RATIO MATHEMATICA 22 (2012) 69-84 ISSN:1592-7415

Hypergroups and Geometric Spaces

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Abstract

We explain some links between hypergrpoups and geometric spaces. We show that for any given hypergroup it is possible to define a particular geometric space and then a canonical homomorphism between the hypergroup and a group.

Key words: hypergroup, geometric space

2000 AMS subject classifications: 22N22.

1 Hypergroups and their properties

A hypergroupoid is a pair $(G, \circ)$ where G is a non-empty set and $\circ : G \times G \to P'(G)$ is a mapping of $G \times G$ into the set of non-empty subsets of G, denoted as $P'(G)$.

A semihypergroup is a hypergroupoid satisfying the following associative property:

$$\forall x, y, z \in G,\ (x \circ y) \circ z = x \circ (y \circ z), \tag{1}$$

where the left hand side of (1) is the set $(x \circ y) \circ z = \bigcup_{u \in x \circ y} u \circ z$ and the right hand side is the set $x \circ (y \circ z) = \bigcup_{v \in y \circ z} x \circ v$. The associative property means that the two set theoretical unions coincide.

We say that $(G, \circ)$ satisfies the reproducibility property (both left and right), if

$$\forall a, b \in G, \quad \exists x \in G : \ b \in a \circ x \quad \text{and} \quad \exists y \in G : \ b \in y \circ a. \tag{2}$$

If (2) is satisfied, the family $B_2 = \{a \circ b : a, b \in G\}$ is a covering of G (the index 2 under B, means that we consider the hyper product of two elements of G).

A hypergroup is an associative hypergroupoid satisfying the reproducibility property.

We remark that a hypergroup $(G, \circ)$ having a single valued product (that is, such that $\forall x, y \in G$, $|x \circ y| = 1$) is a group. This is because the following result holds:

Theorem 1.1. *An associative groupoid* $(G, \circ)$ *is a group if, and only if,*

$$\forall a, b \in G, \quad \exists x \in G: \ ax = b \quad \textit{and} \quad \exists y \in G: \ ya = b. \tag{3}$$

((3) is also called right and left quotient axiom).

Proof. If $(G, \circ)$ is a classical group, (3) obviously holds. Assume that the associative groupoid $(G, \circ)$ satisfies (3). Let us proove that it is a group. Fix $a \in G$, let u be one of the elements $z \in G$ such that: $az = a$ (see (3)). For any $c \in G$, there is $y \in G$ such that $ya = c$. Then we have $au = a \Longrightarrow y(au) = (ya)u = ya \Longrightarrow cu = (ya)u = ya = c \Longrightarrow cu = c$. Hence,

$$\forall c \in G, \ cu = c. \tag{4}$$

Similarly, by (3), we prove that there exists $v \in G$ such that:

$$\forall c \in G, \ vc = c. \tag{5}$$

By (4), for $c = v$ and by (5), for $c = u$, we get $vu = v$, $vu = u$, that is $v = u$ and then $\forall c \in G, \ uc = cu = c$. Therefore the unity of $(G, \circ)$ exists and it is unique.

For any $a \in G$, there is at least an element $a' \in G$ and $a'' \in G$ such that: $aa' = u = a''a$, (see 3). Then $a' = ua' = (a''a)a' = a''(aa') = a''u = a''$, that is in G there is an element $a'(= a'')$, such that $a' = a'a = u$. Such an element a' is obviously unique and it is the inverse, a^{-1}, of a. Then $(G, \circ)$ is a classical group and the theorem is proved. $\square$

A substructure of the hypergroup $(G, \circ)$ is a subset $H(\neq \emptyset)$ such that $\forall x, y \in H, \ x \circ y \in H$. The pair $(H, \circ)$ is a semihypergroup, if it satisfies (1). In particular it is a hypergroup if it satisfies also (2).

Let $\mathcal{F}$ be the set of all the substructures of $(G, \circ)$. Two cases may occur.

$$\bigcap_{T \in \mathcal{F}} T \notin \emptyset \qquad \text{or} \qquad \bigcap_{T \in \mathcal{F}} T = \emptyset.$$

Set $\mathcal{S} = \mathcal{F}$ in the first case and $\mathcal{S} = \mathcal{F} \cup \{\emptyset\}$ in the second. In both cases it is: $G \in \mathcal{S}$ and $\forall i \in I, \ T_i \in \mathcal{S} \Longrightarrow \bigcap_{i \in I} T_i \in \mathcal{S}$, where I is a non empty set of indices. In this way, $\mathcal{S}$ is a closure system of G. Hence, for any $X \subseteq G$, the closure $\bar{X}$ of X in G is:

$$X = \bigcap_{\substack{T \in \mathcal{S}, \\ T \subseteq X}} T$$

If $\bar{X} = \emptyset$, $\bar{X}$ is the least (from the set theoretical perspective) substructure containing X. So the following closure operator is defined as follows.

$$\bar{\ }: X \subseteq G \longrightarrow \bar{X} \in \mathcal{S}. \tag{6}$$

Note that this closure operator satisfies the following properties:

$$X \subseteq \bar{X}; X \subseteq S,\ S \in \mathcal{S} \implies \bar{X} \subseteq S; X = \bar{X} \iff X \in \mathcal{S};$$

$$\forall\, X \subseteq G,\ \bar{X} = \bar{\bar{X}}; \quad \forall\, X, Y \subseteq G,\ X \subseteq Y \implies \bar{X} = \bar{Y};$$

$$\forall\, i \in I,\ X_i \subseteq G \implies \bigcup_{i\in I} \bar{X}_i \subseteq \overline{\bigcup_{i\in I} X_i} \quad \text{and} \quad \overline{\bigcap_{i\in I} X_i} \subseteq \bigcap_{i\in I} \bar{X}_i.$$

