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Directed Graphs representing isomorphism classes of C-Hypergroupoids

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Abstract

We investigate the relation of directed graphs and hyperstructures by virtue of the *graph hyperoperation*. A new class of graphs arises in this way representing isomorphism classes of **C**-hypergroupoids and we present the 17 such graphs that correspond to the 73 **C**-hypergroupoids associated with binary relations on three element sets. As it is shown they constitute an upper semilattice with respect to graph inclusion.

Key words: Hyperoperations, Hypergroupoids, Directed Graphs.

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1 Introduction

The correlation between hyperstructures and binary relations has been intensively investigated in the last 20 years by several researchers ([10], [11], [12], [14], [18], [6], [7], [8]) while of particular importance are the hypergroupoids that derive from binary relations - known as ***C**-hypergroupoids* - which were introduced by Corsini in [9] (see also [19], [20], [22], [21]).

The purpose of the present paper is to further expand the ongoing research on hypergroupoids by employing concepts from Graph Theory. While the foundations of graph theory can be traced back to L. Euler and his "Königsberg bridge problem" [1] (see also [13, 2]), its growth in recent years has been explosive covering a large number of disciplines ranging from mathematical foundations of computer science [3, 4] to physical chemistry [5] and natural language processing [17]. In Section 2, basic concepts and results on directed graphs and hypergroupoids are presented. In Section 3 we define the *graph hyperoperation* which is actually Corsini's hyperoperation applied on graphs. A particular class of directed graphs arises in this way, representing isomorphism classes of **C**-hypergroupoids, namely *Corsini's graphs*. As it is evident from their construction, Corsini's graphs constitute a useful apparatus in order to represent and arrange the hypergroupoid classes they represent which thus results in a hierarchy inside the class of ***C**-hypergroupoids*. We identify and present the 17 Corsini's graphs with 3 nodes and we find that they constitute an upper semilattice with graph inclusion as the partial order.

2 Preliminaries on Hyperstructures and Graph Theory

A partial hypergroupoid is a pair $(H, *)$, where H is a non-empty set, and $*$ is a hyperoperation i.e.

$$* : H \times H \to \mathcal{P}(H), \quad (x, y) \mapsto x * y.$$

If $A, B \in \mathcal{P}(H)$-$\{\emptyset\}$, then

$$A * B = \bigcup_{a \in A, b \in B} a * b.$$

We denote by $a * B$ (respectively, $A * b$) the hyperproduct $A * B$ in the case that the set A (respectively, the set B) is the singleton $\{a\}$ (respectively, $\{b\}$). Moreover, $(H, *)$ is called hypergroupoid if $x * y \neq \emptyset$, for all $x, y \in H$ and it is called a degenerative (respectively, total) hypergroupoid in the case that for all $x, y \in H$, $x * y = \emptyset$ (respectively, $x * y = H$).

Given a binary relation $R \subseteq H \times H$ the *Corsini's hyperoperation* (cf. [9])

$$*_R : H \times H \to \mathcal{P}(H)$$

is defined in the following way:

$$(x, y) \mapsto x *_R y = \{z \in H \mid (x, z) \in R \text{ and } (z, y) \in R\}.$$

The hyperstructure $(H, *_R)$ is called *Corsini's partial hypergroupoid associated with the binary relation* R or simply *partial **C**-hypergroupoid* and is denoted H_R (cf. [19], [20]). In the case that $x *_R y \neq \emptyset$, for all $x, y \in H$, then $(H, *_R)$ is called **C**-hypergroupoid. It can be easily seen that a partial **C**-hypergroupoid H_R is a **C**-hypergroupoid if and only if it holds $R \circ R = H \times H$, where $\circ$ is the usual relation composition. Let $R \subseteq H \times H$ be a binary relation on the set $H = \{x_1, x_2, \ldots, x_n\}$ then the $n \times n$ matrix

$$M_R = [m_{i,j}]_{n \times n}, \text{ with } m_{i,j} = 1 \text{ if } (x_i, x_j) \in R \text{ and } m_{i,j} = 0, \text{ else}$$

is called the boolean matrix of R.

Formally a *concrete directed graph* G is a pair (V_G, E_G) where:

- V_G is a finite set, the elements of which we call vertices and
- $E_G \subseteq V_G \times V_G$ is a set of ordered pairs of V_G the elements of which we call edges.

A vertex is simply drawn as a node and an edge as an arrow connecting two vertices the head and the tail of the edge. A graph $G' = (V_{G'}, E_{G'})$ is a *subgraph* of the graph $G = (V_G, E_G)$ if it holds $V'_G \subseteq V_G$ and $E'_G \subseteq E_G$. In the other direction, a *supergraph* of a graph G is a graph that has G as a subgraph. We say that the graph H is included in the graph G ($H \leq G$) if G has a subgraph that is equal or isomorphic to H. The relation $\leq$, which is called *graph inclusion*, is a graph invariant.

Given a graph $G = (V_G, E_G)$ and $v \in V_G$ the number of edges that "leave" the vertex v is called the *out degree* of v and the number of edges that "enter" the vertex is called the *in degree* of v. Moreover we denote by $degreeout(G)$ the set of all the out degrees of G's vertices and similarly for $degreein(G)$. The *order* of a graph is the number of its vertices, i.e. $|V_G|$, and the size of a graph is the number of its edges, i.e. $|E_G|$. A loop is an edge whose head and tail is the same vertex. An edge is multiple if there is another edge with the same head and the same tail. A graph is called simple if it has no multiple edges. A vertex is called isolated if there is no edge connected to it.

Two graphs G and H are said to be isomorphic if there exists an isomorphism f between the vertices of the two graphs that respects the edges, i.e. it holds $(x, y) \in E_G$ if and only if $(f(x), f(y)) \in E_H$. Since the specific sets V_G, E_G chosen to define a concrete directed graph G are actually irrelevant we don't distinguish between two isomorphic graphs. Hence the following definition of an abstract graph. The equivalence class of a concrete directed graph with respect to isomorphism is called an *abstract directed graph* or simply *graph*. A graph property is called *invariant* if it is invariant under graph isomorphisms. Examples of graph invariants are

- order, size and diameter (the longest of the shortest path lengths between pairs of vertices)
- *vertex (edge) connectivity*, the smallest number of vertices (edges) whose removal disconnects the graph
- *vertex (edge) chromatic number*, the minimum number of colors needed to color all vertices (edges) so that adjacent vertices (edges) have a different color
- *vertex (edge) covering number*, the minimal number of vertices (edges) needed to cover all edges (vertices)

In what follows we consider simple graphs without isolated vertices.

3 The Graph Hyperoperation

Given a concrete directed graph $G = (V_G, E_G)$, we introduce the graph hyperoperation $\circ_G$ defined on the nodes of the graph G:

$$\circ_G : V_G \times V_G \to \mathcal{P}(V_G), \quad (x, y) \mapsto x \circ_G y = \{z \in V_G \mid (x, z), (z, y) \in E_G\}.$$

It can be easily seen that it holds $\circ_G = *_{E_G}$, where $*_{E_G}$ is the Corsini's hyperoperation derived by the binary relation E_G. Hence $\circ_G$ structures the set V_G into a (partial) **C**-hypergroupoid called the *Corsini's (partial) hypergroupoid associated with* G. If the hyperoperation $\circ_G$ structures V_G into a **C**-hypergroupoid then the same holds for every graph G' isomorphic with G, hence this property is a graph invariant. An (abstract) graph is called *Corsini's graph* if the hyperoperation $\circ_G$ structures V_G into a **C**-hypergroupoid.

Remark 3.1 *A hyperoperation for undirected graphs has been presented in [15] by virtue of spanning trees.*

Proposition 3.1 *The Graph hyperoperation associated with $G = (V_G, E_G)$ structures V_G into a **C**-hypergroupoid if and only if, there exists a path of length 2 between every pair of nodes of G.*

Proof. The Corsini's hyperoperation associated with G is a hypergroupoid if and only if $x \circ_G y \neq \emptyset$ for all $x, y \in V_G$ if and only if for every pair of nodes $x, y \in V_G$ there exists a node $z \in V_G$ such that $(x, z), (z, y) \in E_G$, if and only if there exists a path of length 2 between x and y. $\square$

It is easy to prove that there are two Corsini's graphs with two nodes, depicted in the following figure.

We note that the second graph is actually a subgraph of the first.

Remark 3.2 *The adjacency matrix* A_G *of the graph* G *is identical with the boolean matrix* M_{E_G} *of the relation* E_G.

From this remark we obtain an alternative proof of Proposition 3.1. Indeed as it is stated in [9] a hyperoperation derived from the binary relation $R \subseteq H \times H$ structures H into a hypergroupoid if and only if the Boolean matrix M_R of the relation has the property $M_R^2 = \mathbf{1}$, where $\mathbf{1}$ is the matrix that has the unit at every entry. Since in the case of a hyperoperation associated with a graph G this matrix is equal with the adjacency matrix A_G of G it follows that $A_G^2 = \mathbf{1}$. Hence there exists a path of length 2 between any given pair of vertices.

Proposition 3.2 *The number of Corsini's graphs with order* n *is always smaller than the number of different Corsini's hypergroupoids derived from all binary relations* $R \subseteq H \times H$ *with* $card|H| = n$.

Proof. Indeed, although there exist isomorphic graphs with different adjacency matrices, different binary relations $R \subseteq H \times H$ always correspond to different Boolean matrices. Hence there exist different Corsini's hypergroupoids corresponding to two distinct Boolean matrices that represent the same graph up to isomorphism. □

As an example for the above proposition we recall that there are three different Corsini's hypergroupoids deriving from binary relations with 2 elements (cf. [9]) but only two Corsini's graphs as we noted before.

Proposition 3.3 *If* G *is a Corsini's graph then every graph* G' *with* $G \leq G'$ *is also Corsini's.*

Proof. This is obtained by applying Proposition 3.1 since if there exists a path of length 2 between two nodes in a graph G then there exists such a path in every supergraph of G. □

Now let $H = \{1, 2, 3\}$, $R \subseteq H \times H$ and M_R the 3×3 boolean matrix of R. We say that M_R has the form (p_1, p_2, p_3) if

$$|\{j \in H \mid (k, j) \in R\}| = p_k \text{ for all } k \in H$$

or equivalently if, for all $k \in H$, the sum of the elements of the kth line of M_R is p_k. We call the matrix M_R *good* if the Corsini hyperoperation $*_R$ structures $(H, *_R)$ into a **C**-hypergroupoid. As it is shown in [9] there are 30 good matrices with form (p_1, p_2, p_3) such that $p_1+p_2+p_3 = 6$. More precisely there are 12 matrices such that $p_i = 2$, where $i \in H$, and 18 matrices such that $p_i = 1$, $p_j = 2$, $p_k = 3$, where $i, j, k \in H$.

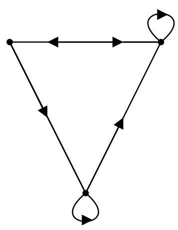
Proposition 3.4 *The 30 good boolean matrices with form (p_1, p_2, p_3), where $p_1 + p_2 + p_3 = 6$, correspond to 7 different non-isomorphic Corsini's graphs.*

Proof. First we examine the 12 good matrices with form $(2, 2, 2)$. The matrices

$$\begin{pmatrix}1&1&0\\1&0&1\\1&0&1\end{pmatrix}, \quad \begin{pmatrix}1&0&1\\1&1&0\\1&1&0\end{pmatrix}, \quad \begin{pmatrix}1&0&1\\1&0&1\\0&1&1\end{pmatrix},$$

$$\begin{pmatrix}1&1&0\\0&1&1\\1&1&0\end{pmatrix}, \quad \begin{pmatrix}0&1&1\\1&1&0\\0&1&1\end{pmatrix}, \quad \begin{pmatrix}0&1&1\\0&1&1\\1&0&1\end{pmatrix},$$

which are respectively the matrices $(1), (3), (5), (8), (10)$ and (12) of [9], all represent Corsini's graphs isomorphic with the following graph.

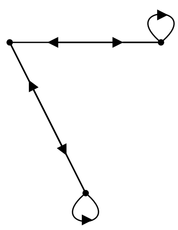


(G_1)

The matrices

$$\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}, \quad \begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}, \quad \begin{pmatrix}0&1&1\\1&1&0\\1&0&1\end{pmatrix},$$

which are respectively the matrices $(2), (6)$ and (9) of [9], correspond to graphs isomorphic with the graph below.

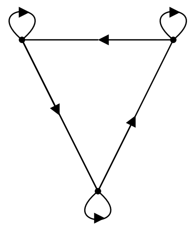


(G_2)

The matrices

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

which are respectively the matrices (4) and (7) of [9], both represent Corsini's graphs isomorphic with the next graph.

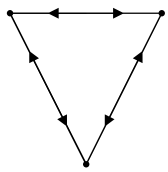


(G_3)

The matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

is the matrix (11) of [9] and represents the following Corsini's graph.



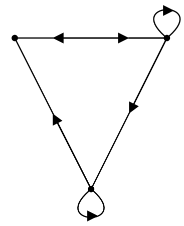
(G_4)

Now we examine the rest 18 good matrices with the desired form. The matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

which are respectively the matrices $(13), (16), (20), (23), (27)$ and (30) of [9], represent Corsini's graphs isomorphic with the following graph.



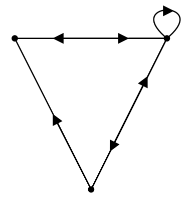
(G_5)

The matrices

$$\begin{pmatrix}0&0&1\\1&0&1\\1&1&1\end{pmatrix}, \quad \begin{pmatrix}0&1&1\\0&0&1\\1&1&1\end{pmatrix}, \quad \begin{pmatrix}0&1&0\\1&1&1\\1&1&0\end{pmatrix},$$

$$\begin{pmatrix}0&1&1\\1&1&1\\0&1&0\end{pmatrix}, \quad \begin{pmatrix}1&1&1\\1&0&0\\1&1&0\end{pmatrix}, \quad \begin{pmatrix}1&1&1\\1&0&1\\1&0&0\end{pmatrix},$$

which are respectively the matrices $(14), (18), (19), (24), (25)$ and (29) of [9], represent Corsini's graphs isomorphic with the following graph.



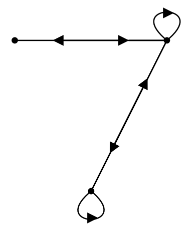
(G_6)

The matrices

$$\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}, \quad \begin{pmatrix}1&0&1\\0&0&1\\1&1&1\end{pmatrix}, \quad \begin{pmatrix}0&1&0\\1&1&1\\0&1&1\end{pmatrix},$$

$$\begin{pmatrix}1&1&0\\1&1&1\\0&1&0\end{pmatrix}, \quad \begin{pmatrix}1&1&1\\1&0&0\\1&0&1\end{pmatrix}, \quad \begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix},$$

which are respectively the matrices $(15), (17), (19), (22), (25)$ and (28) of [9], represent Corsini's graphs isomorphic with the next graph.



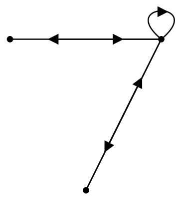
(G_7)

□

Remark 3.3 *A boolean matrix with form* (p_1, p_2, p_3) *corresponds to a graph* G *with a degreeout*$(G) = \{p_1, p_2, p_3\}$*. Hence graphs* G_1*-*G_4 *have degreeout*$(G) = \{2, 2, 2\}$ *and graphs* G_5*-*G_7 *have degreeout*$(G) = \{1, 2, 3\}$*.*

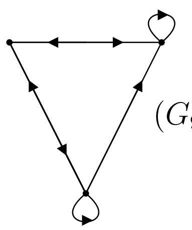
The number of Corsini's graphs with order n is equal with the number of boolean $n \times n$ matrices forming non-isomorphic hypergroupoids. Although Massouros and Tsitouras [16] have calculated this number - for $n = 3$ there are 17 such matrices - no actual representation of these 17 hypergroupoid classes has been demonstrated in [16]. Such a representation is of much greater importance than merely computing their number since it will allow us to compare and correlate these isomorphism classes. In what follows we present the remaining Corsini's graphs and moreover we discover that they constitute an upper semilattice with respect to graph inclusion. This hierarchy actually determines a hierarchy inside the set of 73 **C**-hypergroupoids (cf. [22]) deriving from binary relations on 3 elements.

Since in the previous proposition we found 7 Corsini's graphs with 3 vertices it follows that there are 10 more. From these 1 has *degreeout*$(G) = \{1, 1, 3\}$

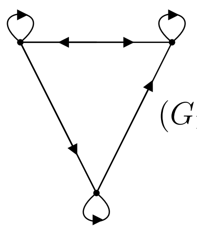


(G_8)

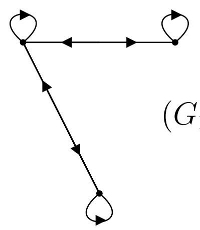
5 have *degreeout*$(G) = \{2, 2, 3\}$



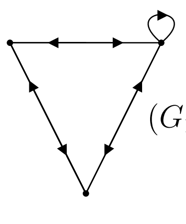
(G_9)



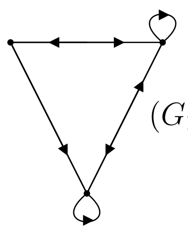
(G_{10})



(G_{11})

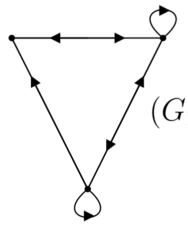


(G_{12})



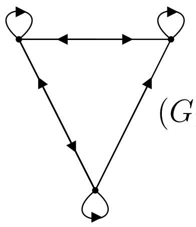
(G_{13})

1 has *degreeout*$(G) = \{1, 3, 3\}$

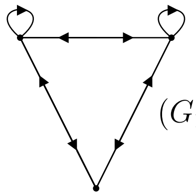


(G_{14})

2 have $degreeout(G) = \{2, 3, 3\}$

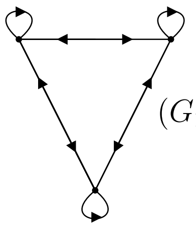


(G_{15})



(G_{16})

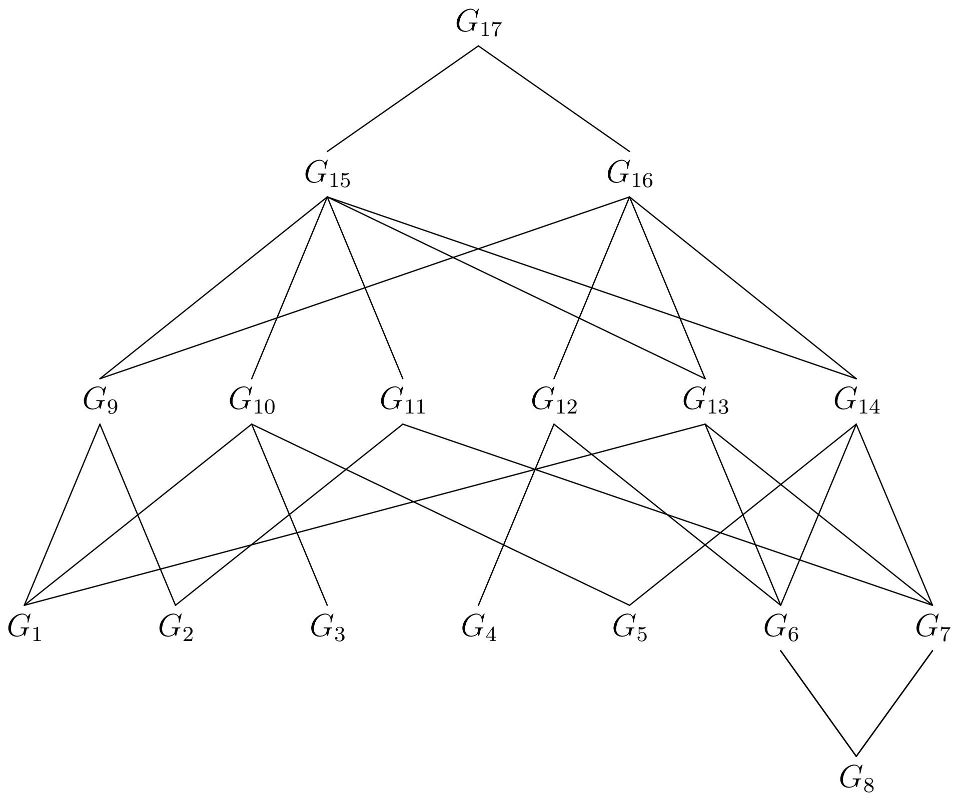
and 1 (the complete graph with 3 nodes) has $degreeout(G) = \{3, 3, 3\}$.



(G_{17})

As it is implied by Proposition 3.3, the graphs G_1-G_{17} form a partially ordered set with respect to graph inclusion. More precisely they form the following upper semilattice as it can be verified by merely inspecting the drawings of graphs G_1-G_{17}.

Directed Graphs representing isomorphism classes of **C**-Hypergroupoids



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