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On Cesàro's Means of First Order of Wavelet Packet Series

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Abstract

Wavelet packets have the capability of partitioning the higher-frequency octaves to yield better frequency localisation. Ahmad and Kumar [2000] have obtained the pointwise convergence of the wavelet packet series. But till now no work seems to have been done to obtain Cesàro summability of order 1 of wavelet packet series. In an attempt to make an advanced study in this direction, a novel theory on $(C, 1)$, Cesàro summability of order 1 of wavelet packet series is obtained in this study.

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1 Introduction

Several researchers, including S. E. Kelly and Raphael [1994a], S. E. Kelly and Raphael [1994b], Kumar and Lal [2013], Meyer [1992], Walter [1992], Walter [1995], Wickerhauser [1994], have investigated the problem of wavelet packet series convergence and demonstrated that wavelet packets are a basic yet effective wavelet and multiresolution analysis extension. Wavelet packet functions are a collection of functions that can be used to create other functions. Wavelet packet

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functions are still time-localized, but they have more versatility in describing diverse types of signals than wavelets. Wavelet packets, in particular, are better at encoding signals with periodic behaviour. Wavelet packets can partition higher-frequency octaves, resulting in more accurate frequency localization.

Working in slight different directions, Ahmad and Kumar [2000] have obtained the pointwise convergence of the wavelet packet series. But till now no work seems to have been done to obtain Cesàro summability of order 1 of wavelet packet series. It is important to note that Cesàro summability is a strong tool to obtain the convergence than that of ordinary convergence. This work establishes a new theory on Cesàro summability of order 1 of wavelet packet series in an attempt to make a more advanced study in this field.

2 Definitions and Preliminaries

Let $L^2(\mathbb{R})$ be the space of measurable and square integrable functions over set of real numbers $\mathbb{R}$. If a function $\phi \in L^2(\mathbb{R})$ generates nested sequences of closed subspaces, it is said to produce an MRA (multiresolution analysis), $Q_\imath = \overline{span}\{\phi_{\imath,\jmath} : \imath, \jmath \in \mathbb{Z}\}$, where $\phi_{\imath,\jmath}(t) = 2^{\imath/2}\phi(2^\imath t - \jmath)$ and $\mathbb{Z}$ is the set of integers , satisfying the following conditions

(i) $... \subset Q_{-2} \subset Q_{-1} \subset Q_0 \subset Q_1 \subset Q_2 \subset ...,\ i.e.\ Q_\imath \subset Q_{\imath+1},\ \imath \in \mathbb{Z};$

(ii) $\overline{(\cup_{\imath\in\mathbb{Z}} Q_\imath)} = L^2(\mathbb{R});$

(iii) $\cap_{\imath\in\mathbb{Z}} Q_\imath = \{0\};$

(iv) $\lambda(t) \in Q_\imath \Leftrightarrow \lambda(2t) \in Q_{\imath+1},\ \imath \in \mathbb{Z}$

such that $\phi_{0,\jmath}$ form a Riesz basis of $\{Q_0\}$. A function ϕ which generates a multiresolution analysis, is called a scaling function . Wavelet packets can be constructed with the help of multiresolution analysis. We know that if $\mathbb{H}$ is a Hilbert space with ONB (orthonormal basis) $\{\epsilon_\jmath\}_{\jmath\in\mathbb{Z}}$ then,

$$\lambda_{2k} = \sqrt{2}\sum_{\jmath\in\mathbb{Z}} \alpha_{2k-\jmath}\epsilon_\jmath,\ \lambda_{2k+1} = \sqrt{2}\sum_{\jmath\in\mathbb{Z}} \beta_{2k-\jmath}\epsilon_\jmath,$$

where $\{\alpha_k\}_{k\in\mathbb{Z}}$ and $\{\beta_k\}_{k\in\mathbb{Z}}$ are in $l^2(\mathbb{Z})$, are orthonormal bases of two orthogonal closed subspaces $\mathbb{H}_1$ and $\mathbb{H}_0$ respectively, such that $\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_0$.

Using the foregoing decomposition strategy, we now build the fundamental wavelet packets connected with the scaling function $\phi \in L^2(\mathbb{R})$ which is already defined in multiresolution analysis.

