ON DAVIDSON'S PROBLEM IN THE COLLECTIVE RISK THEORY

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Abstract. In this paper Davidson's classic problem concerning the solution of an integro-differential equation regarding the collective risk theory with the aim of examining the probability of the failure of an insurance company is further analysed. The validity of a new representation's formula to the solution of the problem is demonstrated after having discussed the question of the existence of that solution.

Keyword. Ruin probability. Integro-differential equation.

1. Introduction.

As it is well known, the collective risk theory, introduced by Lundberg and subsequently developed by various authors during the last hundred years, has been a fundamental contribution to questions concerning the probability of the failure of an insurance company in a finite time.

The usual approach to such problems consists in examining the dynamic over time of the risk reserve's fund, which the company assigns in the starting time to the management of non-life insurance portfolio with homogeneous policies covering repeatable accidents. The topic will now be briefly reviewed in order that the problem in question can then be discussed.

2. Recalls on the collective point of view

With reference to the period $(0,t)$, $t \in R^+$, we put

$W(t)$ = size at time t of the risk reserve's fund, which an insurance company above specified assigns to whole portfolio or its part; in particular: $W(0) = x$.

Z_i = random variable (r.v.) " Company's outlay relative to i-th claim";

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c = flow of the company's receipts without fund's yield, assumed constant over time (Lundberg's hypothesis, 1903);

N(t) = r.v. "number of claims in (0,t)";

S(t) = r.v. "Company's total outlays relative to *N(t)* claims".

Therefore

$$S(t) = \sum_{i=1}^{N(t)} Z_i \tag{2.1}$$

$$W(t) = x + ct - S(t) \tag{2.2}$$

In the sequel we assume the r.v. Z_i independent and identically distributed (i.i.d.) with c.d.f. *P(z)* absolutely continuous, and therefore with p.d.f. *p(z)* continuous, in R_0^+.

With these hypotheses and positions, the two factors which determine the net risk premiums flow are:

i) the average outlay per claim, given by

$$\mu_1 = E(Z_i) = \int_0^{+\infty} z\, dP(z) = \int_0^{+\infty} z\, p(z) dz, \quad \forall i \tag{2.3}$$

ii) the average number $E[N(1)]$ of claims in the unit of time, which, in the hypothesis that *N(t)* follows Poisson distribution with parameter $\nu t = E[N(t)]$, is given by intensity ν.

In the same hypothesis, the r.v. *S(t)* follows compound-Poisson distribution with intensity ν for the arrivals process and its expected value is given by

$$E[S(t)] = \mu_1 \nu\ t \tag{2.4}$$

Owing to loading on the premiums with mean rate λ>0, the parameters must be fixed in such a way that

$$c = \mu_1 \nu\ (1 + \lambda) \tag{2.5}$$

For the sake of exposition's simplicity, the unit of time (or operational time if the arrivals process were non-homogeneous) will be chosen in such a way that ν=1. Moreover, one will be assumed that the average outlay per claim is the unit of amounts, so $\mu_1 = 1$. Therefore (2.5), giving $\mu_1 \nu = 1$, becomes

$$c = 1 + l \tag{2.5'}$$

In the aforementioned paper [4] Davidson dealt with fundamental questions concerning the theory of risk and of ruin in the hypothesis that the safety loading is variable in function of the initial level x of the risk reserve's fund.

He introduces the following values:

y(u, x) = probability that the fund initially at x level falls below x-u *(u ³0)*;

c(u, v, x)dv = probability that the fund, initially at x level, falls below x-u *(u ³0)* and that, when it does so for the first time, its value is between *x-(u+v)* e *x-(u+v+dv)*.

