Ratio Mathematica Volume 41, 2021, pp. 206-222

On the solutions of Pellian equation $\boldsymbol{U^2 - DV^2 = k^2N}$

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Abstract

In this paper we consider a class of Pell's equation $U^2 - DV^2 = k^2N$, where D, N are positive integers, D is non-square and k is any integer. When (u, v) satisfy this equation, we define $\frac{u+v\sqrt{D}}{k}$ to be its *solution*. We first introduce the *class of solutions* of this equation and call $\frac{u+v\sqrt{D}}{k}$ to be the *fundamental solution* of the class, if v is the smallest positive value which occurs in the solutions of that class.
We first derive the necessary and sufficient condition for any two solutions of proposed equation to belong to the same class and the bounds for the values of u, v occurring in the fundamental solution. We also derive an explicit formula which gives all the solutions of this equation. We further present some interesting recurrence relations connecting the values of u, v. Finally, we obtain the results for total number of positive solutions of proposed equation not exceeding any given positive real number Z.

Keywords: Pell's equation, Solutions of Pell's equation, Recurrence relations.
2010 AMS subject classification[‡]**:** 11D09, 11D45.

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[‡]Received on July 15, 2021. Accepted on December 25, 2021. Published on December 31, 2021. doi: 10.23755/rm.v41i0.628. ISSN: 1592-7415. eISSN: 2282-8214.

1. Introduction

In number theory, Pell's equation falls in the category of Diophantine equations and it is considered to be one of the oldest Diophantine equations. Specifically, the term Pell's equation is used to refer any Diophantine equation of the form

$$u^2 - Dv^2 = N, \tag{1.1}$$

where D and N are fixed non-zero integers and we wish to find integers u and v that satisfy the equation. Throughout, we consider the integer D to be positive and non-square. This condition is helpful because only it leaves open the possibility of infinitely many solutions in positive integers u, v.

Pellian equation

$$x^2 - Dy^2 = 1, \tag{1.2}$$

where D is fixed positive non-square integer, attracted attention of early mathematicians. It is known that (1.2) always has infinitude of solutions. We refer to Stolt [11, 12] who nicely studied the equation $x^2 - Dy^2 = \pm 4N$. In this paper we use the notions proposed by Stolt. Many authors considered some specific Pell equations and discussed about their solutions. For completeness we recall that there are many papers which considered different types of Pell's equation. For extensive resources on Pell's equation, one can refer [1] – [14].

In the present paper we consider the Pellian equation

$$U^2 - DV^2 = k^2N, \tag{1.3}$$

where D, N are positive integers, D is non-square and k is any integer. Throughout we assume that (1.3) is solvable. If (u, v) satisfy this equation, we define $\frac{u+v\sqrt{D}}{k}$ to be the *solution* of (1.3). If $\frac{u+v\sqrt{D}}{k}$ is any solution of (1.3), $x + y\sqrt{D}$ is any solution of (1.2) and $x_1 + y_1\sqrt{D}$ is its smallest positive solution, then it is easy to observe that $\frac{u+v\sqrt{D}}{k} \times \left(x + y\sqrt{D}\right)$ is also a solution of (1.3). This solution is said to be *associated* with the solution $\frac{u+v\sqrt{D}}{k}$. Here we note that if two positive solutions $\frac{u_\alpha+v_\alpha\sqrt{D}}{k}$ and $\frac{u_\beta+v_\beta\sqrt{D}}{k}$ are associated then there exists an integer t such that

$$\frac{u_\beta+v_\beta\sqrt{D}}{k} = \frac{u_\alpha+v_\alpha\sqrt{D}}{k} \times \left(x_1 + y_1\sqrt{D}\right)^t; \; t = 0, \pm 1, \pm 2, \dots . \tag{1.4}$$

The set of all solutions associated with other forms a *class of solutions* of (1.3). Among all the solutions $\frac{u+v\sqrt{D}}{k}$ in any given class K we now choose a solution $\frac{u_1+v_1\sqrt{D}}{k}$ in the following way: Let v_1 be the least non-negative value of v which occurs in K. Then in this case, u_1 is also uniquely determined. The solution $\frac{u_1+v_1\sqrt{D}}{k}$ defined in this way is said to be *fundamental solution* of the class. It can be observed that $\frac{u_1+v_1\sqrt{D}}{k}$ is a fundamental solution of class K if $\frac{u_1+v_1\sqrt{D}}{k} \times \left(x_1 - y_1\sqrt{D}\right)$ is not a positive solution of (1.3). Thus if $\frac{u_\alpha+v_\alpha\sqrt{D}}{k}$ is any fixed fundamental solution then all the positive solutions given by (1.4) are associated with each other. Moreover, we observe that if (1.3) is solvable, then it has only finite number of classes of solutions.