For all $X \subseteq G$, we define:

$$X \text{ independent } \overset{\text{def}}{\iff} \forall\, x \in X,\ x \notin \overline{X \smallsetminus \{x\}}, \tag{7}$$

$$X \text{ dependent } \overset{\text{def}}{\iff} \in x \in X,\ x \in \overline{X \smallsetminus \{x\}}, \tag{8}$$

$$X \text{ generator } \overset{\text{def}}{\iff} \bar{X} = G, \tag{9}$$

$$X \text{ base } \overset{\text{def}}{\iff} X \text{ is independent and } \bar{X} = G. \tag{10}$$

The pair $(G, \circ)$ is finitely generated if, and only if, there is a finite subset X of G such that $\bar{X} = G$. We can easily prove that

Theorem 1.2. *If $(G, \circ)$ has a finite generator X then there is a finite base contained in X.*

A hypergroup $(G, \circ)$ is called *monic* if, and only if, it does not contain any substructure different from $(G, \circ)$. If $(G, \circ)$ is a hypergroup and $n \in \mathbb{N}^+ = \mathbb{N} - \{0\}$ then we let

$$\mathcal{B}_n \overset{\text{def}}{=} \{x_1 \circ x_2 \circ \ldots \circ x_n \in G : \ (x_1, x_2, \ldots, x_n) \in G^n\}, \tag{11}$$

$$\mathcal{B} \overset{\text{def}}{=} \{\mathcal{B}_n : \ n \in \mathbb{N}^+\}. \tag{12}$$

We call complete part of the hypergoup $(G, \circ)$ a subset A such that

$$B \in \mathcal{B},\, B \cap A \neq \emptyset \implies B \subseteq A. \tag{13}$$

Obviously, $\emptyset$ and G are complete parts. Furthermore, the union and the intersection of complete parts are complete parts. Moreover the complement of a complete part is a complete part. Hence, the complete parts of $(G, \circ)$ form a topology, where every open set it is also closed. We remark that $(G, \circ)$ is a group if, and only if, every subset is a complete part (getting the discrete topology). We easily prove that

> If $(G, \circ)$ is a hypergroup such that $\forall\ x, y \in G, \in B \in \mathcal{B}$ such that $x, y \in B$ then the only complete parts of $(G, \circ)$ are $\emptyset$ and G (getting the trivial topology). (14)

In Section 5, we characterize the complete parts of $(G, \circ)$ and we prove that the intersection of the subhypergroups which are also complete parts is a subhypergroup which is a complete part. We remark that the intersection of two subhypergroups may not be a subhypergroup in general. As a matter of fact, the intersection of two subhypergroups may be even the emptyset; as we will see in an example down below.

We define heart of $(G, \circ)$ and we denote it by ω, the intersection of all the subhypergroups which are also complete parts of $(G, \circ)$; that is, the least subhypergroup complete part of $(G, \circ)$. We easily prove that (see (14)):

$$\forall x, y \in G,\ \exists B \in \mathcal{B}:\ x, y \in B \implies \omega = G. \tag{15}$$

As a matter of fact, by (14), the only complete parts of $(G, \circ)$ are $\emptyset$ and G and the only subhypergroup which is a complete part is G. The following statements hold:

$$\text{If } G \in \mathcal{B} \text{ then } \omega = G.$$

If $(G, \circ)$ does not contain proper subhypergroups complete parts then $\omega = G$.

$$\text{If } (G, \circ) \text{ is monic then } \omega = G.$$

We define scalar unity of $(G, \circ)$ an element $u \in G$ such that:

$$\forall a \in G,\ a \circ u = u \circ a = \{a\}.$$

Obviously, if a scalar unity in $(G, \circ)$ exists then it is unique. Moreover, $\{u\}$ is a subhypergroup of $(G, \circ)$ and generally it is not a complete part. We get: $\{u\}$ is a complete part $\iff$ $[\forall a, b \in G : u \in a \circ b \Longrightarrow a \circ b = u]$ $\iff$ $\{u\} = \omega$. In Section 5 we prove that $u \in \omega$ in any case. Finally, we remark that the hypergroups having a scalar unity are rather difficult to discover.

2 Examples of hypergroups

Example 2.1. *Let G be any non-empty set. Define the multivalued product $\forall x, y \in G$, $x \circ y = G$. Then $(G, \circ)$ is a hypergroup which we call trivial. This hypergroup is monic, its complete parts are only $\emptyset$ and G and its heart coincides with G.*

Example 2.2. *Let G be any non-empty set. Define the product $\forall x, y \in G$, $x \circ y = \{x, y\}$. The $(G, \circ)$ is called discrete hypergroup. Every non-empty set is a subhypergroup and then there are subhypergroups whose intersection is the empty set. By (14) the only complete parts of $(G, \circ)$ are $\emptyset$ and G and then $\omega = G$.*

Example 2.3. *Let $(G, \circ)$ be a group and N a normal sugroup of $(G, \circ)$. Set $\forall\ x, y \in G$, $x \circ y = xyN$. Then $(G, \circ)$ is a hypergroup. The condition (2) is obvious (it suffices to set $x = a^{-1}b$ and $y = ba^{-1}$); as regard the associativity, we have:*

$$\forall\ a, b \in G, \quad (a \circ b) \circ c = abNc = abcN = a \circ (b \circ c).$$

Moreover, $\mathcal{B}_n$ coincides with the cosets of N in G and then $\omega = N$ and $(N, \circ)$ is the trivial hypergroup (that is $x \circ y = N$).

Example 2.4. *Let $(G_i, \circ_i)_{i \in I}$ be a family of hypergroups (in particular of groups) such that $|I| \geq 2$, $G_i \cap G_j = \emptyset$, $i \neq j$. Set $G = \bigcup_{i \in I} G_i$ and*

$$\forall x, y \in G, \quad x \circ y \overset{def}{=} \begin{cases} x \circ_i y & \text{if } x, y \in (G_i, \circ_i), \\ G & \text{if } x \in G_i \text{ and } x \in G_j. \end{cases}$$

It is easy to prove that the pair $(G, \circ)$ is a hypergroup. For any $i \in I$ we get that $(G_i, \circ_i)$ is a subhypergroup of $(G, \circ)$ and such hypergroups are two by two disjoint. The only complete parts of $(G, \circ)$ are $\emptyset$ and G and then $\omega = G$.

Example 2.5. *Let $(G, \circ)$ be any group with $|G| \geq 3$. Set:*

$$\forall x, y \in G, \quad x \circ y = G \smallsetminus \{xy\}. \tag{16}$$

Let us prove that $(G, \circ)$ is a hypergroup. We have:

$$\forall x, y, z \in G, \quad (x \circ y) \circ z = \bigcup_{t \in G \smallsetminus \{xy\}} G \smallsetminus \{tz\}. \tag{17}$$

Since $|G| \geq 3$, for any $c \in G$ there is an element t' in G such that $t' \in G \smallsetminus \{xy, cz^{-1}\}$. We have $c \in G \smallsetminus \{t'z\}$, with $t' \in G \smallsetminus \{xy\}$, whence, by

(17), $c \in (x \circ y) \circ z$. Therefore $(x \circ y) \circ z = G$. Similarly, we prove that $x \circ (y \circ z) = G$. So, the associative property of $(G, \circ)$ follows.

Now, let us prove the reproducibility property (2). For any $a, b \in G$ there is $x \in G \setminus \{a^{-1}b\}$ and $y \in G \setminus \{ba^{-1}\}$. So, $b \in G \setminus \{ax\} = a \circ x$, $b \in G \setminus \{ya\} = y \circ a$. This implies (2).

Let us prove that $(G, \circ)$ is monic. Let S be a subhypergroup of $(G, \circ)$ and $a \in S$. We have $a \circ a = G \setminus \{a^2\} \subseteq S$, then $|S| \geq 2$ (because $|G| \geq 3$). Now, let $b \in S$, with $b \neq a$. It is $a \circ b = G \setminus \{ab\} \subseteq S$, hence, if $S \neq G$ then $S = G \setminus \{ab\} = G \setminus \{a^2\}$ which is impossible because $a \neq b$. Therefore, $S = G$ and $(G, \circ)$ is monic.

We easily prove that the only complete parts of $(G, \circ)$ are $\emptyset$ and G, and so, $\omega = G$.

Example 2.6. *Let $G = \mathbb{R}^n$. For any $x, y \in \mathbb{R}^n$, with $x \neq y$, set*

$$\begin{aligned} &x \circ x = \{x\}, \\ &x \circ y = \text{the closed interval whose extremal points are } x \text{ and } y. \end{aligned}$$

We prove that $(\mathbb{R}^n, \circ)$ is a hypergroup. In fact, the reproducibility property is obvious and the associativity holds because $(x \circ y) \circ z$ coincides with the triangle (eventually, degenerate), $T(x, y, z)$, with vertices x, y and z. The same happens for $x \circ (y \circ z)$. And so, $(x \circ y) \circ z = T(x, y, z) = x \circ (y \circ z)$.

In $(\mathbb{R}^n, \circ)$ every convex set is a substructure and viceversa. The open convexes are subhypergroups. Therefore, disjoint subhypergroups exist. Moreover, by (14) the only complete parts are $\emptyset$ and $G = \mathbb{R}^n$, and then $\omega = G$.

Example 2.7. *Let $G = P(d, K)$ be the d-dimensional projective space over the field K. For all $x, y \in P(d, K)$, with $x \neq y$, set*

$$\begin{aligned} &x \circ x = \{x\}, \\ &x \circ y = \text{the line through } x \text{ and } y. \end{aligned}$$

It is easy to prove that $(G, \circ)$ is a hypergroup. In fact, the reproducibility property is obvious and the associativity holds because $(x \circ y) \circ z$ coincides with the subspace spanned by x, y and z. The substructures of $(G, \circ)$ are hypergroups and coincide with the subspaces of $G = P(d, K)$. From (14), the only complete parts of $(G, \circ)$ are $\emptyset$ and $G = P(d, K)$, and then $\omega = G$.

Example 2.8. *Let $G = V_K$ be a vector space over the field K. Set*

$$\forall\ x, y \in G,\ x \circ y = \{a(x + y) :\ a \in K\}.$$

We can prove that $(G, \circ)$ is a hypergroup. The substructure of $(G, \circ)$ are the subspaces of V_K and, hence, they are hypergroups. Moreover (14) is satisfied and the only complete parts of $(G, \circ)$ are $\emptyset$ and $G = V_K$. Hence, $\omega = G$.

Example 2.9. *Let* $(G_1, \circ_1)$ *and* $(G_2, \circ_2)$ *be two hypergroups. Set* $G = G_1 \times G_2$*. Furthermore, if* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$ *are two elements of* G *then* $x \circ y \overset{def}{=} (x_1 \circ_1 y_1, x_2 \circ_2 y_2$*. It is easy to prove that such a* $(G, \circ)$ *is a hypergroup which we call the cartesian product of* $(G_1, \circ_1)$ *and* $(G_2, \circ_2)$*. Similarly, we define the cartesian product of a family of hypergroups* $\{(G_i, \circ_i) : i \in I\}$*. In particular, we set*

$$G = \prod_{i \in I} G_i,$$
$$x \circ y = \{x_i \circ_i y_i : i \in I\}, \text{ where } x = \{x_i : i \in I\} \text{ and } y = \{y_i : i \in I\}.$$

We remark that if each $(G_i, \circ_i)$ *has a scalar unity* u_i *then the cartesian product has a scalar unity* $u = \{u_i : i \in I\}$*.*

Example 2.10. *In the examples from 2.1 to 2.8, the hypergroups do not have a scalar unity. Now we give an example of a hypergroup with a scalar unity of order 2. And this is also the simplest hypergroup which is not a group. Consider the set* $G = \{u, a\}$ *and set*

$$u \circ u \overset{def}{=} \{u\}, \; u \circ a \overset{def}{=} a \circ u \overset{def}{=} \{a\}, \; a \circ a \overset{def}{=} \{u, a\}.$$

It can be easily checked that $(G, \circ)$ *is a hypergroup which as* u *as scalar unity. From (14), the only complete parts of* $(G, \circ)$ *are* $\emptyset$ *and* G *because* $a \circ a = G$*. Hence,* $\omega = G$*. The only subhypergroup of* $(G, \circ)$ *is* $\{u\}$ *which is not a complete part. The cartesian product of copies of such hypergroup is a wide class of hypergroups with scalar unity.*

3 Homomorphisms of hypergroups

Let $(G, \circ)$ and $(G', \circ')$ be two hypergroupoids. We call homomorphism of $(G, \circ)$ and $(G', \circ')$ a mapping $f : G \longrightarrow G'$ such that

$$\forall x, y \in G, f(x \circ y) \subseteq f(x) \circ' f(y). \tag{18}$$

We remark that in general $f(G)$ is a substructure of $(G', \circ')$. Let us prove:

Theorem 3.1. *If* $G \longrightarrow G'$ *is a homomorphism of* $(G, \circ)$ *to* $(G', \circ')$*, for any substructure* H' *of* $(G', \circ')$ *such that* $f^{-1}(H') \neq \emptyset$ *then* $H' = f^{-1}(H')$ *is a substructure of* $(G, \circ)$*. Moreover, if* $(G, \circ)$ *satisfies the reproducibility property, we have:*

$$\forall a', b' \in f(G), \exists x' \in f(G) : b' \in a' \circ' x' \Longrightarrow \exists y' \in G' : b' \in y' \circ' a'. \tag{19}$$

Finally, if $(G, \circ)$ *and* $(G', \circ')$ *are semihypergroups and A' is a complete part of* $(G', \circ')$ *then* $A = f^{-1}(A')$ *is a complete part of* $(G', \circ')$ *and* $A = f^{-1}(A')$ *is a complete part of* $(G, \circ)$*.*

Proof. Let $x, y \in H$. It is $x' = f(x), y' = f(y) \in H'$ and then $a' \circ' y' \in H'$. By (18) we get:

$$f(x \circ y) \subseteq f(x) \circ' f(y) = x' \circ' y' \subseteq H', \quad \text{where} \quad x \circ y \subseteq f^{-1}(H') = H.$$

Therefore H is a substructure of $(G, \circ)$.

If $(G, \circ)$ satisfies the reproducibility property we get:

$$\forall a', b' \in f(G) \implies \exists a, b \in G : a = f(a), b' = f(b) \implies$$

$$\exists x \in G : b \in a \circ x, \exists y \in G : b \in y \circ a \implies$$

$$\exists x' = f(x) \in f(G) : b' \in f(a \circ x) \subseteq f(a) \circ' f(x) = a' \circ' x' \quad \text{and}$$

$$\exists y' = f(y) \in f(G) : b' = f(y \circ a) \subseteq f(y) \circ' f(a) = y' \circ' a' \implies$$

$$\exists x' \in f(G), b' \in a' \circ' x', \exists y' \in f(G) : b' \in y' \circ' a'.$$

If $(G, \circ)$ and $(G', \circ')$ are semihypergroups, that is the associativity property holds, for any complete part A' of $(G', \circ')$, setting $A = f^{-1}(A')$, we get, since $f(A) = A'$ and setting $x'_i = f(x_i)$, $x_i \in G_i$:

$$(x_1 \circ x_2 \circ \ldots \circ x_n) \cap A \neq \emptyset \implies f(x_1 \circ x_2 \circ \ldots \circ x_n) \cap f(A) \neq \emptyset \implies$$

$$\emptyset \neq f(x_1 \circ x_2 \circ \ldots \circ x_n) \cap f(A) \subseteq (x'_1 \circ' x'_2 \circ' \ldots \circ' x'_n) \subseteq A' \implies$$

$$(x_1 \circ x_2 \circ \ldots \circ x_n) \subseteq f^{-1}(x'_1 \circ' x'_2 \circ' \ldots \circ' x'_n) \subseteq A \implies (x_1 \circ x_2 \circ \ldots \circ x_n) \subseteq A,$$

therefore, A is a complete part of $(G, \circ)$ and the theorem is proved. $\square$

A homomorphism is called strong if in (18) the equality holds. Obviously, if $(G', \circ')$ is a groupoid (that is, if the operation $\circ'$ is single valued) then every homomorphism between any two hypergroupoids $(G, \circ)$ and $(G', \circ')$ is strong. Now, let us prove

Theorem 3.2. *Let f be a strong homomorphism between the hypergroupoid $(G, \circ)$ and $(G', \circ')$. Then*

$$\mathrm{Im}(f) = f(G) \text{ is a hypergroup;} \tag{20}$$

$$\text{If } H \text{ is a substructure of } (G, \circ) \text{ then } f(H) \text{ is a substructure of } f(G); \tag{21}$$

$$\text{If } H \text{ is a subhypergroup of } (G, \circ) \text{ then } f(H) \text{ is a subhypergroup of } f(G); \tag{22}$$

$$\text{If } H' \text{ is a substructure of } f(G) \text{ then } f^{-1}(H') \text{ is a substructure of } (G, \circ); \tag{23}$$

$$\text{If } A' \text{ is a complete part of } (G', \circ') \text{ then } f^{-1}(A') \text{ is a complete part of } (G, \circ). \tag{24}$$

Proof. Since f is a strong homomorphism, if H is a substructure of $(G, \circ)$ then $\forall x', y' \in f(H) \Longrightarrow \exists x, y \in H$: $x' = f(x)$, $y' = f(y) \Longrightarrow x \circ y \subseteq H$, $x' \circ' y' = f(x) \circ' f(y) = f(x \circ y) \subseteq f(H) \Longrightarrow x' \circ' y' \subseteq f(H) \subseteq f(G)$, that is

$$\text{If } H \text{ is a substructure of } (G, \circ) \text{ then } f(H) \text{ is a substructure of } (G', \circ'). \tag{25}$$

From (25) with $H = G$, we have that $f(G)$ is a substructure of $(G', \circ')$, that is $f(G)$ is closed with respect to the product. Furthermore, from Theorem 3.1 (second part), we have that $(f(G), \circ')$ satisfies the reproducibility property.

Let us prove that $(f(G), \circ')$ is associative. In fact, $\forall x', y', z' \in f(G)$ $\exists x, y, z \in G$: $x' = f(x)$, $y' = f(y)$, $z' = f(z) \Longrightarrow (x' \circ' y') \circ' z' = f((x \circ y) \circ z) = f(x \circ (y \circ z)) = x' \circ' (y' \circ' z')$; and so, $(f(G), \circ')$ is a hypergroup. Hence, the property (20) holds. The property (21) follows from (25) and (20). The property (22) follows from (21) and Theorem 3.1 (second part). The property (23) is trivial. The property (24) follows from Theorem 3.1 (third part). Hence, the Theorem is proved. □

Theorem 3.3. *Let $(G, \circ)$ be a hypergroup, ω be the heart of $(G, \circ)$, $(G', \circ)$ be a group and u be the unity of $(G', \circ)$. If $f : G \longrightarrow G'$ is a homomorphism (necessarily, strong) of $(G, \circ)$ in $(G', \circ)$ then*

1. *If H' is a subgroup of $(G', \circ)$ then $f^{-1}(H')$ is a subhypergroup of $(G, \circ)$;*
2. *$\forall A' \subseteq G'$, $f^{-1}(A')$ is a complete part of $(G, \circ)$;*
3. *$\forall x' \in f(G)$, $f^{-1}(x')$ is a complete part ($\neq \emptyset$) of $(G, \circ)$;*
4. *$\forall B \in \mathcal{B}$ (see (12), $|f(B)| = 1$;*
5. *$\omega \subseteq f^{-1}(u)$.*

Proof. If H' is a subgroup of G', from (20), we have that $H' \cap f(G)$ is a subgroup of $f(G)$ and hence, from (23), $f^{-1}(H' \cap f(G)) = f^{-1}(H')$ is a substructure of $(G, \circ)$. Let us prove that (2) holds for $f^{-1}(H')$. We have that:

$$\forall a, b \in f^{-1}(H'),\ \exists x, y \in G\colon\ b \in a \circ x,\ b \in y \circ a \ \Longrightarrow$$

$$f(a), f(b) \in H',\ f(b) = f(a) \cdot f(x) = f(y) \cdot f(a) \ \Longrightarrow$$

$$f(x) = (f(a))^{-1} \cdot f(b) \in H',\ f(y) = f(b) \cdot (f(a))^{-1} \in H' \ \Longrightarrow$$

$$\exists x, y \in f^{-1}(H')\colon\ b \in a \circ x,\ b \in y \circ a.$$

This implies that (2) holds for $f^{-1}(H')$. Hence (3.3,1) holds.

Since every subset A' of the group G' is a complete part in G', from Theorem 3.1 (third part), we have that $(3.3, 2)$ and $(3.3, 3)$ hold.

For all $B = (x_1 \circ x_2 \circ \ldots \circ x_n) \in \mathcal{B}$, let $x \in B$. If $x' \stackrel{\text{def}}{=} f(x) \in f(B)$ then $x \in f^{-1}(x')$ and hence $b \cap f^{-1}(x') \neq \emptyset$. But, $f^{-1}(x')$ is a complete part of $(G, \circ)$, hence $B \subseteq f^{-1}(x')$, and so $f(B) = \{x'\}$ and $|f(B)| = 1$. This implies $(3.3, 4)$.

The set $\{u\}$ is a subgroup of G', so from $(3.3, 1)$, $f^{-1}(u)$ is a subhypergroup of $(G, \circ)$ which is also a complete part because of $(3.3, 3)$. This implies $\omega \subseteq f^{-1}(u)$ because ω is the intersection of all the subhypergroups complete parts of $(G, \circ)$. This implies $(3.3, 5)$ and hence the theorem. $\square$

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The composition of two homomorphisms is a homomorphism, likewise the composition of two strong homomorphisms is a strong homomorphism. Furthermore the identity is a strong homomorphism. Hence, the hypergroups form a category with respect to the homomorphisms and a category with respect to the strong homomorphisms which is a subcategory of the first category.

4 Geometric spaces

A geometric space is a pair $(P, \mathcal{B})$, where P is a non-empty set, whose elements we call points and $\mathcal{B}$ is a family of parts of P, whose elements we call blocks. The set P is called the support of the geometric space and $\mathcal{B}$ is called the geometric structure. Let $(P, \mathcal{B})$ and $(P', \mathcal{B}')$ be two geometric spaces. We call isomorphism between them a bijection $f\colon P \to P'$, such that

$$\forall B \in \mathcal{B},\ f(B) \in \mathcal{B}', \quad \forall B' \in \mathcal{B}',\ f^{-1}(B') \in \mathcal{B}.$$

