. *Then there is a bijection between the set of all closed prime ideals of M and the set of all weak ultrafilters on* Ω*.*

Proof. Let $\mathcal{P}_c(M)$ to denote the set of all closed prime ideals of M. Define two mappings:

$\Phi : \mathcal{P}_c(M) \longrightarrow W(\Omega)$ defined by $\Phi(I) = \mathbf{F}_I$ and $\Gamma : W(\Omega) \longrightarrow \mathcal{P}_c(M)$ defined by $\Gamma(F) = \mathbf{I}_F$.

The mappings Φ and Γ are well-defined by Proposition 4.4, Proposition 3.4 (3) and the definition of weak ultrafilters.

For any $I \in \mathcal{P}_c(M)$ and $F \in W(\Omega)$, we have $\Gamma\Phi(I) = \Gamma(\mathbf{F}_I) = \mathbf{I}_{\mathbf{F}_I} = I$ and $\Phi\Gamma(F) = \Phi(\mathbf{I}_F) = \mathbf{F}_{\mathbf{I}_F} = F$. So $\Phi\Gamma$ and $\Gamma\Phi$ are identical mappings. Hence, Φ is a bijection.□

Lemma 4.1. *Let M be a semisimple EMV-algebra such that* $M \subseteq [0,1]^\Omega$. *Then there is a topology on the space* $W(\Omega)$ *which has* $\mathcal{B}_w \triangleq \{\mathcal{U}_w(f) \mid f \in M\}$ *as a basis, where* $\mathcal{U}_w(f) = \{F \in W(\Omega) \mid f \notin \mathbf{I}_F\}$ *for* $f \in M$*.*

Proof. For any $F \in W(\Omega)$, there is $f \in M \backslash \mathbf{I}_F$ such that $F \in \mathcal{U}_w(f) \in \mathcal{B}_w$ since $\mathbf{I}_F$ is prime.

Furthermore, for all $f, g \in M$, suppose that $F \in \mathcal{U}_w(f) \cap \mathcal{U}_w(g)$. Then $f \notin \mathbf{I}_F$ and $g \notin \mathbf{I}_F$. We have $f \wedge g \notin \mathbf{I}_F$ since $\mathbf{I}_F$ is a prime ideal of M, which follows that $\mathcal{U}_w(f) \cap \mathcal{U}_w(g) \subseteq \mathcal{U}_w(f \wedge g)$. For any $F \in \mathcal{U}_w(f \wedge g)$, we have $f \wedge g \notin \mathbf{I}_F$. It implies that $D(f \wedge g, \varepsilon_0) \notin F$ for some $\varepsilon_0 > 0$. It follows from $D(f, \varepsilon_0), D(g, \varepsilon_0) \subseteq D(f \wedge g, \varepsilon_0) \notin F$ and $F \in W(\Omega)$ that $f \notin \mathbf{I}_F$ and $g \notin \mathbf{I}_F$. Then $\mathcal{U}_w(f \wedge g) \subseteq \mathcal{U}_w(f) \cap \mathcal{U}_w(g)$. So $\mathcal{U}_w(f \wedge g) = \mathcal{U}_w(f) \cap \mathcal{U}_w(g)$. That is, for any $F \in \mathcal{U}_w(f) \cap \mathcal{U}_w(g)$, there is $\mathcal{U}_w(f \wedge g) \in \mathcal{B}_w$ such that $F \in \mathcal{U}_w(f \wedge g) \subseteq \mathcal{U}_w(f) \cap \mathcal{U}_w(g)$.

We have shown that the sets $\mathcal{U}_w(f)$ form a basis of the topology on $W(\Omega)$.□

From Lemma 4.1, we get a space $W(\Omega)$ whose topology is the topology generated by $\mathcal{B}_w$. The open sets on $W(\Omega)$ are sets $\bigcup\limits_{\mathcal{U}_w(f) \in \mathcal{B}_w{}'} \mathcal{U}_w(f)$, where $\mathcal{B}_w{}' \subseteq \mathcal{B}_w$ and $f \in M$. When we refer to the topological space $W(\Omega)$, it will be with reference to the topology $\{\bigcup\limits_{\mathcal{U}_w(f) \in \mathcal{B}_w{}'} \mathcal{U}_w(f) \mid \mathcal{B}_w{}' \subseteq \mathcal{B}_w\}$ ([Munkres, 2000]).

Lemma 4.2. *Let M be a semisimple EMV-algebra and* $M \subseteq [0,1]^{\Omega}$. *The sets* $\mathcal{U}_c(f), f \in M$ *form a basis of the topology on* $\mathcal{P}_c(M)$*, where* $\mathcal{U}_c(f) = \{I \in \mathcal{P}_c(M) \mid f \notin I\}$ *for* $f \in M$.

Proof. We denote $\mathcal{B}_c = \{\mathcal{U}_c(f) \mid f \in M\}$.

For any $I \in \mathcal{P}_c(M)$, there is $f \in M \backslash I$ such that $I \in \mathcal{U}_c(f) \in \mathcal{B}_c$ since I is proper.

It is obvious that $\mathcal{U}_c(f) \cap \mathcal{U}_c(g) \subseteq \mathcal{U}_c(f \wedge g)$. Suppose that $I \in \mathcal{U}_c(f \wedge g)$. Then $f \wedge g \notin I = \mathbf{I}_{\mathbf{F}_I}$, where $f, g \in M$. Similar to Lemma 4.1, we have $f \notin \mathbf{I}_{\mathbf{F}_I} = I$ and $g \notin \mathbf{I}_{\mathbf{F}_I} = I$. It implies that $\mathcal{U}_c(f \wedge g) \subseteq \mathcal{U}_c(f) \cap \mathcal{U}_c(g)$. So $\mathcal{U}_c(f \wedge g) = \mathcal{U}_c(f) \cap \mathcal{U}_c(g)$. That is, for any $I \in \mathcal{U}_c(f) \cap \mathcal{U}_c(g)$, there is $\mathcal{U}_c(f \wedge g) \in \mathcal{B}_c$ such that $I \in \mathcal{U}_c(f \wedge g) \subseteq \mathcal{U}_c(f) \cap \mathcal{U}_c(g)$.

Hence, we have shown that $\mathcal{B}_c$ as the basis of the topology on $\mathcal{P}_c(M)$.□

By Lemma 4.2 and Munkres [2000], the topology on $\mathcal{P}_c(M)$ is the topology generated by $\mathcal{B}_c$ where the open sets are sets $\bigcup_{\mathcal{U}_c(f) \in \mathcal{B}_c{}'} \mathcal{U}_c(f)$, where $\mathcal{B}_c{}' \subseteq \mathcal{B}_c$ and $f \in M$.

Theorem 4.3. *Let M be a semisimple EMV-algebra such that* $\{0,1\}^{\Omega} \subseteq M \subseteq [0,1]^{\Omega}$. *Then the two topological spaces* $\mathcal{P}_c(M)$ *and* $W(\Omega)$ *are homeomorphic.*

Proof. Consider the two well-defined bijections Φ and Γ defined by Theorem 4.2.

(1) Φ is continuous. Without lost of generality, we shall prove that the preimage of any $\mathcal{U}_w(f)$ in $W(\Omega)$ is open in $\mathcal{P}_c(M)$. We have $\Phi^{-1}(\mathcal{U}_w(f)) = \Gamma(\mathcal{U}_w(f)) = \{\mathbf{I}_F \mid f \notin \mathbf{I}_F\}$. For any $\mathbf{I}_F \in \Gamma(\mathcal{U}_w(f))$, where $F \in W(\Omega)$ and $f \notin \mathbf{I}_F$, by Proposition 3.4 (3), we have $\mathbf{I}_F \in \mathcal{P}_c(M)$. Then $\mathbf{I}_F \in \mathcal{U}_c(f)$. So $\Gamma(\mathcal{U}_w(f)) \subseteq \mathcal{U}_c(f)$. Moreover, for any $I \in \mathcal{U}_c(f)$, then $I \in \mathcal{P}_c(M)$ and $f \notin I$. We have $\mathbf{F}_I \in W(\Omega)$ and $f \notin I = \mathbf{I}_{\mathbf{F}_I}$. It implies that $I \in \Gamma(\mathcal{U}_w(f))$. So $\mathcal{U}_c(f) \subseteq \Gamma(\mathcal{U}_w(f))$. Hence, $\Phi^{-1}(\mathcal{U}_w(f)) = \Gamma(\mathcal{U}_w(f)) = \mathcal{U}_c(f)$ is an open set in $\mathcal{P}_c(M)$.

(2) Γ is continuous. We shall prove $\Gamma^{-1}(\mathcal{U}_c(f)) = \mathcal{U}_w(f)$. We have $\Gamma^{-1}(\mathcal{U}_c(f)) = \Phi(\mathcal{U}_c(f)) = \{\mathbf{F}_I \mid f \notin I\}$. For any $F \in \mathcal{U}_w(f)$, we get $F \in W(\Omega)$ and $f \notin \mathbf{I}_F$. By Proposition 3.4 (3), we see that $\mathbf{I}_F \in \mathcal{P}_c(M)$ and $F = \mathbf{F}_{\mathbf{I}_F} \in \Phi(\mathcal{U}_c(f))$. So $\mathcal{U}_w(f) \subseteq \Phi(\mathcal{U}_c(f))$. For each $\mathbf{F}_I \in \Phi(\mathcal{U}_c(f))$, where $I \in \mathcal{P}_c(M)$ and $f \notin I = \mathbf{I}_{\mathbf{F}_I}$. It follows that $\mathbf{F}_I \in \mathcal{U}_w(f)$. So $\Phi(\mathcal{U}_c(f)) \subseteq \mathcal{U}_w(f)$. Thus $\Gamma^{-1}(\mathcal{U}_c(f)) = \Phi(\mathcal{U}_c(f)) = \mathcal{U}_w(f)$ is an open set in $W(\Omega)$.

We have shown that Φ is a homeomorphism between $\mathcal{P}_c(M)$ and $W(\Omega)$.□

Example 4.1. *There exist non-closed ideals.*

Let M *be a semisimple EMV-algebra such that* $M \subseteq [0,1]^{\Omega}$. *Suppose that I is an ideal of* M. *It is obvious that* $\mathbf{I}_{\mathbf{F}_I} = \{f \in M \mid \forall \varepsilon > 0, \exists \delta > 0$ *and* $g \in I$ *such that* $g^{-1}([0,\delta)) \subseteq f^{-1}([0,\varepsilon))\}$. *In fact, for each* $f \in \mathbf{I}_{\mathbf{F}_I}$, *we have* $D(f,\varepsilon) \in \mathbf{F}_I$

for all $\varepsilon > 0$. So there exist $g \in I$ and $\delta > 0$ such that $D(g,\delta) \subseteq D(f,\varepsilon)$. It follows that $g^{-1}([0,\delta)) \subseteq f^{-1}([0,\varepsilon))$.

Let $M = [0,1]^{\mathbb{Z}^+}$, where all operations given by Definition 2.3 and Dvurečenskij and Zahiri [2019, Proposition 4.10]. Let $I = \{f \in M \mid$ for all but finitely many $n \in \mathbb{Z}^+$ such that $f(n) = 0\}$. It follows from $(f \oplus g)(n) = \min\{f(n)+g(n), a(n)\}$ and simple exercises that I is an ideal of M, where $f, g \in I$ and $a \in M$ is a 0-1-valued function such that $f(n), g(n) \leq a(n)$ for all $n \in \mathbb{Z}^+$.

Consider f given by $f(n) = \frac{n+1}{n^2+1}$ ($n \in \mathbb{Z}^+$). Clearly, $f \in M \backslash I$. It is easy to see that $f(n) \to 0$ when $n \to \infty$. That is, for all $\varepsilon > 0$, there is $N \in \mathbb{Z}^+$ such that $f(n) < \varepsilon$ when $n > N$. Now we consider $g \in M$ defined by

$$g(n) = \begin{cases} \frac{1}{n} & 1 \leq n \leq N, \\ 0 & n > N. \end{cases}$$

Then $g \in I$ and $D(g,\delta) \subseteq D(f,\varepsilon)$ for $\delta = \min\{\frac{1}{N+1}, \varepsilon\}$. It implies that $g^{-1}([0,\delta)) \subseteq f^{-1}([0,\varepsilon))$. So $f \in \mathbf{I}_{\mathbf{F}_I}$. We have shown that I *is a non-closed ideal.*

Definition 4.2. *Let M be an EMV-algebra and I be an ideal of M. Then I is called radical if $I = Rad(M)$, where $Rad(M)$ is the radical of M.*

Proposition 4.5. *Let M be a semisimple EMV-algebra such that $M \subseteq [0,1]^\Omega$. The following conditions are satisfied:*
(1) The intersection of closed ideals of M is also a closed ideal.
(2) An ideal I of M is closed if I is radical.

Proof. (1) Let $\{I_\alpha \mid \alpha \in \Lambda\}$ be a family of closed ideals of M. For each $\beta \in \Lambda$, it follows from $\bigcap_{\alpha\in\Lambda} I_\alpha \subseteq I_\beta$ that $(\bigcap_{\alpha\in\Lambda} I_\alpha)^c \subseteq {I_\beta}^c = I_\beta$. Then $(\bigcap_{\alpha\in\Lambda} I_\alpha)^c \subseteq \bigcap_{\beta\in\Lambda} I_\beta = \bigcap_{\alpha\in\Lambda} I_\alpha$. Since $\bigcap_{\alpha\in\Lambda} I_\alpha \subseteq (\bigcap_{\alpha\in\Lambda} I_\alpha)^c$, we have $(\bigcap_{\alpha\in\Lambda} I_\alpha)^c = \bigcap_{\alpha\in\Lambda} I_\alpha$. So $\bigcap_{\alpha\in\Lambda} I_\alpha \in \mathcal{C}(M)$.

(2) Suppose that *I* is radical. It implies that $I = \cap\{K \mid K \in MaxI(M)\}$. So by Proposition 4.2 and (1), I is closed.□

5 Conclusion

For a semisimple EMV-algebra M such that $M \subseteq [0,1]^\Omega$, we introduce the notion of limits along a filter on Ω, which is unique if it exists. For all ultrafilters U on Ω and for all $f \in M$, we give an EMV-homomorphism Φ_U with kernel equal to $\mathbf{I}_U$, which is defined by $\Phi_U(f) = lim_U f$. We study connections between ideals of M and filters on Ω. We define closure operations and closed ideals on EMV-algebras. We show that there is a bijection between the set of all closed ideals of M and the set of all filters on Ω. We show that there is a homeomorphism between the topological space $\mathcal{P}_c(M)$ and the topological space $W(\Omega)$. We give an example of a non-closed ideal and some properties of closed ideals.

Assume that F is a filter of the proper EMV-algebra M and I is an ideal of M. We can show that $\mathbf{I}_F = \{\lambda_a(x) \mid x \in F, a \in \mathcal{I}(M), x \leq a\}$ is an ideal of M. If F is a maximal filter of M, $\mathbf{I}_F$ is a maximal ideal of M can be proved. We can also get that $\mathbf{F}_I = \{\lambda_a(x) \mid x \in I, a \in \mathcal{I}(M) \backslash I, x < a\}$ is a filter of M under the assumption that $\forall a \in \mathcal{I}(M), a \notin I \Longrightarrow (\forall b \in \mathcal{I}(M), a < b)\lambda_b(a) \in I$.

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