Therefore, $\forall\, u \geq 0$ it results

$$y(u,x) = \int_0^{+\infty} \chi(u,v,x)\,dv\,.$$

That given, Davidson analyses all the mutually exclusive events whose probability is *c(u, v, x)dv*. By means of differential arguments, he obtains the equation

$$\frac{\partial}{\partial u}\chi(u,v,x) - \frac{\partial}{\partial v}\chi(u,v,x) = \chi(u,0,x)\;\chi(0,v,x-u) \tag{2.6}$$

By evaluating the risk reserve through the various possibilities about the number of claims in (0,t), Davidson demonstrates that the process is regulated by the following differential equation

$$\frac{\partial}{\partial x}\nu(v,x) + \frac{\partial}{\partial v}\nu(v,x) + \left[\nu(0,x) - \frac{1}{1+\lambda(x)}\right]\nu(v,x) + \frac{1}{1+\lambda(x)}\,p(v) = 0 \tag{2.7}$$

where *n(v, x) =c(0, v, x)dv* and being *l(x)* the safety loading rate, which is supposed a function of the initial level x of the risk reserve.

Laurin and Lundberg (see [11], [14]) had already obtained (2.7) via other methods. From (2.7) the integro-differential equation in the unknown *y(u, x)*

$$[1+\lambda(x)]\,[\psi_u(u,x)+\psi_x(u,x)] - \psi(u,x) + \int_0^u \psi(u-z,x-z)\,p(z)\,dz + $$
$$+1-P(u)=0 \tag{2.8}$$

follows.

The (2.8), with the positions

$$\xi = x - u \tag{2.9}$$

$$\overline{\psi}(u,\xi) = \psi(u,\xi+u) \qquad (2.10)$$

becomes

$$\overline{\psi_u}(u,\xi) = \frac{1}{1+\lambda(\xi+u)}\left[\, \overline{\psi}(u,\xi) - \int_0^u \overline{\psi}(z,\xi)\cdot\cdot\, p(u-z)\, dz - 1 + P(u)\,\right] \qquad (2.11)$$

that Davidson, in the aforementioned work, resolved by means of a procedure based on the theory of integral equations, assuming the initial condition

$$\overline{\psi}(0,\xi) = \psi(0,\xi) \qquad (2.12)$$

The (2.11) with the condition (2.12) is often cited in the literature as "Davidson's problem".

Let us remark that, put

$$\overline{\psi}(u,\xi) = 1 - f(u,\xi) \qquad (2.13)$$

(2.11) leads to

$$f\,(u,\xi) - [\, 1+\lambda(u+\xi)\,]\, f_u(u,\xi) = = \int_0^u f\,(u-z,\xi)\, p(z)\, dz \qquad (2.14)$$

with the initial condition

$$f\,(0,\xi) = 1 - \overline{\psi}(0,\xi) \equiv f_0 \qquad (2.15)$$

in which f_0 is a constant suitably assigned .

Moreover, the assignment of f_0 gives rise to some difficulties. Really, remembering (2.9) and the meaning of $y(u,x)$, if one puts $u = x$ or $\xi = 0$, $y(u,x)$ signifies the asymptotic probability of ruin in proper sense, or rather that the risk reserve, initially at x level, will sooner or later fall to zero. In such a case, let us write y in the form $y(x,x) = y^*(x)$ and put $f(x) = 1 - y^*(x)$, that is the asymptotic probability of non-ruin when x is the initial fund. That stated, it results that the constant f_0, which appears in (2.11), supposing a variable loading, must be fixed in such a way as to satisfy the condition: $f(+\infty) = 1$ (obviously non-ruin is assured if the initial reserve is infinitely large). Due to this problem, a resolving procedure has not yet been found in the case of an infinitely large initial fund. Algorithms of asymptotic calculus of the constant f_0 can, however, be applied with reference to the similar problem $f(k) = 1$, for a sufficiently large k (see [2], [10]).

3. New thoughts on Davidson's problem.

Let us consider the following integro-differential problem

$$\begin{cases} f(u,\xi) - [\, 1 + \lambda(u+\xi)\,] \dfrac{\partial f(u,\xi)}{\partial u} = \displaystyle\int_0^u f(u-z,\xi)\, p(z)\, dz \\ \quad f(0,\xi) = f_o \qquad\qquad\qquad\qquad f \in C^1(R_0^+) \end{cases} \tag{3.1}$$

where: $0 < f_0 < 1$, $C^1(R_0^+)$ is the class of continuous functions with continuous derivatives in R_0^+ e *P(z)*, *p(z)* are defined as in § 2. Besides, it results:

$$\int_0^\infty [\, 1 - P(z)\,]\, dz = 1$$

taking as amount’s unit the mean outlay per claim.