2. The equation $U^2 - DV^2 = k^2N$

It is easy to observe that if $\frac{u+v\sqrt{D}}{k}$ and $\frac{u'+v'\sqrt{D}}{k}$ are any two integer solutions of (1.3) such that $u \equiv u'$ and $v \equiv v' (mod\ N)$, then $\left(\frac{uu'-Dvv'}{kN}, \frac{uv'-u'v}{kN}\right)$ is the positive solution of the equation $R^2 - DS^2 = k^2$. We first present the necessary and sufficient condition for any two solutions of (1.3) to be associated with each other.

Theorem 2.1. If $\frac{u+v\sqrt{D}}{k}$ and $\frac{u'+v'\sqrt{D}}{k}$ are any two solutions of (1.3) then the necessary and sufficient condition for these two solutions to be associated with each other is that $\frac{uv'-u'v}{kN}$ is an integer.

Proof. Since $\frac{u+v\sqrt{D}}{k}$ and $\frac{u'+v'\sqrt{D}}{k}$ are solutions of (1.3), we get $u^2 - Dv^2 = k^2N$ and $u'^2 - Dv'^2 = k^2N$. Multiplying these two equations we get $\left(\frac{uu'-Dvv'}{kN}\right)^2 - D\left(\frac{uv'-u'v}{kN}\right)^2 = k^2$. We first show that the solutions $\frac{u+v\sqrt{D}}{k}$ and $\frac{u'+v'\sqrt{D}}{k}$ of the Pellian equation (1.3) are associated with each other if and only if both $\frac{uu'-Dvv'}{kN}$ and $\frac{uv'-u'v}{kN}$ are integers.

If two numbers $\frac{uu'-Dvv'}{kN}$ and $\frac{vu'-uv'}{kN}$ are integers, then $\left(\frac{uu'-Dvv'}{kN}\right)^2 - D\left(\frac{uv'-u'v}{kN}\right)^2 = k^2$. Thus $\frac{\frac{uu'-Dvv'}{kN} + \frac{uv'-u'v}{kN}\sqrt{D}}{k}$ is a solution of Pellian equation $R^2 - DS^2 = k^2$. Now

$$\frac{\frac{u+v\sqrt{D}}{k}}{\frac{u'+v'\sqrt{D}}{k}} = \frac{u+v\sqrt{D}}{u'+v'\sqrt{D}} = \left(u + v\sqrt{D}\right) \times \frac{u'-v'\sqrt{D}}{\pm k^2N} = \frac{uu'-Dvv'}{k^2N} - \frac{uv'-u'v}{k^2N}\sqrt{D}.$$

Clearly this is a solution of (1.2). Thus, both the solutions $\frac{uu'-Dvv'}{kN}$ and $\frac{vu'-uv'}{kN}$ of (1.3) are associated.

Conversely let two solutions $\frac{u+v\sqrt{D}}{k}$ and $\frac{u'+v'\sqrt{D}}{k}$ of (1.3) be associated with each other. Then clearly, they should lie in the same solution class of (1.3). Then we have $\frac{\frac{u+v\sqrt{D}}{k}}{\frac{u'+v'\sqrt{D}}{k}} = \frac{uu'-Dvv'}{k^2N} - \frac{uv'-u'v}{k^2N}\sqrt{D}$, which should be some solution of (1.2). In this case we have $u^2 - Dv^2 = k^2N$ and ${u'}^2 - D{v'}^2 = k^2N$. This gives $\left(\frac{uu'-Dvv'}{kN}\right)^2 - D\left(\frac{vu'-uv'}{kN}\right)^2 = k^2$. Thus,

$$k \times \left(\frac{uu'-Dvv'}{k^2N}, \frac{uv'-u'v}{k^2N}\right) = \left(\frac{uu'-Dvv'}{kN}, \frac{uv'-u'v}{kN}\right)$$

is an integer solution of $R^2 - DS^2 = k^2$. Hence $\frac{uu'-Dvv'}{kN}$ and $\frac{uv'-u'v}{kN}$ are integers.