The composition of two isomorphisms is an isomorphism and the identity is an isomorphism. It follows that within the geometric spaces the relation of isomorphism is an equivalence relation. So, we study the equivalence classes of such spaces. The isomorphisms of $(P, \mathcal{B})$ onto itself are called automorphisms. The automorphisms of $(P, \mathcal{B})$ form a group under the composition which is called $Aut(P, \mathcal{B})$. This gives rise to a geometry of the geometric space $(P, \mathcal{B})$. More precisely, two subsets F and F' of $(P, \mathcal{B})$ are called "equal", if there is an automorphism of $(P, \mathcal{B})$ which changes F onto F'. Such an equality relation is an equivalence relation. The geometry of $(P, \mathcal{B})$ is the study of the properties of the subsets of $(P, \mathcal{B})$ which are invariant under the group $Aut(P, \mathcal{B})$. Two geometric spaces $(P, \mathcal{B})$ and $(P', \mathcal{B}')$ are called equivalent, if, and only if,

$$Aut(P, \mathcal{B}) = Aut(P', \mathcal{B}'); \tag{26}$$

that is, if the geometry determined by $Aut(P, \mathcal{B})$ coincides with that arising from $Aut(P', \mathcal{B}')$.

We remark that, given a geometric space $(P, \mathcal{B})$, if $\mathcal{B}'$ consists of the complements of the elements of $\mathcal{B}$ in P, then $(P, \mathcal{B})$ and $(P, \mathcal{B}')$ are ,because of (26), two distinct geometric spaces, which have the same geometry.

Example 4.1. *Let $(S, \mathcal{A})$ be a topological space whose open sets are such that $\emptyset \in \mathcal{A}$, $S \in \mathcal{A}$, $A_{i i \in I}, A_i \in \mathcal{A} \implies \bigcup_{i \in I} A_i \in \mathcal{A}$, and $A_1, A_2 \in \mathcal{A} \implies A_1 \cap A_2 \in \mathcal{A}$.*

The complements of the open sets are the closed sets of the topology. We denote by $\mathcal{C}$ the family of the closed sets. The two structures $(P, \mathcal{A})$ and $(P, \mathcal{C})$ are two distinct geometric spaces, admitting the same geometry.

Example 4.2. *Let $\mathcal{R}$ be the family of the lines of the real plane $\mathbb{R}^2$. The pair $(\mathbb{R}^2, \mathcal{R})$ is a geometric space which is called real affine plane. The group $Aut(\mathbb{R}^2, \mathcal{R})$ is called the group of the affinities of $(\mathbb{R}^2, \mathcal{R})$ and we prove that every affinity is an invertible linear transformation; that is:*

$$x' = ax + by + c, \quad y' = a_1 x + b_1 y + c_1, \ \textit{with } ab_1 - a_1 b \neq 0.$$

Example 4.3. *Let $\mathcal{C}$ be the family of the circles of $\mathbb{R}^2$. The pair $(\mathbb{R}^2, \mathcal{C})$ is a geometric space. An automorphism of such a space is a bijection which changes circles to circles and therefore changes also lines to lines (because it changes three collinear points to three collinear points, and conversely). It follows that an automorphism of $(\mathbb{R}^2, \mathcal{C})$ is an affinity which changes circles to circles and therefore it is a similitude. It follows that $Aut(\mathbb{R}^2, \mathcal{C})$ is the group of the similitudes of the real plane and the geometry of $(\mathbb{R}^2, \mathcal{C})$ is the similitude geometry.*

Example 4.4. *Let $\mathcal{C}_1$ be the family of the circles with radius 1 in the real plane $\mathbb{R}^2$. The automorphisms of the geometric space $(\mathbb{R}^2, \mathcal{C}_1)$ is the set of the bijections changing the circles of radius 1 to themselves, We can prove that such bijections change circles to circles and, then, lines to lines; hence, $Aut(\mathbb{R}^2, \mathcal{C}_1)$ is the group of the movements in the plane and the geometry of $(\mathbb{R}^2, \mathcal{C}_1)$ is the euclidean geometry.*

Example 4.5. *Let K be a field and $P(r, K)$ be the r-dimensional projective space over the field K. Its points are the $(r+1)$-tuples not all zero in K, defined up to a non-zero multiplicative factor. The lines consist of those* *points* $x = (x_0, x_1, \ldots, x_r) \in P(r, K)$ *each of which is the linear combination of two distinct fixed points* $y = (y_0, y_1, \ldots, y_r)$, $z = (z_0, z_1, \ldots, z_r) \in P(r, K)$*:*

$$x = \lambda y + \mu z \iff x_i = \lambda y_i + \mu z_i, \qquad i = 0, 1, \ldots, r$$

Let $\mathcal{L}$ *be the family of the lines of* $P(r, K)$*. The geometric space* $(P(r, K), \mathcal{L})$ *is the* r*-dimensional projective space over the field* K*. We prove that the automorphisms of such a space are of the form*

$$x_i' = \sum_{j=0}^{r} a_{ij} \vartheta x_j, \qquad i = 0, 1, \ldots, r,$$

where $\det(a_{i,j}) \neq 0$ *and* ϑ *is a collineation (that is, an automorphism* $\vartheta : K \to K$*. The geometry of* $(P(r, K), \mathcal{L})$ *is called projective geometry.*

Let $(P, \mathcal{B})$ be any geometric space. An n-tuple of blocks $(B_1, B_2, \ldots, B_n)$ is called polygonal of $(P, \mathcal{B})$ if, and only if, $B_i \cap B_{i+1} \neq \emptyset$, $i = 1, 2, \ldots, n-1$.

Assume that $\mathcal{B}$ is a covering of P. In P the following relation γ is defined (connectedness by polygonals):

$$\forall\, x, y \in P, \qquad x \,\gamma\, y \iff \begin{array}{l}\text{there is a polygonal } (B_1, B_2, \ldots, B_n) \\ \text{such that } x \in B_1 \text{ and } y \in B_n.\end{array} \tag{27}$$

The relation γ is an equivalence. In fact, γ is reflexive because $\mathcal{B}$ is a covering of P, and it is also obviously symmetric and transitive. For any $x \in P$, the equivalence class $\gamma(x)$ of x is the union of the polygonals through x and it is called connected component of x. If $\gamma(x) = P$ then the space $(P, \mathcal{B})$ is connected by polygonals. Note that, in any geometric space $(P, \mathcal{B})$, for all given $x \in P$:

$$\forall\, B \in \mathcal{B}, \qquad B \cap \gamma(x) \neq \emptyset \implies B \subseteq \gamma(x).$$

This implies that if $\mathcal{B}_{\gamma(x)}$ indicates the family of blocks contained in $\gamma(x)$ then the pair $(\gamma(x), \mathcal{B}_{\gamma(x)}$ is a connected geometric space. This space is called connected component of $(P, \mathcal{B})$. Note that $(P, \mathcal{B})$ is the disjoint union of its connected components.

Example 4.6. *Let* Ω *be a non-empty open set in* $\mathbb{R}^n$ *and let* $\mathcal{B}$ *be the family of the closed segments contained in* Ω*. Consider the geometric space* $(\Omega, \mathcal{B})$*. Actually, every classical polygonal is a polygonal according to the above definition. Conversely, every above polygonal contains a classical polygonal. Hence, the connected components of* Ω *in the classical sense coincide with the connected components of* Ω *previously defined.*

Example 4.7. *Let* (V, E) *be a graph. This is a geometric space* $(P, \mathcal{B})$ *in which* $P = V$ *and* $\mathcal{B} = V \cup E$*. Note that every block has cardinality which is less than or equal to* 2*. The classical notion of polygonal (or, path) of a graph coincide with the above definition of polygonal. Also, the connected components of* (V, E) *according to the classical definition coincide with the connected components of* $(P = V, \mathcal{B} = V \cup E)$*.*

Example 4.8. *A semilinear space* $(P, \mathcal{L})$*, where the elements of* $\mathcal{L}$ *are called lines, is a geometric space such that every line has at least two points and through two distinct points there is at most a line. For instance, every graph without loops is a semilinear space. Another example is given by any ruled algebraic variety of* $P(r, K)$*. The notion of polygonal actually coincide with the classical one (a* n*-tuple of lines* $(l_1, l_2, \ldots, l_n)$ *such that* $l_i \cap l_{i+1} \neq \emptyset$*) and then the notion of connected component of a semilinear space just given, coincide with the classical one.*

5 Hypergroups and geometric spaces

Let $(G, \circ)$ be a hypergroupoid and let $\mathcal{B}$ be the family of parts of G consisting of all the hyper products of more than one element in G (see (11) and (12). Then the geometric space $(P = G, \mathcal{B})$ remains defined. If in $(G, \circ)$ the reproducibility property (2) holds then $\mathcal{B}$ is a covering of G. If $x, y \in G$, x is in relation τ with y if, and only if, there is an element $B \in \mathcal{B}$ containing x and y. Equivalently, $x \,\tau\, y \overset{\text{def}}{\iff}$ there exists an hyper product containing both x and y. The relation τ is reflexive because $\mathcal{B}$ is a covering of G and obviously symmetric. However, τ may not be transitive in general.

We recall that a relation ρ defined in G can be regarded as a subset (called graph of ρ) of the cartesian product $G \times G$. This implies that it is possible to define a partial ordering relation in the set of all the relations defined in G given by the usual set-theoretical inclusion. Moreover, if $\{\rho_i : \; i \in I\}$ is a family of equivalence relations in G then $\rho \overset{\text{def}}{=} \bigcap_{i \in I} \rho_i$ is an equivalence relation defined in G. This is because if $a, b, c \in G$, then

$$a \,\rho\, b \quad \iff \quad a \,\rho_i\, b, \; \forall \; i \in I;$$

and so, ρ is reflexive, symmetric, and transitive. This implies that the equivalences in G form a closure system (also because the relation whose graph is $G \times G$ is an equivalence relation). Now, let τ^* be the intersection of all the equivalences containing τ. Note that τ^* is the smallest equivalence relation (and hence, transitive) which contains the possibly non-transitive relation τ. For this reason, τ^* is called transitive closure of τ. As a matter of fact, if G

is an hypergroup then $\tau^* = \tau$ as the following theorem states; and hence, τ is transitive.

Theorem 5.1. *Let G be a hypergroup,$N^* = N - 0$, z be an element of the heart ω of G and*

$$P(z) \overset{def}{=} \{A \in P'(G) : \ z \in A, \ \exists m \in N^*, \ \exists (a_1, a_2, ..., a_m) \in G^m : \ A = \prod_{j=1}^{m} a_j\}.$$

Set $M = \bigcup_{a \in P(z)} A$, then $\omega = M$, $\tau^* = \tau$.

Proof. First we prove that M is a complete part of G. Let $(z_1, z_2, ..., z_n) \in G^n$ such that $\prod_{i \in I} z_\cap M \neq \emptyset$. If $a \in \prod_{i \in I} z_i \cap M$ then a product $A \in P(z)$ such that $a \in A$ exists. From the reproducibility property of G, a pair (w, b) of elements of G exists such that $z_n \in wz$ and $z \in ab$. Hence:

$$z \in ab \subset \prod_{i=1}^{n-1} z_i b = \prod_{i=1}^{n-1} z_i z_n b \subset \prod_{i=1}^{n-1} z_i wzb \subset \prod_{i=1}^{n-1} z_i wAb;$$

and so $\prod_{i=1}^{n-1} z_i wAb \subset M$. It then follows that:

$$\prod_{i=1}^{n} z_i = \prod_{i=1}^{n-1} z_i z_n \subset \prod_{i=1}^{n-1} z_i wz \subset \prod i = 1^{n-1} z_i wab \subset \prod_{i=1}^{n-1} z_i wAb \subset M,$$

and therefore M is a complete part of G.

For any non-empty subset A of G,we denote by $C(A)$ the complete closure of A; that is, the intersection of all the complete parts of G containing A. Now, since $z \in \omega \cap M$ and M is a complete part of G, we have $\omega = C(z) \subset C(M) = M$. Moreover, for any product A of $P(z)$, it is $z \in \omega \cap A$ and, as ω is a complete part of G, then $A \subset \omega$ and consequently the inclusion $M \subset \omega$ holds; hence, $M = \omega$.

Now, let $x \, \tau^* \, y$. We have $x \in C(y) = y\omega = yM$ and so a product $A \in P(z)$ exists such that $x = yA$. By the reproducibility property of ω, there exists $b \in \omega$ such that $z \in bz$. Since $b \in \omega = M$, there exist a product $B \in P(z)$ such that $b \in B$. By the reproducibility property of G, there is $v \in G$ such that $y = vz$. Now, from $z \in A$, $\{b, z\} \subset B$, $y \in vz$, $z \in bz$ and $x \in yA$, we get:

$$x \in yA \subset vzA \subset vbzA \subset vbBA, \text{ and } y \in vz \subset vbz \subset vbbz \subset vbBA.$$

Consequently, it follows $\{x, y\} \subset vbBA$ and then $x \, \tau \, y$. So $\tau = \tau^*$. We note that a somewhat similar argument can be found in [1]. $\square$

Note that, the above theorem does not hold in the more general case of G being a semihypergroup, as the following example shows.

Example 5.1. *Let G be a set such that $|G| \geq 4$ and let a, b, c, d be four distinct elements of G. Let $a \circ a = \{b, c\}$ and, for any pair $(x, y) \in G \times G$, with $(x, y) \neq (a, a)$, let $x \circ y = \{b, d\}$. It can be easily shown that the set G equipped with the hyperproduct just defined is a semihypergroup such that, for all $n \in \mathbb{N}$, $n \geq 3$ and for all $(x_1, x_2, \ldots, x_n) \in G^n$, $\prod_{i=1}^{n} x_i = \{b, d\}$. Furthermore, $c\,\tau\,b$ because $a \circ a = \{b, c\}$, and $b\,\tau\,d$ because, say, $a \circ b = \{b, d\}$. This implies $c\,\tau^*\,b$. However, $c\,\tau\,d$ does not hold.*

If ρ is an equivalence relation in G containing τ (that is, such that $x\,\tau\,y \Longrightarrow x\,\rho\,y$) then ρ contains the connectedness by polygonals relation γ defined in (27); that is, $\rho \supseteq \tau \Longrightarrow \rho \supset \gamma$. In fact, since $\rho \supseteq \tau$ (that is, $x\,\tau\,y \Longrightarrow x\,\rho\,y$), if $x\,\gamma\,y \Longrightarrow \exists\ (B_1, B_2, \ldots, B_m)$: $B_i \in \mathcal{B}$, $B_i \cap B_{i-1} \neq \emptyset$, $x \in B_1$ and $y \in B_m \Longrightarrow x\,\tau\,x_1$, $x_1 \in B_2 \cap B_1$, $x_1\,\tau\,x_2$, $x_2 \in B_3 \cap B_2$, $\ldots$, $x_{m-1}\,\tau\,y$, $x_1 \in B_{m-1} \cap B_m \neq \emptyset \Longrightarrow x\,\rho\,x_1$, $x_1\,\rho\,x_2$, $\ldots$, $x_{m-1}\,\rho\,y \Longrightarrow x\,\rho\,y$ because ρ is an equivalence and, hence, it is transitive. It then follows that

$$\gamma = \tau^*. \tag{28}$$

Note that $\forall\, x, y \in G$, $x \circ y \in \mathcal{B}$. Hence, all the elements of $x \circ y$ belong to the same connected component of $(G, \mathcal{B})$ which we denote by $\gamma(x \circ y)$. If $x\,\tau\,x'$ and $y\,\tau\,y'$ then $x, x' \in B_1 \in \mathcal{B}$ and $y, y' \in B_2 \in \mathcal{B}$, then $x \circ y \subseteq B_1 \circ B_2$ and $x' \circ y' \subseteq B_1 \circ B_2$ and then $\gamma(x \circ y) = \gamma(x' \circ y')$. This proves,

$$x\,\tau\,x' \text{ and } y\,\tau\,y' \implies \gamma(x \circ y) = \gamma(x' \circ y'). \tag{29}$$

In the set $G/\gamma = G/\tau^*$ (see (28)) of the connected components of $(G, \mathcal{B})$ it is then possible to define the following single valued product.

$$\forall\ \gamma(x), \gamma(y) \in G/\gamma, \text{ where } x, y \in G, \qquad \gamma(x) \cdot \gamma(y) \overset{\text{def}}{=} \gamma(x \circ y). \tag{30}$$

The pair $(G/\gamma, \cdot)$ just defined is a groupoid. Furthermore, the mapping

$$\begin{array}{rcl} \varphi : (G, \circ) & \to & G/\gamma \\ x & \to & \gamma(x), \end{array} \tag{31}$$

is a surjective homomorphism because of (30). This implies that $(G/\gamma, \cdot)$ is a group, because the associativity and (2) hold in G. Let u be the unity of $(G/\gamma, \cdot)$. It can be proved that every complete part A of $(G, \circ)$ is the counterimage through φ defined in (31) of a subset of $(G/\gamma, \cdot)$, and so A is the union of connected components. Moreover, the image under φ of every subhypergroup complete part of $(G, \circ)$ is a subgroup of $(G/\gamma, \cdot)$, and viceversa.

This implies that the intersection of a set of some subhypergroups complete parts of $(G, \circ)$ is a subhypergroup complete part of $(G, \circ)$. Now, since the heart ω of $(G, \circ)$ is the intersection of all the subhypergroups complete parts of $(G, \circ)$ it follows that $\omega = \varphi^{-1}(u)$.

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