It is known that the integro-differential problem (3.1) admits only one solution. With the parameter ξ fixed, the existence of the solution can be demonstrated by using the successive approximation method, which however allows to find a representation’s formula for the solution, performed in the following § 4. To this aim, we observe that the first of (3.1) yields:

$$\frac{\partial f(u,\xi)}{\partial u} = \frac{1}{1+\lambda(u+\xi)} f(u,\xi) - \frac{1}{1+\lambda(u+\xi)} \int_0^u f(u-z)\, p(z)\, dz \tag{3.2}$$

which, integrated between 0 and u, becomes:

$$f(u,\xi) = f(0) + \int_0^u \frac{f(\tau,\xi)}{[1+\lambda(\tau+\xi)]}\, d\tau +$$
$$- \int_0^u \frac{1}{[1+\lambda(\tau+\xi)]} \int_0^\tau f(\tau - z,\xi)\, p(z)\, dz\, d\tau. \tag{3.3}$$

Putting:

$$f_0 = f(0,\xi) \tag{3.4}$$

$$f_n(u,\xi) = f(0,\xi) + \int_0^u \frac{f_{n-1}(\tau,\xi)}{[1+\lambda(\tau+\xi)]}\, d\tau +$$

$$-\int_0^u \frac{1}{[1+\lambda(\tau+\xi)]}\int_0^\tau f_{n-1}(\tau-z,\xi)\, p(z)\, dz\, d\tau\,; \qquad (n=1,2,3,...) \tag{3.5}$$

$$\beta_n(u,\xi)=f_n(u,\xi)\text{ - }f_{n-1}(u,\xi)\;\;;\quad (n=1,2,3,\dots) \tag{3.6}$$

$$\beta_0 = f_0 \;\; . \tag{3.7}$$

one obtains

$$f_n(u,\xi)=f(0,\xi)+\beta_1(u,\xi)+\beta_2(u,\xi)+...+\beta_n(u,\xi)\,;\;(n=1,2,3,\dots) \tag{3.8}$$

and also

$$\beta_{n+1}(u,\xi)=\int_0^u \frac{\beta_n(\tau,\xi)}{[1+\lambda(\tau+\xi)]}\,d\tau-\int_0^u \frac{1}{[1+\lambda(\tau+\xi)]}\cdot\int_0^\tau \beta_n(\tau-z,\xi)\,p(z)\,dz\,d\tau\;\;; \quad (n=1,2,3,\dots) \tag{3.9}$$

We now prove that:

$$\beta_n(u,\xi) \;=\; f_0\int_0^u \frac{1}{1+\lambda(\tau_1+\xi)}\cdot\int_0^{\tau_1}\frac{1}{1+\lambda(\tau_2+\xi)}\cdots\int_0^{\tau_{n-2}}\frac{1}{1+\lambda(\tau_{n-1}+\xi)}\cdot \cdots[1-P(\tau_1-\tau_2)]\,d\tau_n\cdot d\tau_{n-1}\cdots d\tau_1 \tag{3.10}$$

Dim.:

Proceeding by induction, for n=1 one obtains

$$\boldsymbol{b}_1(u,\boldsymbol{x}) \;=\; \int_0^u \frac{f_0}{1+\boldsymbol{l}(\boldsymbol{t}_1+\boldsymbol{x})} \;-\; \int_0^u\frac{1}{1+\boldsymbol{l}(\boldsymbol{t}_1+\boldsymbol{x})}\int_0^{t_1} f_0\,p(z)\,dz\,d\boldsymbol{t}_1 \;=$$

$$= f_0\int_0^u \frac{1-P(\boldsymbol{t}_1)}{1+\boldsymbol{l}(\boldsymbol{t}_1+\boldsymbol{x})}\,d\boldsymbol{t}_1 \tag{3.11}$$

being $P(0)=0$.