Thus, it is now sufficient to show that $\frac{uu'-Dvv'}{kN}$ is an integer when $\frac{uv'-u'v}{kN}$ is an integer; and $\frac{uv'-u'v}{kN}$ is not an integer when $\frac{uu'-Dvv'}{kN}$ is not an integer. Since $\frac{u+v\sqrt{D}}{k}$ and $\frac{u'+v'\sqrt{D}}{k}$ are solutions of (1.3), we have $u^2 - Dv^2 = \pm k^2N$, ${u'}^2 - D{v'}^2 = \pm k^2N$. Multiplying these two equations we get

$$(uu' - vv'D)^2 - D(uv' - vu')^2 = k^2(kN)^2. \tag{2.1}$$

It is obvious from (2.1) that $uu' - Dvv'$ is divisible by kN when $uv' - u'v$ is divisible by kN. That is, $\frac{uu'-Dvv'}{kN}$ is an integer when $\frac{uv'-u'v}{kN}$ is an integer. Conversely, if $\frac{uu'-Dvv'}{kN}$ is not an integer, then there exists an integer d such that $d \mid kN$ but $d \nmid (uu' - Dvv')$. Now from (2.1) it is seen that if $d \mid (uv' - u'v)$, then $d \mid (uu' - vv'D)$ too, which is contrary to the assumption. Thus $d \nmid (uv' - u'v)$, that is $\frac{uv'-u'v}{kN}$ is not an integer, as required. This proves the required result.

We now derive the bounds for the values of u, v occurring in the fundamental solutions.

Theorem 2.2. If $\frac{u+v\sqrt{D}}{k}$ is the smallest fundamental solution of the class K of the Pellian equation (1.3) and if $x_1 + y_1\sqrt{D}$ is the fundamental solution of (1.2), then $0 < |u| \leq k\sqrt{\frac{(x_1+1)N}{2}}$ and $0 \leq v \leq ky_1\sqrt{\frac{N}{2(x_1+1)}}$.

Proof. If both the inequalities to be proved are true for a class K, then they are also true for the conjugate class $\bar{K}$. Thus, we can assume that u is positive. Note that $\frac{ux_1-Dvy_1}{k} = \frac{ux_1}{k} - \left(y_1\sqrt{D}\right)\left(\frac{v\sqrt{D}}{k}\right) = \frac{ux_1}{k} - \sqrt{x_1^2 - 1}\sqrt{\frac{u^2-k^2N}{k^2}}$

$$= \frac{ux_1}{k^2} - \sqrt{(x_1^2 - 1)\left(\frac{u^2}{k^2} - N\right)} > 0.$$

Now consider the solution $\left(\frac{u+v\sqrt{D}}{k}\right)\left(x_1 - y_1\sqrt{D}\right) = \frac{ux_1 - Dvy_1 + (x_1v - y_1u)\sqrt{D}}{k}$ of (1.3) which belongs to the same class as $\frac{u+v\sqrt{D}}{k}$. But $\frac{u+v\sqrt{D}}{k}$ is the fundamental solution of the class and by above $\frac{ux_1 - Dvy_1}{k}$ is positive. Thus, we must have $\frac{ux_1 - Dvy_1}{k} \geq \frac{u}{k}$, as $\frac{u}{k}$ occurs in fundamental solution of (1.3). From this inequality it now follows that $ux_1 - Dvy_1 \geq u$. This gives $u(x_1 - 1) \geq Dvy_1$, which gives $u^2(x_1 - 1)^2 \geq D^2v^2{y_1}^2 = (u^2 - k^2N)(x_1^2 - 1)$. This can be written as $u^2 \frac{x_1 - 1}{x_1 + 1} \geq u^2 - k^2N$, which eventually gives $u^2 \leq \frac{(x_1+1)k^2N}{2}$. This proves the first of required inequality.

Again by (1.3) we have $Dv^2 = u^2 - k^2N$. Using above inequality, we get $Dv^2 \leq \frac{(x_1+1)k^2N}{2} - k^2N = \frac{(x_1-1)k^2N}{2}$. This gives

$$v \leq k\sqrt{\frac{(x_1-1)N}{2D}} = k\sqrt{\frac{(x_1^2-1)N}{2D(x_1+1)}}.$$

Since $x_1 + y_1\sqrt{D}$ is the solution of (1.2), we thus get $0 \leq v \leq ky_1\sqrt{\frac{N}{2(x_1+1)}}$, as required.

We now present an illustration to demonstrate the results of this theorem.

Illustration. Consider the Pellian equation $U^2 - 7V^2 = 18$. Then clearly, $D = 7, k = 3$ and $N = 2$. If we consider the equation $x^2 - 7y^2 = 1$, then it is easy to see that $8 + 3\sqrt{7}$ is its fundamental solution. This gives $x_1 = 8, y_1 = 3$. Now if $\frac{u+v\sqrt{7}}{3}$ is the smallest fundamental solution of equation $U^2 - 7V^2 = 18$, then by above theorem we should have

$$0 < |u| \leq 3 \times \sqrt{\frac{9\times2}{2}} \text{ and } 0 \leq v \leq 3 \times 3\sqrt{\frac{2}{2\times9}}.$$

This gives $0 < |u| \leq 9$ and $0 \leq v \leq 3$. These are indeed true as $\frac{5+\sqrt{7}}{3}$ is the smallest fundamental solution of the equation $U^2 - 7V^2 = 18$.

We now prove a very important result which produces all the fundamental solutions of (1.3).

Theorem 2.3. If $u_i + v_i\sqrt{D}$ runs through all the fundamental solutions of (1.1) and $\frac{r_j+s_j\sqrt{D}}{k}$ runs through all the fundamental solutions of $R^2 - DS^2 = k^2$, then all the fundamental solutions of $U^2 - DV^2 = k^2N$ are covered by

$$\frac{A_{ij}+B_{ij}\sqrt{D}}{k} = \left(\frac{r_j+s_j\sqrt{D}}{k}\right)\left(u_i \pm v_i\sqrt{D}\right). \qquad (2.2)$$

Proof. If we multiply the surd conjugate of (2.2) with (2.2) then we get $\frac{A_{ij}^2 - DB_{ij}^2}{k^2} = \left(\frac{r_j^2 - Ds_j^2}{k^2}\right)(u_i^2 - Dv_i^2)$. Since $u_i^2 - Dv_i^2 = N$ and $r_j^2 - Ds_j^2 = k^2$, we get $A_{ij}^2 - DB_{ij}^2 = k^2N$. This shows that $\frac{A_{ij} + B_{ij}\sqrt{D}}{k}$ defined by (2.2) are the solutions of (1.3).

We next show that the solutions $\frac{A_{ij} + B_{ij}\sqrt{D}}{k}$ defined by (2.2) covers all the fundamental solutions of (1.3). On the contrary assume that there exists some positive solution, say $\frac{X + Y\sqrt{D}}{k}$ of (1.3) which is not covered by (2.2). Then this solution will lie between any two successive solutions of (1.3) of some fixed class generated by $\frac{r_j + s_j\sqrt{D}}{k}$. This means for some fixed j, we have

$$\left(\frac{r_j + s_j\sqrt{D}}{k}\right)(u_i \pm v_i\sqrt{D}) \le \frac{X + Y\sqrt{D}}{k} < \left(\frac{r_j + s_j\sqrt{D}}{k}\right)(u_{i+1} \pm v_{i+1}\sqrt{D}).$$

Then $u_i \pm v_i\sqrt{D} \le \left(\frac{X + Y\sqrt{D}}{k}\right)\left(\frac{k}{r_j + s_j\sqrt{D}}\right) < u_{i+1} \pm v_{i+1}\sqrt{D}$. This gives

$$u_i \pm v_i\sqrt{D} \le \left(\frac{X + Y\sqrt{D}}{k}\right)\left(\frac{r_j - s_j\sqrt{D}}{k}\right) < u_{i+1} \pm v_{i+1}\sqrt{D}.$$

We denote

$$\epsilon + \delta\sqrt{D} = \left(\frac{X + Y\sqrt{D}}{k}\right)\left(\frac{r_j - s_j\sqrt{D}}{k}\right). \tag{2.3}$$

Then

$$u_i \pm v_i\sqrt{D} \le \epsilon + \delta\sqrt{D} < u_{i+1} \pm v_{i+1}\sqrt{D}. \tag{2.4}$$

To prove the required result, it is sufficient to prove that

(i) $\epsilon + \delta\sqrt{D}$ is a solution of (1.1), and

(ii) $\epsilon > 0$ and $\delta > 0$ or $\delta < 0$ depending on the sign of $u_i \pm v_i\sqrt{D}$.

This will produce one positive solution of (1.1) between two consecutive fundamental solutions of (1.1) for any fixed class j, which is a contradiction.

To prove (i), we multiply surd conjugate of (2.3) with (2.3). This gives $\epsilon^2 - D\delta^2 = \left(\frac{X^2 - DY^2}{k^2}\right)\left(\frac{r_j^2 - Ds_j^2}{k^2}\right)$. Since $X^2 - DY^2 = k^2N$ and $r_j^2 - Ds_j^2 = k^2$, we get $\epsilon^2 - D\delta^2 = N$, which proves the first part.

Next, we show that ϵ and δ defined by (2.3) are positive. Now $r_j^2 - Ds_j^2 = k^2$ implies $\left(\frac{r_j + s_j\sqrt{D}}{k}\right)\left(\frac{r_j - s_j\sqrt{D}}{k}\right) = 1$ and $\frac{X + Y\sqrt{D}}{k} > 1$. Since $0 < \frac{r_j + s_j\sqrt{D}}{k} < \infty$, clearly $0 < \frac{r_j - s_j\sqrt{D}}{k} < \infty$. Since $\epsilon + \delta\sqrt{D} = \left(\frac{X + Y\sqrt{D}}{k}\right)\left(\frac{r_j - s_j\sqrt{D}}{k}\right)$, we get $0 < \epsilon + \delta\sqrt{D} < \infty$. Also, since $\epsilon^2 - D\delta^2 = N$, we get $0 < \frac{N}{\epsilon - \delta\sqrt{D}} < \infty$, that is $0 < \frac{\epsilon - \delta\sqrt{D}}{N} < \infty$. This gives $0 < \epsilon - \delta\sqrt{D} < \infty$. Adding this result with $0 < \epsilon + \delta\sqrt{D} < \infty$ proves that $\epsilon > 0$.

We further observe by (2.4) that

$$1 < u_i \pm v_i\sqrt{D} \leq \epsilon + \delta\sqrt{D} < u_{i+1} \pm v_{i+1}\sqrt{D} < \infty.$$

By considering the '+' sign, we get $0 < \frac{1}{\epsilon+\delta\sqrt{D}} \leq \frac{1}{u_i+v_i\sqrt{D}} < 1$. Then $0 < \frac{\epsilon-\delta\sqrt{D}}{N} \leq \frac{u_i-v_i\sqrt{D}}{N}$, which gives $0 < \epsilon - \delta\sqrt{D} \leq u_i - v_i\sqrt{D}$. Subsequently we get

$$2\delta\sqrt{D} = \left(\epsilon + \delta\sqrt{D}\right) - \left(\epsilon - \delta\sqrt{D}\right) \geq \left(u_i + v_i\sqrt{D}\right) - \left(u_i - v_i\sqrt{D}\right) = 2v_i\sqrt{D}.$$

This gives $\delta \geq v_i > 0$.

Also, if we select '−' sign in $u_i \pm v_i\sqrt{D}$, then we have

$$\epsilon + \delta\sqrt{D} < u_{i+1} - v_{i+1}\sqrt{D} < \infty,$$

which implies that $0 < \frac{1}{u_{i+1}-v_{i+1}\sqrt{D}} < \frac{1}{\epsilon+\delta\sqrt{D}}$. Then $0 < \frac{u_{i+1}+v_{i+1}\sqrt{D}}{N} < \frac{\epsilon-\delta\sqrt{D}}{N}$.

This gives $\epsilon - \delta\sqrt{D} > u_{i+1} + v_{i+1}\sqrt{D}$. Thus, we get

$$2\delta\sqrt{D} = \left(\epsilon + \delta\sqrt{D}\right) - \left(\epsilon - \delta\sqrt{D}\right) < \left(u_{i+1} - v_{i+1}\sqrt{D}\right) - \left(u_{i+1} + v_{i+1}\sqrt{D}\right)$$
$$= -2v_{i+1}\sqrt{D}.$$

This gives $\delta < -v_{i+1} < 0$. Hence $\delta > 0$ or $\delta < 0$ depends on the sign of $u_i \pm v_i\sqrt{D}$, which proves (ii). Hence all the fundamental solutions $\frac{A_{ij}+B_{ij}\sqrt{D}}{k}$ of (1.3) are covered by (2.2).

We illustrate this by an example which justifies the meaning of "… covered by (2.2)".

Illustration. As earlier we once again consider the equation $U^2 - 7V^2 = 18$. Then we have $D = 7, k = 3$ and $N = 2$. Next, we consider the Pellian equation $R^2 - 7S^2 = 9$ and if $\frac{r_j+s_j\sqrt{7}}{3}$ runs through all of its fundamental solutions, then it can be observed that $\frac{r_1+s_1\sqrt{7}}{3} = \frac{4+\sqrt{7}}{3}$, $\frac{r_2+s_2\sqrt{7}}{3} = \frac{11+4\sqrt{7}}{3}$, $\frac{r_3+s_3\sqrt{7}}{3} = \frac{24+9\sqrt{7}}{3}$. Also $u_i + v_i\sqrt{7}$ runs through all the fundamental solutions of $u^2 - 7v^2 = 2$, then we have $u_1 \pm v_1\sqrt{7} = 3 \pm \sqrt{7}$. Then above theorem claims that all the fundamental solutions of $U^2 - 7V^2 = 18$ are covered by

$$\frac{A_{1j}+B_{1j}\sqrt{2}}{3} = \left(\frac{r_j+s_j\sqrt{D}}{3}\right)\left(3 \pm \sqrt{7}\right); j = 1, 2, 3.$$

Thus, $\frac{A_{11}+B_{11}\sqrt{2}}{3} = \left(\frac{4+\sqrt{7}}{3}\right)\left(3 \pm \sqrt{7}\right) = \frac{19+7\sqrt{7}}{3}, \frac{5-\sqrt{7}}{3}$;

$\frac{A_{12}+B_{12}\sqrt{2}}{3} = \left(\frac{11+4\sqrt{7}}{3}\right)\left(3 \pm \sqrt{7}\right) = \frac{61+23\sqrt{7}}{3}, \boldsymbol{\frac{5+\sqrt{7}}{3}}$ and

$\frac{A_{13}+B_{13}\sqrt{2}}{3} = \left(\frac{24+9\sqrt{7}}{3}\right)\left(3 \pm \sqrt{7}\right) = \frac{135+51\sqrt{7}}{3}, \boldsymbol{\frac{9+3\sqrt{7}}{3}}$.

It can be observed that $\frac{5+\sqrt{7}}{3}, \frac{9+3\sqrt{7}}{3}$ and $\frac{19+7\sqrt{7}}{3}$ are the only three fundamental solutions of the equation $U^2 - 7V^2 = 18$. Thus $\frac{A_{1j}+B_{1j}\sqrt{2}}{3}; j = 1,2,3$ covers all the fundamental solutions of $U^2 - 7V^2 = 18$.

Here the smallest fundamental solution of $U^2 - 7V^2 = 18$ is $\frac{5+\sqrt{7}}{3}$ and $x_1 + y_1\sqrt{D} = 8 + 3\sqrt{7}$ is the fundamental solution of $x^2 - 7y^2 = 1$. Then every fundamental solution of $U^2 - 7V^2 = 18$ should be smaller than $\frac{5+\sqrt{7}}{3} \times \left(8 + 3\sqrt{7}\right) = \frac{61+23\sqrt{7}}{3}$. This is indeed true as the only fundamental solutions of $U^2 - 7V^2 = 18$ are $\frac{A_{11}+B_{11}\sqrt{2}}{3} = \frac{19+7\sqrt{7}}{3}, \frac{A_{12}+B_{12}\sqrt{2}}{3} = \frac{5+\sqrt{7}}{3}$ and $\frac{A_{13}+B_{13}\sqrt{2}}{3} = \frac{9+3\sqrt{7}}{3}$.

We next derive an explicit formula which produces all the positive solutions of (1.3).

Theorem 2.4. If $x_1 + y_1\sqrt{D}$ is the smallest positive solution of (1.2) and $\frac{A_{ij}+B_{ij}\sqrt{D}}{k}$ defined by (2.2) runs through all the fundamentals solutions of (1.3), then all the integer solutions $\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k}$ of $U^2 - DV^2 = k^2N$ are given by

$$\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k} = \left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right)\left(x_1 + y_1\sqrt{D}\right)^n; n \geq 0. \tag{2.5}$$

Proof. If we consider the surd conjugate of (2.5) and multiply with (2.5), we get $\frac{u_{ij,n}^2-Dv_{ij,n}^2}{k^2} = \left(\frac{A_{ij}^2-DB_{ij}^2}{k^2}\right)(x_1^2 - Dy_1^2)^n$. Since $A_{ij}^2 - DB_{ij}^2 = k^2N$ and $x_1^2 - Dy_1^2 = 1$, we get $u_{ij,n}^2 - Dv_{ij,n}^2 = k^2N$. Thus $\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k}$ defined by (2.5) are the solutions of (1.3).

We next show that the solutions $\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k}$ defined by (2.5) gives all the positive solutions of (1.3). On the contrary assume that there exists some positive solution, say $\frac{X+Y\sqrt{D}}{k}$ of (1.3) which is not covered by (2.5). Then this solution will lie between any two successive solutions of (1.3) of some classes generated by $\frac{A_{ij}+B_{ij}\sqrt{D}}{k}$ (for a fixed i, j). This means for some fixed i, j and for some fixed m, we have

$$\left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right)\left(x_1 + y_1\sqrt{D}\right)^m \leq \frac{X+Y\sqrt{D}}{k} < \left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right)\left(x_1 + y_1\sqrt{D}\right)^{m+1}.$$

Then $\left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right) \leq \left(\frac{X+Y\sqrt{D}}{k}\right)\left(x_1 - y_1\sqrt{D}\right)^m < \left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right)\left(x_1 + y_1\sqrt{D}\right)$.

We denote

$$\frac{\epsilon+\delta\sqrt{D}}{k}=\left(\frac{X+Y\sqrt{D}}{k}\right)\left(x_1-y_1\sqrt{D}\right)^m. \tag{2.6}$$

Thus,

$$\left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right)\leq\frac{\epsilon+\delta\sqrt{D}}{k}<\left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right)\left(x_1+y_1\sqrt{D}\right). \tag{2.7}$$

To prove the required result, it is sufficient to prove that

(i) $\frac{\epsilon+\delta\sqrt{D}}{k}$ is a solution of (1.3)

(ii) $\epsilon>0,\delta>0$.

(iii) $\frac{\epsilon+\delta\sqrt{D}}{k}$ is the solution of (1.3) smaller than all its fundamental solutions for the fixed classes i,j. This will contradict the fact that $\frac{A_{ij}+B_{ij}\sqrt{D}}{k}$ runs through every fundamental solution of (1.3) for some fixed value of i,j.

To prove (i), we take surd conjugate of (2.6) and multiply it with (2.6). This gives $\frac{\epsilon^2-D\delta^2}{k^2}=\left(\frac{X^2-DY^2}{k^2}\right)(x_1^2-Dy_1^2)^m$. Since $X^2-DY^2=k^2N$ and $x_1^2-Dy_1^2=1$, we get $\epsilon^2-D\delta^2=k^2N$, as required.

Again, since $1<\left(x_1+y_1\sqrt{D}\right)^m<\infty$ and $\frac{X+Y\sqrt{D}}{k}>1$, we have $0<\left(x_1-y_1\sqrt{D}\right)^m<1$. Hence, we get $0<\frac{\epsilon+\delta\sqrt{D}}{k}<\infty$, as $\frac{\epsilon+\delta\sqrt{D}}{k}=\left(\frac{X+Y\sqrt{D}}{k}\right)\left(x_1-y_1\sqrt{D}\right)^m$. This gives $0<\frac{\epsilon-\delta\sqrt{D}}{kN}<\infty$, that is $0<\frac{\epsilon-\delta\sqrt{D}}{k}<\infty$, as $\epsilon^2-D\delta^2=k^2N$. Adding this result with $0<\frac{\epsilon+\delta\sqrt{D}}{k}<\infty$ proves that $\epsilon>0$.

We further observe by (2.7) that $1<\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\leq\frac{\epsilon+\delta\sqrt{D}}{k}<\infty$. Taking reciprocal, we get $0<\frac{k}{\epsilon+\delta\sqrt{D}}\leq\frac{k}{A_{ij}+B_{ij}\sqrt{D}}<1$. