

Codes on s -periodic errors

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Abstract

In this paper, we study linear codes capable of detecting and correcting s -periodic errors. Lower and upper bounds on the number of parity check digits required for codes detecting such errors are obtained. Another bound on codes correcting such errors is also obtained. An example of a code detecting such errors is provided.

Key words: parity check matrix, syndromes, standard array, periodic error.

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1 Introduction

Investigations in coding theory have been made in several directions but one of the most important directions has been the detection and correction of errors. It began with Hamming codes[9] for single errors, Golay codes[10, 11] for double and triple random errors and thereafter BCH codes[12, 13, 14] were studied for multiple error correction. There is a long history towards the growth of the subject and many of the codes developed have found applications in numerous areas of practical interest. One of the areas of practical importance in which a parallel growth of the subject took place is that of burst error detecting and correcting codes. It has also been observed that in many communication channels, burst errors occur more frequently than random errors. A burst of length b may be defined as follows:

Definition 1.1. *A burst of length b is a vector whose only non-zero components are among some b consecutive components, the first and the last of which is non zero.*

Extending the work of Hamming[9], Abramson[1] developed codes which dealt with the correction of single and double adjacent errors. The work due to Fire[8] depicted a more general concept of burst errors.

Stone[19], and Bridwell and Wolf[4] considered multiple bursts. It was noted by Chien and Tang[5] that in several channels errors do occur in the form of a burst but not near the end of the vector. Channels due to Alexander, Gryb and Nast[2] fall in this category. In this light, Chien and Tang proposed a modification in the definition of a burst, now known as CT burst, according to which a CT burst of length b is defined as follows:

Definition 1.2. *A CT burst of length b is a vector whose only non zero components are confined to some b consecutive positions, the first of which is non-zero.*

Recently a new kind of error, known as repeated burst, has been observed by Berardi, Dass and Verma[3]. For further work on this type of error, one may refer to [6, 7, 18] and references therein.

It is very clear that the nature of error differ from channel to channel depending upon the behaviour of channels or the kind of errors which occur during the process of transmission. There is a need to deal with many types of error patterns and accordingly codes are to be constructed to combat such error patterns. Though the errors are generally classified mainly in two categories - random errors and burst errors, it has also been observed that the occurrence of errors may follow a pattern, different from random and burst. In certain communication channel like Astrophotography[21], small mechanical error occurs periodically in the accuracy of the tracking in a motorized mount that results small movements of the target that can spoil long-exposure images, even if the mount is perfectly polar-aligned and appears to be tracking perfectly in short tests. It repeats at a regular interval - the interval being the amount of time it takes the mount's drive gear to complete one revolution. This type of error pattern is termed as *periodic or alternate pattern*. It was in this spirit that the codes correcting s -alternate errors were developed by Tyagi and Das [20]. An s -periodic error is defined as follows:

Definition 1.3. *An s -periodic error is an n -tuple whose non zero components are located at a gap of s positions where $s = 1, 2, 3, \dots, (n - 1)$ and the number of its starting positions is among the first $s + 1$ components.*

For $s=1$, the 1-periodic error vectors are the ones where error may occur in 1st, 3rd, 5th...positions or 2nd, 4th, 6th,... positions. For example, in a vector of length 8, 1-periodic error vectors are of the type 10101000, 00101000, 0010101, 10101010, 10001010, 01010101, 01000101, 00000101, 00000001 etc.

For $s=2$, the 2-periodic error vectors are those where error may occur in 1st, 4th, 7th,... positions or 2nd, 5th, 8th,...positions or 3rd, 6th, 9th,... positions. The 2-periodic error vectors may look like 10010010, 10000010, 00010010, 01000001, 01000000, 00001001, etc in a vector of length 8.

For $s=3$, in a code length 8, the 3-periodic errors are 10001000, 01000100, 00100010, 00010001, 10000000, 01000000 etc.

In what follows a linear code will be considered as a subspace of the space of all n -tuples over $GF(q)$. The distance between two vectors shall be considered in the Hamming sense.

The rest of the paper is organized as follows:

In section 2, we study the linear codes that detect any s -periodic error. We obtain lower and upper bounds on the parity check digits for codes detecting such errors. It is followed by an example of such a code. In section 3, we give a bound (based on Reiger's bound[16]) on codes correcting such errors .

2 Codes detecting s -periodic errors

We consider the linear codes that are capable of detecting any s -periodic error. Clearly, the patterns to be detected should not be code words. In other words we consider codes that have no s -periodic error as a code word. Firstly, we obtain a lower bound over the number of parity-check digits required for such a code. The proof is based on the technique used in theorem 4.13, Peterson and Weldon [15].

Theorem 2.1. *Any (n, k) linear code over $GF(q)$ that detects any s -periodic error must have at least $\left\lceil \frac{n}{s + 1} \right\rceil$ parity-check digits.*

Proof. The result will be proved on the basis that no detectable error vector can be a code word.

Let V be an (n, k) linear code over $GF(q)$. Consider a set X of all those vectors such that the non-zero components are located at the first position and thereafter a gap of s positions.

We claim that no two vectors of the set X can belong to the same coset of the standard array; else a code word shall be expressible as a sum or difference of two error vectors.

Assume on the contrary that there is a pair, say x_1, x_2 in X belonging to the same coset of the standard array. Their difference viz. $x_1 - x_2$ must be a code vector. But $x_1 - x_2$ is a vector all of whose non-zero components are located at the 1st position or after a gap of s position and so is a member of X , i.e., $x_1 - x_2$ is an s -periodic error, which is a contradiction. Thus all the vectors in X must belong to distinct cosets of the standard array. The number of such vectors over $GF(q)$ is clearly q^p , where $p = \left\lceil \frac{n}{s+1} \right\rceil$.

The theorem follows since there must be at least this number of cosets. \square

In the following, an upper bound on the number of check digits required for the construction of a linear code discussed in theorem 2.1 is provided. This bound assures the existence of such a linear code and has been obtained by constructing a matrix under certain constraints. The proof is based on the well known technique used in Varshomov-Gilbert Sacks bound (refer Sacks[17], also theorem 4.7 Peterson and Weldon [15]).

Theorem 2.2. *There exists an (n, k) linear code over $GF(q)$ that has no s -periodic error as a code word provided that*

$$n - k \geq \left\lceil \frac{n}{s+1} \right\rceil.$$

Proof. The existence of such a code will be shown by constructing an appropriate $(n - k) \times n$ parity-check matrix H . The requisite parity-check matrix H shall be constructed as follows.

Select any non-zero $(n - k)$ -tuples as the first $j - 1$ columns h_1, h_2, \dots, h_{j-1} ; the j^{th} ($j > s + 1$) column h_j is added provided that

$$h_j \neq \sum_{i=1}^p u_i h_{j-i(s+1)}$$

where $u_i \in GF(q)$ and $p = \left\lceil \frac{j}{s+1} \right\rceil - 1$.

This condition ensures that no s -periodic error will be a code word. The number of ways in which the coefficients u_i can be selected is clearly q^p .

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At worst, all these linear combinations might yield a distinct sum.

Therefore a column h_j can be added to H provided that

$$q^{n-k} > q^p.$$

or,

$$n - k \geq \left\lceil \frac{j}{s+1} \right\rceil.$$

For a code of length n , replacing j by n gives the result. \square

Remark: The above two theorems can be combined as follows:

For detecting s -periodic errors in a linear code of length n , $\left\lceil \frac{n}{s+1} \right\rceil$ parity check symbols are necessary and sufficient.

Example 2.1. Consider a (7, 4) binary code with parity check matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

This matrix has been constructed by the synthesis procedure, outlined in the proof of Theorem 2.2, by taking $s = 2$ and $n = 7$. It can be seen from Table 1 that the syndromes of the different 2-periodic errors are nonzero, showing thereby that the code that is the null space of this matrix can detect all 2-periodic errors.

Table 1

Error patterns	Syndromes
100000	100
000100	010
000001	001
100100	110
100001	101
0001001	011
1001001	111
0100000	110
0000100	011
0100100	101
0010000	101
0000010	111
0010010	010

3 Codes correcting s -periodic errors

The following theorem gives a bound on the number of parity-check digits for a linear code that corrects s -periodic errors. The proof is based on the technique used to establish Reiger's bound[16] (also refer Theorem 4.15, Peterson and Weldon [15]) for correction of s -periodic errors.

Theorem 3.1. *An (n, k) linear code over $GF(q)$ that corrects all t -periodic errors, $t = 2s + 1$ must have at least $\left\lceil \frac{n}{s+1} \right\rceil$ parity-check digits.*

Proof. Any vector that has the form of an s -periodic error can be expressible as a sum or difference of two vectors, each of which is an t -periodic error. These component vectors must belong to different cosets of the standard array because both such errors are correctable errors. Accordingly, such a vector viz. s -periodic error can not be a code vector. In view of Theorem 2.1, such a code must have at least $\left\lceil \frac{n}{s+1} \right\rceil$ parity-check digits.

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