Let us now verify that (3.10), supposed to be true for $n=k$, is also valid for $n=k+1$. In fact :

$$\beta_{k+1}(u,\xi)=\int_0^u \frac{\beta_k(\tau_1,\xi)}{1+\lambda(\tau_1+\xi)}\,d\tau_1-\int_0^u \frac{1}{1+\lambda(\tau_1+\xi)}\int_0^{\tau_1}\beta_k(\tau_1-z,\xi)\,p(z)\,dz\,d\tau_1=$$

$$=f_0\int_o^u \frac{1}{1+\lambda(\tau_1+\xi)}\int_0^{\tau_1}\frac{1}{1+\lambda(\tau_2+\xi)}\int_0^{\tau_2}\frac{1}{1+\lambda(\tau_3+\xi)}\cdots\int_0^{\tau_{k-1}}\frac{1}{1+\lambda(\tau_k+\xi)}\cdot$$

$$\cdots[1-P(\tau_2-\tau_3)]\,d\tau_{k+1}\,d\tau_k\cdots d\tau_1\quad d\boldsymbol{t}_{k+1}\,d\boldsymbol{t}_k\cdots\,d\boldsymbol{t}_1\quad -$$

$$\int_0^{u}\frac{d\boldsymbol{t}_1}{1+\boldsymbol{l}(\boldsymbol{t}_1+\boldsymbol{x})}\int_0^{t_1}p(z)\int_0^{t_1-z}\frac{\mathrm{d}\boldsymbol{t}_2}{1+\boldsymbol{l}(\boldsymbol{t}_2+\boldsymbol{x})}\int_0^{t_2}\ldots\ldots \qquad (3.12)$$

$$\int_0^{\tau_k}[1-P(\tau_{k+1})]\prod_{i=3}^{k+1}\frac{1}{1+\lambda(\tau_i+\xi)}\prod_{j=2}^{k}[1-P(\tau_j-\tau_{j+1})]\;d\tau_3\ldots d\tau_{k+1}\Big\}$$

As a consequence of

$$\int_o^{\tau_1}p(z)\int_o^{\tau_1-z}\left\{\frac{1}{1+\lambda(\tau_2+\xi)}\int_o^{\tau_2}\frac{1}{1+\lambda(\tau_3+\xi)}\cdots\int_o^{\tau_{k-1}}[1-P(\tau_k-\tau_{k+1})]\cdot\right.$$

$$\left.\cdot[1-P(\tau_{k-1}-\tau_k)]\cdots[1-P(\tau_2-\tau_3)]\,d\,\tau_{k+1}\cdot d\tau_k\cdots d\tau_3\right\}d\tau_2\,dz=$$

$$=\int_0^{\tau_1}\left\{\frac{1}{1+\lambda(\tau_2+\xi)}\int_0^{\tau_2}\frac{1}{1+\lambda(\tau_3+\xi)}\cdots\int_o^{\tau_{k-1}}[1-P(\tau_k-\tau_{k+1})]\cdot\right.$$

$$\left.\cdot[1-P(\tau_{k-1}-\tau_k)]\cdots[1-P(\tau_2-\tau_3)]\,d\tau_{k+1}\cdots d\tau_3\right\}\int_0^{\tau_1-\tau_2}p(z)\,dz\,d\tau_2 \quad (3.13)$$

we obtain

$$b_{k+1}(u,x)=f_0\left\{\int_0^u\int_0^{t_k}\ldots\int_0^{t_k}[1-P(t_{k+1})]\;\prod_{i=1}^{k+1}\frac{1}{1+l(t_i+x)}\right.\;\cdot$$

$$\cdot\prod_{j=2}^{k}[1-P(\tau_j-\tau_{j+1})\;\;]\;d\tau_{k+1}\;d\tau_k\ldots d\tau_1+$$

$$-\int_0^u\int_0^{\tau_1}\ldots\int_0^{\tau_k}[1-P(\tau_{k+1})]\;P(\tau_1-\tau_2)\prod_{i=1}^{k+1}\frac{1}{1+\lambda(\tau_i+\xi)}\cdot$$

