Ratio Mathematica Volume 41, 2021, pp. 7-18

On the stability of a multiplicative type sum form functional equation

Surbhi Madan*
Shveta Grover†
Dhiraj Kumar Singh‡§

Abstract

In this paper we intend to discuss the stability of a sum form functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right) = \sum_{i=1}^{n} k\left(p_i\right)\sum_{j=1}^{m} q_j^{\beta}$$

where f, k are real valued mappings each having the domain I; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ are fixed integers and β is a fixed positive real power different from 1 satisfying the conventions $0^{\beta} := 0$ and $1^{\beta} := 1$.

Keywords: Additive mapping; bounded mapping; functional equation; stability of a sum form functional equation.
2020 AMS subject classifications: 39B52, 39B82. [1]

*Department of Mathematics, Shivaji College (University of Delhi), Raja Garden, Ring Road, New Delhi-110027, India; surbhimadan@gmail.com, surbhi@shivaji.du.ac.in.

†Department of Mathematics, University of Delhi, Delhi 110007, India; srkgrover9@gmail.com.

‡Department of Mathematics, Zakir Husain Delhi College (University of Delhi), Jawaharlal Nehru Marg, Delhi 110002, India; dhiraj426@rediffmail.com, dksingh@zh.du.ac.in.

§Corresponding author

[1]Received on November 23, 2021. Accepted on December 27, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.690. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers; $\mathbb{R}$ denotes the set of real numbers and I denotes the closed unit interval $[0, 1]$. For $n \in \mathbb{N}$,

$$\Gamma_n = \left\{ (p_1, \ldots, p_n);\ p_i \geq 0,\ i = 1, \ldots, n;\ \sum_{i=1}^{n} p_i = 1 \right\}$$

denotes the set of all finite n-component complete discrete probability distributions.

A mapping $a : I \to \mathbb{R}$ is said to be additive on I or on the unit triangle $\Delta = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1\}$ if it satisfies the equation $a(x + y) = a(x) + a(y)$ for all $(x, y) \in \Delta$. Further, a mapping $A : \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if it satisfies the equation $A(x + y) = A(x) + A(y)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$. An interesting relation between these two additive mappings was established by Daróczy and Losonczi (2). They proved that additive mapping $a : I \to \mathbb{R}$ can be uniquely extended to the set of real numbers.

The stability of functional equations has intrigued mathematicians for more than eight decades now. The problem primarily aims to study the functional inequality corresponding to the given functional equation. Thereafter, it focuses on examining the proximity of their solutions. It needs to be remarked that without knowing the general solution of an equation we can not discuss its stability.

The seminal problem entered the corpus of sum form functional equations with the paper of Maksa (9). One of the interesting aspect explored in this field is to obtain *general solution* and discuss the *stability* of those sum form functional equations which are useful in characterizing entropies. Working in this direction, Nath and Singh (15; 16; 18); Singh and Grover (25) have recently addressed few sum form functional equations. These equations were useful in characterizing the Shannon entropies (19) defined as:

$$H_n(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i \qquad (\text{with } 0 \log_2 0 := 0) \tag{1.1}$$

where $H_n : \Gamma_n \to \mathbb{R}$, $n \in \mathbb{N}$ and $(p_1, \ldots, p_n) \in \Gamma_n$.

With the aim to further delve in the area of entropies, Havrda and Charvát (4) generalized the Shannon entropies (1.1) by introducing entropies of degree β defined as:

$$H_n^{\beta}(p_1, \ldots, p_n) = (1 - 2^{1-\beta})^{-1} \left[1 - \sum_{i=1}^{n} p_i^{\beta} \right] \tag{1.2}$$

where $H_n^\beta : \Gamma_n \to \mathbb{R}$, $n \in \mathbb{N}$; $(p_1, \ldots, p_n) \in \Gamma_n$ and β is a fixed positive real power different from 1, such that $0^\beta := 0$ and $1^\beta := 1$.

Losonczi and Maksa (8) were first to address the sum form functional equation that characterized the entropies (1.2) by considering the multiplicative type functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right) = \sum_{i=1}^{n} f\left(p_i\right)\sum_{j=1}^{m} f\left(q_j\right) \tag{1.3}$$

where $f : I \to \mathbb{R}$; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ and $n \geq 3, m \geq 3$ are fixed integers. Thereafter, Nath and Singh have analysed Pexiderized forms of (1.3), containing two and three unknown mappings for $n \geq 3, m \geq 3$ being fixed integers in (11) and (12).