Let $\{\xi_k,\ k=0,1,2,...,\}$ denote a wavelet packet family that corresponds to the scaling function ϕ which is orthonormal. Consider $\xi_0=\phi$. Recursively, the wavelet packets $\xi_k,\ k=0,1,2,...,$ are defined by

$$\begin{cases} \xi_{2k}(t)=\sqrt{2}\sum_{\jmath\in\mathbb{Z}} h_\jmath \xi_k(2t-\jmath) \\ \xi_{2k+1}(t)=\sqrt{2}\sum_{\jmath\in\mathbb{Z}} g_\jmath \xi_k(2t-\jmath). \end{cases} \tag{1}$$

As a result, the $\{\xi_k\}$ family is a generalisation of the orthonormal wavelet $\xi_1=\psi$, often known as the mother wavelet. For the Hilbert space $L^2(\mathbb{R})$, the set $\{\xi_k(t-\jmath) : k=0,1,2,...,\ \jmath\in\mathbb{Z}\}$ form an ONB.

Consider the family of subspaces of $L^2(\mathbb{R})$ as

$$P_\imath^k=\text{span}\{2^\imath\xi_k(2^\imath t-\jmath):\jmath\in\mathbb{Z}\}, \imath\in\mathbb{Z}, \tag{2}$$

formed by the family of wavelet packets $\{\xi_k\}$ for each $k=0,1,2,....$

Observe that $P_\imath^0=Q_\imath$ and $P_\imath^1=W_\imath$, where $\{Q_\imath\}$ is the multiresolution analysis of $L^2(\mathbb{R})$ produced by $\xi_0=\phi$ and $\{W_\imath\}$ is the sequence of orthogonal complimentary subspaces generated by the wavelet $\xi_1=\psi$. The orthogonal decomposition $Q_{\imath+1}=Q_\imath\oplus W_\imath, \imath\in\mathbb{Z}$ can then be expressed as

$$P_{\imath+1}^0=P_\imath^0\oplus P_\imath^1. \tag{3}$$

As follows, this orthogonal decomposition can be extended from $k=0$ to any $k=1,2,3,...$ in the form of

$$P_{\imath+1}^k=P_\imath^{2k}\oplus P_\imath^{2k+1},\ \imath\in\mathbb{Z}. \tag{4}$$

Now we'll state a result that will be employed in the theorem's proof.

The decomposition trick (4) produces

$$\begin{aligned} W_\imath &= P_\imath^1=P_{\imath-1}^2\oplus P_{\imath-1}^3 \\ &= P_{\imath-2}^4\oplus P_{\imath-2}^5\oplus P_{\imath-2}^6\oplus P_{\imath-2}^7 \\ &\ \ \vdots \\ &= P_{\imath-\jmath}^{2^\jmath}\oplus P_{\imath-\jmath}^{2^\jmath+1}\oplus ...\oplus P_{\imath-\jmath}^{2^{\jmath+1}-1} \\ &\ \ \vdots \\ &= P_0^{2^\imath}\oplus P_0^{2^\imath+1}\oplus ...\oplus P_0^{2^{\imath+1}-1}, \end{aligned} \tag{5}$$

for each $\imath=1,2,...,$ where (2) declares $P_\imath^k$. Furthermore, the family $\left\{2^{\frac{\imath-\jmath}{2}}\xi_r(2^{\imath-\jmath}t-l):l\in\mathbb{Z}\right\}$ is an ONB of $P_{\imath-\jmath}^r$, where $r=2^\jmath+\mu$ for each $\mu=0,1,2,...,2^\jmath-1$, $\jmath=1,2,...\imath$;

and $\imath = 1, 2, ...$ All of the elements of this base, however, have the same basic shape:

$$\xi_{\imath,k,\jmath}(t) = 2^{\imath/2}\xi_k(2^\imath t - \jmath). \tag{6}$$

Let $\lambda \in L^2(\mathbb{R})$, then the function λ can be approximated by a wavelet packet series as follows:

$$\lambda(t) \sim \sum_{\imath\in\mathbb{Z}} \sum_{k=2^r}^{2^{r+1}-1} \sum_{\jmath\in\mathbb{Z}} C_{l,k,\jmath}\xi_{l,k,\jmath}(t), \tag{7}$$

where $l = \imath - r,\ r = 0$ if $\imath < 0$ and $r = 0, 1, 2, ..., \imath$ if $\imath \geq 0$; and the coefficients $C_{l,k,\jmath}$ defined by

$$C_{l,k,\jmath} = \langle \lambda, \xi_{l,k,\jmath} \rangle\,, \tag{8}$$

are called the wavelet packet coefficients.