$$\left.\cdot\prod_{j=2}^{k}[1-P(\tau_j-\tau_{j+1})]\quad d\tau_{k+1}\;d\tau_k\ldots d\tau_1\right\}=$$

$$=f_o\int_o^u\frac{1}{1+\lambda(\tau_1+\xi)}\int_0^{\tau_1}\frac{1}{1+\lambda(\tau_2+\xi)}\int_o^{\tau_2}\frac{1}{1+\lambda(\tau_3+\xi)}\cdots$$

$$\prod_{j=1}^{k}[1-P(t_j-t_{j+1})]\,d\tau_{k+1}\,d\tau_k\cdots d\tau_1 \quad . \qquad (3.14)$$

Let us now prove

$$\beta_n(u,\xi) \le f_0 \frac{u^n}{n!} \qquad (n = 1,2,3,\ldots)$$

(3.15)

Dim:

Proceeding also here by induction's process, owing to (3.11) it results

$$\boldsymbol{b}_1(u,\boldsymbol{x}) \le f_0 \int_0^{\mathrm{u}} \frac{1}{1+\boldsymbol{l}(\boldsymbol{t}_1+\boldsymbol{x})} d\boldsymbol{t}_1 \le f_0\, u \qquad (3.16)$$

Moreover (3.15), supposed to be true for n=k, is also valid for n=k+1. In fact, we obtain

$$\beta_{k+1} \le \int_o^u \frac{\beta_k(\tau_1,\xi)}{1+\lambda(\tau_1+\xi)} d\tau_1 \le f_0 \int_o^u \frac{\tau_1{}^k}{k!} d\tau_1 = f_0 \frac{u^{k+1}}{(k+1)!}$$

Because of (3.15), the series $\sum_{n=1}^{\infty} \boldsymbol{b}_n(u,\boldsymbol{x})$ for each fixed $\boldsymbol{x}$ is absolutely and uniformly convergent in every limited interval $I \subset R_0^+$ and therefore the succession $\{f_n(u,\xi)\}_{n\in N}$ converges uniformly to a function $f(u,\xi)$.

Considering now in (3.5) the limit for $n \to \infty$, one obtains

$$f(u,\xi) = f(0) + \int_0^u \frac{f(\tau)}{1+\lambda(\tau+\xi)} d\tau - \int_0^u \frac{1}{1+\lambda(\tau+\xi)} \cdot \int_o^{\tau} f(\tau-z,\xi)\, p(z)\, dz\, d\tau \qquad (3.17)$$

Then the existence of the solution of (3.1) (Davidson's problem) is proved.

About the uniqueness of the solution, see [5].

4. On the representation of the solution to Davidson's problem.

A representation's formula for the function $f(u,\boldsymbol{x})$ of problem (3.1) and therefore for the asymptotic probability of non-ruin will now be evaluated.

Substituting (3.10) into (3.8) we obtain:

$$\frac{\partial}{\partial u} f(u,\xi) = f_0\Big[1+\int_0^u \frac{1}{1+\lambda(\tau_1+\xi)}\Big\{1-P(\tau_1) +$$

$$+\sum_{k=2}^{n} \int_0^{\tau_1} \cdot\cdot \int_0^{\tau_{k-1}} \prod_{i=2}^{k} \frac{1}{1+\lambda(\tau_i+\xi)}\left[1-P(\tau_k)\right] \cdot$$

$$\cdot\cdot \prod_{j=1}^{k-1} \left[1-P\left(\tau_j-\tau_{j+1}\right)\right] d\tau_k \cdots d\tau_2 \Big\} d\tau_1 \Big] \qquad (4.1)$$

Setting

$$\alpha_1 = 1 - P(\tau_1) \qquad (4.2)$$

and, more generally,

$$\alpha_k = \int_0^{\tau_1} \cdots \int_o^{\tau_{k-1}} \left[1 - P(\tau_k)\right] \prod_{i=2}^{k} \frac{1}{1+\lambda(\tau_i+\xi)} \cdot$$

$$\cdot \prod_{j=1}^{k-1} \left[1-P\left(\tau_j-\tau_{j+1}\right)\right] d\tau_k \ldots\ldots d\tau_2 \qquad (k=2,3,\ldots\ldots) \qquad (4.3)$$

we observe that, being

$$\alpha_k(\tau_1) \geq 0 \qquad \forall \tau_1 \geq 0 \qquad (4.4)$$

the series $\sum_k \alpha_k(\tau_1)$ results to be regular and, according to the theorem of monotonous convergence, calculating the limit under the sign of integral in (4.1) one obtains the solution

$$f(u,\xi) = f_0\left[1+\int_0^u \frac{1}{1+\lambda(\tau_1+\xi)} \sum_{k=1}^{\infty} \gamma_k \, d\tau_1\right] \qquad (4.5)$$

The foregoing results can be summed up as follows.

Theorem: *Let $\lambda(z)$ be a continuous and positive function in $[0,+\infty]$ and $p(z)$ be a continuous and non-negative function in $[0,+\infty]$. Given*

$$P(z) = \int_0^z p(t)\,dt$$

the solution f of the problem (3.1) is provided by:

$$f(u,\xi) = f_0(\xi)\left[1+\int_0^u \frac{1}{1+\lambda(\tau_1+\xi)} \sum_{k=1}^{\infty} \alpha_k \, d\tau_1\right] \qquad (4.6)$$

where α_k *is given by (4.2) and (4.3).*

REFERENCES

1 ARFWEDSON G., *Some Problems in the Collective Theory of Risk*, Skand. AktuarTidskr, (1950).

2 BADOLATI E., *Sulla probabilità di rovina in ipotesi di caricamenti variabili*, Pubbl. n. 26, Istituto Statistica e Matematica, Ist. Univ. Navale, Napoli, (1987).

3 BADOLATI E., *Una nuova equazione integro-differenziale della teoria del rischio;* Pubbl. n. 28, Istituto Statistica e Matematica, Ist. Univ. Navale, Napoli, (1988).

4 DAVIDSON A., *On the Ruin Problem in the Collective Theory of Risk;* Skand. AktuarTidskr. , 3-4 Supplement, (1969).

5 DI LORENZO E., *Su un'equazione integro-differenziale della teoria collettiva del rischio in ipotesi di caricamenti variabili;* G.I.I.A, (1989).

6 DI LORENZO E., *Una formula di rappresentazione per la probabilità asintotica di non rovina;* Pubbl. Ist. Statistica, Univ. Salerno, (1990).

7 FAVATI L., GOSIO C., LISEI G., *Su una successione di approssimanti per difetto della probabilità di rovina nel caso di investimento della riserva;* Pubbl. Ist. Matem. Finanz., Univ. Genova, (1988).

8 FELLER W., *An introduction to Probability Theory; Wiley,* (1971).

9 GERBER H.U., *An Introduction to Mathematical Risk Theory,* Huebner Found., Irwin; Philadelphia, (1979).

10 GOSIO C., *Alcune limitazioni per la probabilità di rovina nel caso di caricamento non costant*e; Pubbl. Ist. Matem. Finanz., Univ. Genova, (1989).

11 LAURIN I., *An Introduction into Lundberg's Theory of Risk;* Skand. AktuarTidskr., 1-2, (1930).

12 LISEI G., *Su un confine inferiore della probabilità di rovina nell'ambito della teoria collettiva del rischio;* G.I.I.A., (1983).

13 LISEI G., *Una successione di approssimanti per difetto della probabilità di rovina;* Pubbl. Ist. Matem. Finanz., Univ. Genova, (1988).

14 LUNDBERG F., *Forsakrangsteknisk riskutjamming.* I. Teori; Stockholm, (1926).

15 MARINA BORGHESANI M.E., *Alcune osservazioni sulla probabilità di rovina nel caso di caricamento non costante;* Pubbl. Ist. Matem. Finanz., Univ. Genova, (1988).