It needs to be highlighted that many research papers in reference to the sum form functional equations characterizing several entropies have been written. In brief, these papers reflected upon: some generalizations; Pexiderized forms; importance and applications. Some significant contributions are: Nath and Singh (10; 17); Singh and Dass (20); Singh and Grover (21; 22; 23). The primary focus of these authors had been to obtain the general solution (or solutions) of the sum form functional equations for fixed integers $n \geq 3$, $m \geq 3$ or $n \geq 3$, $m \geq 2$. The stability problem for some of these sum form functional equations has been discussed by the authors but for most of them it remains unaddressed.

One of the interesting equation which motivated us is the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right) = \sum_{i=1}^{n} f\left(p_i\right)\sum_{j=1}^{m} q_j^\beta \tag{1.4}$$

where $f : I \to \mathbb{R}$; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ and β is a fixed positive real power different from 1 satisfying $0^\beta := 0$ and $1^\beta := 1$. In the recent past, Nath and Singh (13) have studied the equation (1.4) and obtained its general solutions for $n \geq 3, m \geq 3$ being fixed integers. The authors have further explained the relation of these solutions with entropies (1.2). The stability of (1.4) is established by Singh and Grover (24) for $n \geq 3, m \geq 3$ being fixed integers. Indeed, with the goal of getting a deeper insight of the equation (1.4), Nath and Singh (14); Garg, Grover and Singh (3) have recently studied a Pexiderized form (1.4), that is

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(p_i q_j\right) = \sum_{i=1}^{n} k\left(p_i\right)\sum_{j=1}^{m} q_j^\beta \tag{1.5}$$

where $f : I \to \mathbb{R}$, $k : I \to \mathbb{R}$; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ and β is a fixed positive real power different from 1 satisfying $0^\beta := 0$ and $1^\beta := 1$. In (14) and (3), the authors have obtained the general solutions of (1.5) for $n \geq 3, m \geq 3$ and $n \geq 3, m \geq 2$. The authors have further reflected upon the significance of general solutions in the light of entropies of degree β and diversity index. So far the stability problem remains unaddressed. The objective of this paper is to discuss the stability of functional equation (1.5).

For the problem of stability concerning functional equations, we refer to the survey paper of Hyers and Rassias (5) and Hyers, Isac and Rassias (6). The problem of stability of the functional equation (1.5) is given along the following lines: *Let $n \geq 3$, $m \geq 3$ be fixed integers; $0 \leq \varepsilon \in \mathbb{R}$ be fixed. Find all the mappings $f : I \to \mathbb{R}$, $k : I \to \mathbb{R}$ satisfying the functional inequality*

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) - \sum_{i=1}^{n} k(p_i) \sum_{j=1}^{m} q_j^{\beta} \right| \leq \varepsilon \tag{1.6}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$.

Below we will provide some known results.

Lemma 1.1 ((8)). *Suppose a mapping $\phi : I \to \mathbb{R}$ satisfies the functional equation $\sum\limits_{i=1}^{n} \phi(p_i) = c_1$ for all $(p_1, \ldots, p_n) \in \Gamma_n$, $n \geq 3$ a fixed integer and c_1 a real constant. Then there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ such that $\phi(p) = a(p) - \frac{1}{n}a(1) + \frac{c_1}{n}$ for all $p \in I$.*

Lemma 1.2 ((9)). *Let $0 \leq \varepsilon \in \mathbb{R}$, $n \geq 3$ be fixed integer and $\psi : I \to \mathbb{R}$ be a mapping which satisfies the functional inequality $\left| \sum\limits_{i=1}^{n} \psi(p_i) \right| \leq \varepsilon$ for all $(p_1, \ldots, p_n) \in \Gamma_n$. Then there exist an additive mapping $A_1 : \mathbb{R} \to \mathbb{R}$ and a mapping $B_1 : \mathbb{R} \to \mathbb{R}$ such that $|B_1(p)| \leq 18\varepsilon$ for all $p \in \mathbb{R}$, $B(0) = 0$ and $\psi(p) - \psi(0) = A_1(p) + B_1(p)$ for all $p \in I$.*

Lemma 1.3 ((7)). *Let $A_2 : \mathbb{R} \to \mathbb{R}$ be an additive mapping, $M : I \to \mathbb{R}$ a multiplicative mapping, $B_2 : \mathbb{R} \to \mathbb{R}$ a bounded mapping and $c_2 \in \mathbb{R}$. If $A_2(p) = M(p) + c_2$ for all $p \in I$, then $A_2(p) = dp, p \in \mathbb{R}$ for some $d \in \mathbb{R}$ and $M(p) = 0$ or $M(p) = p$, $p \in I$. Also if $A_2(p) = M(p) + B_2(p)$ for all $p \in I$, then $A_2(p) = dp$, $p \in \mathbb{R}$ for some $d \in \mathbb{R}$ and $M(p) = 0$ or $M(p) = p^{\alpha}$, $p \in I$ for some $0 \leq \alpha \in \mathbb{R}$.*

Lemma 1.4 ((26)). *If f is a solution of the functional equation $f(x+y) = f(x) + f(y)$ which is bounded over an interval $[a, b]$, then it is of the form $f(x) = c_3 x$ for some real number c_3.*

Lemma 1.5 ((3)). *Let $n \geq 3$, $m \geq 2$ be fixed integers; β be fixed positive real power different from 1 satisfying the conventions $0^{\beta} := 0$, $1^{\beta} := 1$ and $f : I \to \mathbb{R}$,*

$k : I \to \mathbb{R}$. *The pair* (f, k) *satisfies* (1.5) *if and only if there exist the additive mappings* $a_1, a_2 : \mathbb{R} \to \mathbb{R}$ *and* $c \in \mathbb{R}$ *such that*

$$\left.\begin{array}{ll}\text{(i)} & f(p) = cp^\beta + a_1(p) - \frac{1}{nm}a_1(1), \\ \text{(ii)} & k(p) = cp^\beta + a_2(p) - \frac{1}{n}a_2(1).\end{array}\right\}$$

2 The stability of the functional equation (1.5)

In this section our primary aim is to find the solutions of inequality (1.6). Thereafter, we need to observe: *What is the difference between these solutions and the solutions (given by Lemma 1.5) of equation* (1.5)*?* In the sense of Hyers, Isac and Rassias (6), if the difference is only a bounded mapping, we would say that functional equation (1.5) is stable. Following this we establish the the main result as follows:

Theorem 2.1. *Let* $n \geq 3$*,* $m \geq 3$ *be fixed integers;* β *be fixed positive real power different from 1 satisfying the conventions* $0^\beta := 0$ *and* $1^\beta := 1$*;* ε *be a nonnegative real constant and let* $f : I \to \mathbb{R}$*,* $k : I \to \mathbb{R}$ *be real valued mappings. Suppose the pair (f,k) satisfies* (1.6)*, then there exist the additive mappings* $a_1, a_2 : \mathbb{R} \to \mathbb{R}$*, the bounded mappings* $b_1, b_2 : \mathbb{R} \to \mathbb{R}$ *and* $0 \neq \overline{c}, c \in \mathbb{R}$ *such that*

$$\left.\begin{array}{ll}\text{(i)} & f(p) - f(0) = cp^\beta + a_1(p) + b_1(p), \\ \text{(ii)} & k(p) - k(0) = cp^\beta + a_2(p) + b_2(p)\end{array}\right\} \tag{2.1}$$

with

$$\left.\begin{array}{ll}\text{(i)} & |b_1(p)| \leq 1296\varepsilon\overline{c}\,(2m+1),\ b_1(0) = 0, \\ \text{(ii)} & |b_2(p)| \leq 1296\varepsilon\overline{c}\,(2m+1) + 18\varepsilon,\ b_2(0) = 0.\end{array}\right\} \tag{2.2}$$

Proof. Let us put $q_1 = 1$, $q_2 = \ldots = q_m = 0$ in (1.6). We get

$$\left|\sum_{i=1}^{n}\left[f(p_i) + (m-1)f(0) - k(p_i)\right]\right| \leq \varepsilon$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$. By Lemma 1.2, there exists an additive mapping $A_1 : \mathbb{R} \to \mathbb{R}$ and a mapping $B_1^* : \mathbb{R} \to \mathbb{R}$ with $|B_1^*(p)| \leq 18\varepsilon$, $B_1^*(0) = 0$, such that

$$f(p) - k(p) - f(0) + k(0) = A_1(p) + B_1^*(p)$$

for all $p \in I$. From this, we obtain the expression

$$k(p) = f(p) - A_1(p) - B_1(p) \tag{2.3}$$

where $B_1 : \mathbb{R} \to \mathbb{R}$, defined as $B_1(x) = B_1^*(x) + f(0) - k(0)$ is a bounded mapping. With the aid of (2.3), inequality (1.6) can be written as

$$\left| \sum_{i=1}^{n} \left[\sum_{j=1}^{m} f(p_i q_j) - [f(p_i) - A_1(1)p_i - B_1(p_i)] \sum_{j=1}^{m} q_j^\beta \right] \right| \leq \varepsilon$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. By Lemma 1.2, there exists a mapping $A_2 : \mathbb{R} \times \Gamma_m \to \mathbb{R}$, additive in the first variable and a mapping $B_2 : \mathbb{R} \times \Gamma_m \to \mathbb{R}$, bounded in the first variable by 18ε with $B_2(0; q_1, \ldots, q_m) = 0$, such that

$$\begin{aligned} \sum_{j=1}^{m} f(pq_j) - [f(p) - A_1(1)p - B_1(p) - f(0) + B_1(0)] \sum_{j=1}^{m} q_j^\beta - mf(0) \\ = A_2(p; q_1, \ldots, q_m) + B_2(p; q_1, \ldots, q_m). \end{aligned} \tag{2.4}$$

Let $x \in I$ and $(r_1, \ldots, r_m) \in \Gamma_m$ be an arbitrary probability distribution. Now, replacing p successively by xr_t, $t = 1, \ldots, m$ in (2.4); summing the resulting m equations so obtained and using the additivity of the mapping $A_2 : \mathbb{R} \times \Gamma_m \to \mathbb{R}$ in the first variable, we have

$$\begin{aligned} \sum_{t=1}^{m}\sum_{j=1}^{m} f(xr_t q_j) - \left[\sum_{t=1}^{m} f(xr_t) - A_1(1)x - \sum_{t=1}^{m} B_1(xr_t) - mf(0) + mB_1(0) \right] \sum_{j=1}^{m} q_j^\beta \\ - m^2 f(0) = A_2(x; q_1, \ldots, q_m) + \sum_{t=1}^{m} B_2(xr_t; q_1, \ldots, q_m) \end{aligned} \tag{2.5}$$

for all $x \in I$, $(q_1, \ldots, q_m) \in \Gamma_m$ and $(r_1, \ldots, r_m) \in \Gamma_m$. Now for $p = x$ and $q_1 = r_1, \ldots, q_m = r_m$, functional equation (2.4) gives

$$\begin{aligned} \sum_{t=1}^{m} f(xr_t) = [f(x) - A_1(1)x - B_1(x) - f(0) + B_1(0)] \sum_{t=1}^{m} r_t^\beta + mf(0) \\ + A_2(x; r_1, \ldots, r_m) + B_2(x; r_1, \ldots, r_m). \end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we get

$$\begin{aligned} \sum_{t=1}^{m}\sum_{j=1}^{m} f(xr_t q_j) - [f(x) - A_1(1)x - B_1(x) - f(0) + B_1(0)] \sum_{t=1}^{m} r_t^\beta \sum_{j=1}^{m} q_j^\beta \\ - m^2 f(0) = \Bigg[A_2(x; r_1, \ldots, r_m) + B_2(x; r_1, \ldots, r_m) - A_1(1)x - \sum_{t=1}^{m} B_1(xr_t) \\ + mB_1(0) \Bigg] \sum_{j=1}^{m} q_j^\beta + A_2(x; q_1, \ldots, q_m) + \sum_{t=1}^{m} B_2(xr_t; q_1, \ldots, q_m). \end{aligned} \tag{2.7}$$

We see that, the left hand side of (2.7) is commutative in r_t and q_j, $t = 1, \dots, m$; $j = 1, \dots, m$ (Aczél (1)). So, the commutativity on the right hand side implies

$$\begin{aligned}
&A_2(x; r_1, \dots, r_m)\left[1 - \sum_{j=1}^{m} q_j^{\beta}\right] - A_2(x; q_1, \dots, q_m)\left[1 - \sum_{t=1}^{m} r_t^{\beta}\right] \\
&= \sum_{t=1}^{m} B_2(xr_t; q_1, \dots, q_m) - \sum_{j=1}^{m} B_2(xq_j; r_1, \dots, r_m) \\
&+ \left[B_2(x; r_1, \dots, r_m) - A_1(1)x - \sum_{t=1}^{m} B_1(xr_t) + mB_1(0)\right]\sum_{j=1}^{m} q_j^{\beta} \\
&- \left[B_2(x; q_1, \dots, q_m) - A_1(1)x - \sum_{j=1}^{m} B_1(xq_j) + mB_1(0)\right]\sum_{t=1}^{m} r_t^{\beta}.
\end{aligned}$$

For fixed $(q_1, \dots, q_m) \in \Gamma_m$ and $(r_1, \dots, r_m) \in \Gamma_m$, the right hand side of the above equation is bounded on I while the left hand side is additive in $x \in I$. So, by Lemma 1.4, it follows that

$$\begin{aligned}
&[A_2(x; r_1, \dots, r_m) - xA_2(1; r_1, \dots, r_m)]\left[1 - \sum_{j=1}^{m} q_j^{\beta}\right] \\
&= [A_2(x; q_1, \dots, q_m) - xA_2(1; q_1, \dots, q_m)]\left[1 - \sum_{t=1}^{m} r_t^{\beta}\right]. \qquad (2.8)
\end{aligned}$$

Now, we assert that for $m \geq 3$ and fixed positive real power $\beta \neq 1$, $1 - \sum\limits_{t=1}^{m} r_t^{\beta}$ does not vanish identically on Γ_m. To the contrary, suppose $1 - \sum\limits_{t=1}^{m} r_t^{\beta}$ vanishes identically on Γ_m. Then, $1 = \sum\limits_{t=1}^{m} r_t^{\beta}$ for all $(r_1, \dots, r_m) \in \Gamma_m$. By Lemma 1.1, there exists an additive mapping $A : \mathbb{R} \to \mathbb{R}$ such that $r^{\beta} = A(r)$ with $A(1) = 1$. By Lemma 1.3, $r^{\beta} = 0$ or $r^{\beta} = r$ for all $r \in I$. This gives a contradiction as for the former case, our supposition $1 = \sum\limits_{t=1}^{m} r_t^{\beta}$ is contradicted while for the latter case, our assumption $\beta \neq 1$ is contradicted. This proves our assertion and so there exists a probability distribution $(r_1^*, \dots, r_m^*) \in \Gamma_m$ such that $1 - \sum\limits_{t=1}^{m} r_t^{*\beta} \neq 0$. Therefore with the substitution $r_t = r_t^*$, $t = 1, \dots, m$, functional equation (2.8) reduces to

$$A_2(x; q_1, \dots, q_m) = A_0(x)\left[1 - \sum_{j=1}^{m} q_j^{\beta}\right] + x\, A_2(1; q_1, \dots, q_m) \qquad (2.9)$$

where $A_0 : \mathbb{R} \to \mathbb{R}$ defined as

$$A_0(x) = \left[1 - \sum_{t=1}^{m} r_t^{*^\beta}\right]^{-1} [A_2(x; r_1^*, \ldots, r_m^*) - x\, A_2(1; r_1^*, \ldots, r_m^*)]$$

is an additive mapping with $A_0(1) = 0$. Further from (2.4), we have

$$A_2\,(1; q_1, \ldots, q_m) = \sum_{j=1}^{m} f(q_j) - [f(1) - A_1(1) - B_1(1) - f(0) + B_1(0)] \sum_{j=1}^{m} q_j^\beta \\ - mf(0) - B_2\,(1; q_1, \ldots, q_m)\,. \tag{2.10}$$

From (2.4), (2.9), (2.10) with $A_0(1) = 0$, we gather that

$$\sum_{j=1}^{m} [f(pq_j) - f(0) - A_0(pq_j) - (f(1) - f(0))pq_j] - [f(p) - f(0) - A_0(p) \\ - p(f(1) - f(0))] \sum_{j=1}^{m} q_j^\beta - p \sum_{j=1}^{m} [f(q_j) - f(0) - A_0(q_j) - (f(1) - f(0))q_j] \\ = [B_1(0) - B_1(p) + pB_1(1) - pB_1(0)] \sum_{j=1}^{m} q_j^\beta - pB_2\,(1; q_1, \ldots, q_m) \tag{2.11}$$

for all $p \in I$ and $(q_1, \ldots, q_m) \in \Gamma_m$.

Now, define a mapping $\overline{F} : I \to \mathbb{R}$ as

$$\overline{F}(x) = f(x) - f(0) - A_0(x) - (f(1) - f(0))x \tag{2.12}$$

for all $x \in I$. With the aid of (2.12), equation (2.11) can be written as

$$\sum_{j=1}^{m} \overline{F}(pq_j) - \overline{F}(p) \sum_{j=1}^{m} q_j^\beta - p \sum_{j=1}^{m} \overline{F}(q_j) \\ = [B_1(0) - B_1(p) + pB_1(1) - pB_1(0)] \sum_{j=1}^{m} q_j^\beta - pB_2\,(1; q_1, \ldots, q_m)\,. \tag{2.13}$$

It clearly follows from (2.12), that $\overline{F}(0) = 0$. Also we observe, the right hand side of (2.13) is bounded by $18\varepsilon(2m + 1)$, consequently by Lemma 1.2, along with $\overline{F}(0) = 0$, there exists a mapping $A_3 : I \times \mathbb{R} \to \mathbb{R}$, additive in the second variable and a mapping $B_3 : I \times \mathbb{R} \to \mathbb{R}$, bounded in the second variable by $324\varepsilon(2m+1)$ with $B_3(p; 0) = 0$, such that

$$\overline{F}(pq) - q^\beta\, \overline{F}(p) - p\, \overline{F}(q) = A_3(p; q) + B_3(p; q). \tag{2.14}$$

Define a mapping $K : I \times I \to \mathbb{R}$ as

$$K(p;q) = \overline{F}(pq) - q^{\beta}\,\overline{F}(p) - p\,\overline{F}(q) \tag{2.15}$$

for all $p \in I$, $q \in I$. With the help of (2.15), it can be verified easily that

$$\begin{aligned}\overline{F}(pqr) - pq\,\overline{F}(r) - q^{\beta}r^{\beta}\,\overline{F}(p) - pr^{\beta}\,\overline{F}(q) &= K\left(pq;r\right) + r^{\beta}K\left(p;q\right)\\ &= K\left(p;qr\right) + pK(q;r) \end{aligned} \tag{2.16}$$

for all $p \in I$, $q \in I$ and $r \in I$. From (2.14), (2.15) and (2.16), it follows that

$$\begin{aligned}&A_3\left(p;qr\right) + pA_3\left(q;r\right) - A_3\left(pq;r\right)\\ &= B_3\left(pq;r\right) + r^{\beta}A_3\left(p;q\right) + r^{\beta}B_3\left(p;q\right) - B_3\left(p;qr\right) - pB_3\left(q;r\right).\end{aligned} \tag{2.17}$$

Apparently, the left hand side of (2.17) is additive in $r \in I$, while its right hand side is bounded on I. Consequently by Lemma 1.4, its left hand side must be linear therefore, we get

$$A_3\left(p;qr\right)+pA_3\left(q;r\right)-A_3\left(pq;r\right)=r\left[A_3\left(p;q\right) + pA_3\left(q;1\right)-A_3\left(pq;1\right)\right]. \tag{2.18}$$

Now, for the substitution $r = 1$, equation (2.17) gives

$$pA_3\left(q;1\right) - A_3\left(pq;1\right) = B_3\left(pq;1\right) - pB_3(q;1)\,. \tag{2.19}$$

From (2.17), (2.18) and (2.19), we obtain

$$\begin{aligned}\left(r^{\beta} - r\right) A_3\left(p;q\right) = rB_3\left(pq;1\right) - rpB_3(q;1) - B_3\left(pq;r\right)&\\ - r^{\beta}B_3\left(p;q\right) + B_3\left(p;qr\right) + pB_3\left(q;r\right).&\end{aligned} \tag{2.20}$$

Since for fixed positive real number $\beta \neq 1$, we have '$r^{\beta} - r$' does not vanish identically on I, there exists some $r^* \in I$, such that

$$r^{*\beta} - r^* \not\equiv 0. \tag{2.21}$$

Using this in (2.20), it follows that

$$\begin{aligned}A_3\left(p;q\right) = \left(r^{*\beta} - r^*\right)^{-1}\Big\{&r^*B_3\left(pq;1\right) - r^*pB_3(q;1) - B_3\left(pq;r^*\right)\\ &- r^{*\beta}B_3\left(p;q\right) + B_3\left(p;qr^*\right) + pB_3\left(q;r^*\right)\Big\}\end{aligned}$$

for all $p \in I$, $q \in I$. This yield that additive mapping $A_3(p;q)$ is bounded in the second variable also. Hence by Lemma 1.4, it must be linear therein. So,

$$A_3\left(p;q\right) = qA_3(p;1) \tag{2.22}$$

for all $p \in I$, $q \in I$. From (2.19), with the substitution $q = 1$ it follows that

$$A_3\,(p;1) = pA_3\,(1;1) - B_3\,(p;1) + pB_3(1;1) \tag{2.23}$$

for all $p \in I$. From (2.22) and (2.23), it can be concluded that mapping $A_3(p;q)$ is bounded. Moreover, from (2.14) and (2.23) with $\overline{F}(1) = 0$ we obtain, $A_3(p;1) = -B_3(p;1)$. Consequently we get $|A_3(p;q)| \leq 324\varepsilon(2m+1)$. Hence, using this in (2.14), it follows that

$$|\overline{F}(pq) - q^{\beta}\,\overline{F}(p) - p\,\overline{F}(q)| \leq 648\varepsilon(2m+1) \tag{2.24}$$

for all $p \in I$, $q \in I$. Now on interchanging the places of p and q in the functional inequality (2.24), we have

$$|\overline{F}(pq) - p^{\beta}\,\overline{F}(q) - q\,\overline{F}(p)| \leq 648\varepsilon(2m+1). \tag{2.25}$$

Applying triangle inequality to functional inequalities (2.24) and (2.25), we obtain

$$|(q^{\beta} - q)\,\overline{F}(p) - (p^{\beta} - p)\,\overline{F}(q)| \leq 1296\varepsilon(2m+1) \tag{2.26}$$

where $p \in I$, $q \in I$ and β is a fixed positive real power different from 1. With the aid of (2.21) we get, $q^{*\beta} - q^{*} \not\equiv 0$ for some $q^{*} \in I$. On taking $\overline{c} := (q^{*\beta} - q^{*})^{-1} \in \mathbb{R}$; $c := \overline{F}(q^{*})(q^{*\beta} - q^{*})^{-1} \in \mathbb{R}$ in (2.26), it follows that there exist a mapping $b_1 : \mathbb{R} \to \mathbb{R}$ such that (2.2)(i) holds for all $p \in \mathbb{R}$ and

$$\overline{F}(p) = cp^{\beta} - cp + b_1(p) \tag{2.27}$$

for all $p \in I$. Thus, the solution (2.1)(i) follows from (2.12) and (2.27) by defining additive mapping $a_1 : \mathbb{R} \to \mathbb{R}$ as $a_1(x) = A_0(x) + (f(1) - f(0))x - cx$. Further, the solution (2.1)(ii) with (2.2)(ii) follows from (2.1)(i), (2.2)(i) and (2.3) by defining additive mapping $a_2 : \mathbb{R} \to \mathbb{R}$ as $a_2(x) = a_1(x) - A_1(x)$ and bounded mapping $b_2 : \mathbb{R} \to \mathbb{R}$ as $b_2(x) = b_1(x) - B_1^{*}(x)$. This completes the proof. □

3 Acknowledgements

The third author is grateful for the support from the SERB-MATRICS scheme (MTR/2020/000508) of the Department of Science and Technology, Government of India.

References

[1] Aczel 1966 J. Aczél. *Lectures on Functional Equations and Their Applications*, Academic Press, New York and London, 1966.

[2] Z. Daróczy and L. Losonczi. Über die Erweiterung der auf einer Punktmenge additiven Funktionen, *Publicationes Mathematicae* (*Debrecen*), **14** (1967), 239 – 245.

[3] P. Garg, S. Grover, and D.K. Singh. On a functional eqution related to diversity, To appear in *Montes Taurus Journal of Pure and Applied Mathematics*.

[4] J. Havrda and F. Charvát. Quantification method of classification processes. Concept of structural α-entropy, *Kybernetika*, **3** (1967), 30 – 35.

[5] D.H. Hyers and T.M. Rassias. Approximate homomorphisms, *Aequationes Mathematicae*, **44(2-3)** (1992), 125 – 153.

[6] D.H. Hyers, G. Isac, and Th.M. Rassias. *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, Basel-Berlin, 1998.

[7] I. Kocsis. On the stability of a sum form functional equation of multiplicative type, *Acta Acad. Paed. Agriensis, Sectio Mathematicae*, **28** (2001), 43 – 53.

[8] L. Losonzi and Gy. Maksa. On some functional equations of the information theory, *Acta Mathematica Academiae Scientiarum Hungarica*, **39(1–3)** (1982), 73 – 82.

[9] Gy. Maksa. On the stability of a sum form equation, *Results in Mathematics*, **26(3–4)** (1994), 342 – 347.

[10] P. Nath and D.K. Singh. On a sum form functional equation and its role in information theory, *Proc. of National Conference on 'Information Technology: Setting Trends In Modern Era' and 8^{th} Annual Conference of Indian Society of Information Theory and Applications*, (March 18–20, 2006), 88 – 94.

[11] P. Nath and D.K. Singh. On a multiplicative type sum form functional equation and its role in information theory, *Applications of Mathematics*, **51(5)** (2006), 495 – 516.

[12] P. Nath and D.K. Singh. Some sum form functional equations containing at most two unknown mappings, *Proc. of 9^{th} National Conference of Indian Society of Information Theory and Applications (ISITA) on "Information Technology and Operational Research Applications"*, (December 8–9, 2007) (2008), 54 – 71.

[13] P. Nath and D.K. Singh. On a sum form functional equation related to entropies of type (α, β), *Asian-European Journal of Mathematics*, **6(2)** (2013), 13 pages.

[14] P. Nath and D.K. Singh. On a sum form functional equation related to various nonadditive entropies in information theory, *Tamsui Oxford Journal of Information and Mathematical Sciences*, **30** (2014), 23 – 43.

[15] P. Nath and D.K. Singh. Some functional equations and their corresponding sum forms, *Sarajevo Journal of Mathematics*, **12(24)** (2016), 89 – 106.

[16] P. Nath and D.K. Singh. On a sum form functional equation containing five unknown mappings, *Aequationes Mathematicae*, **90** (2016), 1087 – 1101.

[17] P. Nath and D.K. Singh, On an equation related to nonadditive entropies in information theory, *Mathematics for Applications*, **6(1)** (2017), 31 – 41.

[18] P. Nath and D.K. Singh. On the stability of a functional equation, *Palestine Journal of Mathematics*, **6(2)** (2017), 573 – 579.

[19] C.E. Shannon. A mathematical theory of communication, *Bell System Technical Journal*, **27** (1948), 379 – 423; 623 – 656.

[20] D.K. Singh and P. Dass. On a functional equation related to some entropies in information theory, *Journal of Discrete Mathematical Sciences and Cryptography*, **21(3)** (2018), 713 – 726.

[21] D.K. Singh and S. Grover. On the stability and general solution of a sum form functional equation emerging from information theory, *Bulletin of Mathematical Analysis and Applications*, **12(3)** (2020), 1 – 18.

[22] D.K. Singh and S. Grover. On the stability of a nonmultiplicative type sum form functional equation, *Italian Journal of Pure and Applied Mathematics*, **45** (2021), 360 – 374.

[23] D.K. Singh and S. Grover. On a sum form functional equation emerging from statistics and its applications, *Results in Nonlinear Analysis*, **4(2)** (2021), 65 – 76.

[24] D.K. Singh and S. Grover. On the stability of a sum form functional equation related to entropies of type (α, β), *Journal of Nonlinear Sciences and Applications*, **14(3)** (2021), 168 – 180.

[25] D.K. Singh and S. Grover. On the stability of a sum form functional equation related to nonadditive entropies, *Journal of Mathematics and Computer Science*, **23(4)** (2021), 328 – 340.

[26] G.S. Young. The linear functional equation, *The American Mathematical Monthly*, **65(1)** (1958), 37 – 38.