Wavelet packets are a scalable time signal analysis method that combines the advantages of windowed Harmonic and wavelet processing. Wavelet bundles, which are periodic as well, offer a fascinating supplement to Fourier series.

Using the periodization techniques for period 1 on the basis functions, an MRA for $L^2(\mathbb{R})$ can be transformed into an MRA for $L^2(0,1)$. Let $\{\xi_k : k \in \mathbb{Z}\}$ denote the family of wavelet packets presented previously which is nonstationary in nature. Define general periodic wavelet packets $\xi^{per}_{k,\imath,\jmath}$ by

$$\xi^{per}_{k,\imath,\jmath} = \sum_{l\in\mathbb{Z}} 2^{\imath/2}\xi_k(2^\imath(t+l) - \jmath)$$

for $0 \leq \jmath < 2^\imath$ and $k, \imath = 1, 2, 3, \cdots$.

With ξ^{per}_k, We now define an operator $S_\nu\lambda$ as follows:

$$(S_\nu\lambda)(t) = \sum_{k=2^r}^{2^{r+1}-1} \sum_{\jmath=0}^{\nu} \left\langle \lambda, \xi^{per}_{l,k,\jmath} \right\rangle \xi^{per}_{l,k,\jmath}(t). \tag{9}$$

Let $s_k = \sum_{\nu=0}^{k} a_\nu$ be the k^{th} partial sum of an infinite series $\sum_{k=0}^{\infty} a_k$.

If $\sigma_k = \frac{1}{k+1}\sum_{\nu=0}^{k} s_\nu \to s$ as $k \to \infty$ then the series $\sum_{k=0}^{\infty} a_k$ is called summable to s by $(C,1)$ i.e. Cesàro means of order 1 (Titchmarsh Titchmarsh [1939]).

Let $D_\mu(\mu = 1, 2, 3, \cdots)$ be the collection of constant dyadic step functions on the intervals $[\jmath 2^{-\mu}, (\jmath+1)2^{-\mu})$; $0 < \jmath \leq 2^\mu$. Let $D = \cup_{\mu=1}^{\infty} D_\mu$. Let $\mathbb{B}$ be a Banach space and σ_ζ be a bounded linear functional on $\mathbb{B}$ which must be generated by any function $\zeta \in D$ as

$$\sigma_\zeta\lambda = \int_0^1 \lambda\zeta \text{ for } \lambda \in \mathbb{B}.$$

We have

$$|\sigma_\zeta \lambda| \leq \|\zeta\|_\infty \|\lambda\|_{\mathbb{B}}.$$

Now if we take $\mathbb{B} = L^q$ and define

$$\|\zeta\|_r = \|\sigma_\zeta\| = \sup_{\|\lambda\|_q \leq 1} \int_0^1 \lambda\zeta \quad \text{for any } \zeta \in D. \tag{10}$$

Then clearly

$$\left|\int_0^1 \lambda\zeta\right| \leq \|\lambda\|_q \|\zeta\|_r, \lambda \in L^q, \zeta \in D. \tag{11}$$

Let us write

$$\begin{aligned} \Pi_\imath \lambda(t) &= \sum_{\mu=0}^{2^\imath - 1} \left(\frac{1}{\mu+1} \sum_{\nu=0}^{\mu} (S_\nu \lambda)(t) \right) \delta_{[\mu 2^{-\imath}, (\mu+1)2^{-\imath})} \\ &= \sum_{\mu=0}^{2^\imath - 1} \sigma_\mu \lambda(t) \delta_{[\mu 2^{-\imath}, (\mu+1)2^{-\imath})} \end{aligned}$$

and

$$A_\imath = \sum_{\mu=0}^{2^\imath - 1} C_{l,k,\jmath}^{per} \delta_{[\mu 2^{-\imath}, (\mu+1)2^{-\imath})},$$

where (9) defines $S_\nu \lambda$ and δ_I is the characteristic function on $I \subset \mathbb{R}$.

We're going to define an operator now

$$\begin{aligned} T_\imath(t,x) &= 2^{-\imath} \sum_{\jmath=0}^{2^\imath - 1} C_{l,o,\jmath}^{per} \phi_{\imath,\jmath}^{per}(t) \overline{\phi_{\imath,\jmath}^{per}(x)} \\ &= 2^{-\imath} \sum_{k=2^r}^{2^{r+1}-1} \sum_{\mu<\imath} \sum_{\jmath=0}^{2^\imath - 1} \xi_{l,k,\jmath}^{per}(t) \overline{\xi_{l,k,\jmath}^{per}(x)}, \end{aligned}$$

where $l = \mu - r,\ r = 0$ if $\mu < 0$ and $r = 0, 1, 2, ..., \mu$ if $0 \leq \mu < \imath$.

In this paper, an estimate for the Cesàro summability of wavelet packet series has been determined in the following form:

Theorem 2.1. *Let λ be 1-periodic continuous function. Then*

$$\left\| \left(2^{-\imath} \sum_{\mu=0}^{2^\imath - 1} \left| \frac{1}{\mu+1} \sum_{\nu=0}^{\mu} S_\nu \lambda \right|^r \right)^{1/r} \right\|_\infty \leq C \|\lambda\|_\infty \tag{12}$$

if and only if

$$\|T_\imath\|_1 \leq C \|A_\imath\|_q, \tag{13}$$

where $C > 0,$ *a constant and* $1 < r < \infty.$

Furthermore,

$$\lim_{\imath\to\infty} \|\Pi_\imath \lambda(t) - \lambda(t)\|_r = 0$$

uniformly in $[0, 1]$.

Proof. By equation 12 we have

$$\begin{aligned}
\left(2^{-\imath}\sum_{\mu=0}^{2^\imath-1}\left|\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}S_\nu\lambda\right|^r\right)^{\frac{1}{r}} &= \|\Pi_\imath\lambda(t)\|_r = \sup_{\|A_\imath\|_q\leq 1} 2^{-\imath}\sum_{\mu=0}^{2^\imath-1}C^{per}_{l,k,\jmath}\sigma_\mu\lambda(t)\\
&= \sup_{\|A_\imath\|_q\leq 1} 2^{-\imath}\sum_{\mu=0}^{2^\imath-1}\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}C^{per}_{l,k,\jmath}S_\nu\lambda(t)\\
&= \sup_{\|A_\imath\|_q\leq 1}\int_0^1 2^{-\imath}\sum_{\mu=0}^{2^\imath-1}\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}C^{per}_{l,k,\jmath}K_\nu(t,x)\lambda(x)dx\\
&\leq \|\lambda\|_\infty \sup_{\|A_\imath\|_q\leq 1}\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}\|T_\nu(t,x)\|_1\\
&\leq \|\lambda\|_\infty \sup_{\|A_\imath\|_q\leq 1}\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}\left(C\|A_\imath\|_q\right), \qquad \text{by (13)}\\
&= \|\lambda\|_\infty \sup_{\|A_\imath\|_q\leq 1} C\|A_\imath\|_q \leq C\|\lambda\|_\infty,
\end{aligned}$$

where

$$K_\imath(t,x) = \sum_{\jmath=0}^{2^\imath-1}\phi^{per}_{\imath,\jmath}(t)\overline{\phi^{per}_{\imath,\jmath}(x)} = \sum_{k=2^r}^{2^{r+1}-1}\sum_{\mu<\imath}\sum_{\jmath=0}^{2^\imath-1}\xi^{per}_{l,k,\jmath}(t)\overline{\xi^{per}_{l,k,\jmath}(x)}.$$

If, on the other hand, (12) is true, we have

$$\begin{aligned}
\|T_\imath(t,x)\|_1 &= \sup_{\|\lambda\|_\infty\le 1}\int_0^1 2^{-\imath}\sum_{\mu=0}^{2^\imath-1}\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}C_{l,k,\jmath}^{per}T_\nu(0,x)\lambda(x)dx \\
&= \sup_{\|\lambda\|_\infty\le 1}\int_0^1 2^{-\imath}\sum_{\mu=0}^{2^\imath-1}\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}C_{l,k,\jmath}^{per}(2^{-\nu}\sum_{k=2^r}^{2^{r+1}-1}\sum_{\jmath=0}^{2^\nu-1}\xi_{l,k,\jmath}^{per}(0)\overline{\xi_{l,k,\jmath}^{per}(x)})\lambda(x)dx \\
&= \sup_{\|\lambda\|_\infty\le 1}2^{-\imath}\sum_{\mu=0}^{2^\imath-1}\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}C_{l,k,\jmath}^{per}(2^{-\nu}\sum_{k=2^r}^{2^{r+1}-1}\sum_{\jmath=0}^{2^\nu-1}\xi_{l,k,\jmath}^{per}(0))\int_0^1\lambda(x)\overline{\xi_{l,k,\jmath}^{per}(x)}dx \\
&= \sup_{\|\lambda\|_\infty\le 1}2^{-\imath}\sum_{\mu=0}^{2^\imath-1}\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}C_{l,k,\jmath}^{per}(2^{-\nu}\sum_{k=2^r}^{2^{r+1}-1}\sum_{\jmath=0}^{2^\nu-1}\left\langle\lambda,\xi_{l,k,\jmath}^{per}\right\rangle\xi_{l,k,\jmath}^{per}(0)) \\
&= \sup_{\|\lambda\|_\infty\le 1}2^{-\imath}\sum_{\mu=0}^{2^\imath-1}\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}C_{l,k,\jmath}^{per}(S_\nu\lambda)(0) \\
&= \sup_{\|\lambda\|_\infty\le 1}2^{-\imath}\sum_{\mu=0}^{2^\imath-1}C_{l,k,\jmath}^{per}(\sigma_\mu\lambda)(0) \\
&= \sup_{\|\lambda\|_\infty\le 1}\int_0^1\Pi_\imath\lambda(0)A_\imath \\
&\le \sup_{\|\lambda\|_\infty\le 1}\|A_\imath\|_q\|\Pi_\imath\lambda(0)\|_r \\
&\le \|A_\imath\|_q\sup_{\|\lambda\|_\infty\le 1}\left\|\left(2^{-\imath}\sum_{\mu=0}^{2^\imath-1}\left|\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}(S_\nu\lambda)(0)\right|^r\right)^{1/r}\right\|_\infty \\
&= \|A_\imath\|_q\sup_{\|\lambda\|_\infty\le 1}\left\|\left(2^{-\imath}\sum_{\mu=0}^{2^\imath-1}|(\sigma_\mu\lambda)(0)|^r\right)^{1/r}\right\|_\infty, \quad \text{by (12)} \\
&\le \|A_\imath\|_q\sup_{\|\lambda\|_\infty\le 1}C\|\lambda\|_\infty \\
&\le C\|A_\imath\|_q.
\end{aligned}$$

Now

$$\begin{aligned}
\Pi_l\lambda(t)-\lambda(t) &= \sum_{\mu=0}^{M}((\sigma_\mu\lambda)(t)-\lambda(t))\,\delta_{[\mu2^{-l},(\mu+1)2^{-l})} \\
&= \sum_{\mu=0}^{M}\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}((S_\nu\lambda)(t)-\lambda(t))\,\delta_{[\mu2^{-l},(\mu+1)2^{-l})}
\end{aligned}$$

for any $l \geq M \geq 2^{\imath}$. As a result,

$$\begin{aligned}\|\Pi_l\lambda(t)-\lambda(t)\| &\leq \sum_{\mu=0}^{M}\left\|\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}((S_\nu\lambda)(t)-\lambda(t))\right\|_\infty \left\|\delta_{[0,2^{-l})}\right\|_r \\ &\leq \sum_{\mu=0}^{M}\frac{1}{\mu+1}\sum_{\nu=0}^{\mu}\|S_\nu\lambda-\lambda\|_\infty \left\|\delta_{[0,2^{-l})}\right\|_r,\end{aligned}$$

since the limit of the characteristic function of $[0, 2^{-\imath})$ in all L^r-space $(1 < r < \infty)$ is 0 and thus the ultimate result is fallowed.

The theorem's proof is now complete.□

3 Conclusions

The estimate for the Cesàro summability of order 1 of wavelet packet series has been determined in the form of

$$\lim_{\imath\to\infty}\|\Pi_\imath\lambda(t)-\lambda(t)\|_r = 0$$

uniformly in $[0, 1]$.

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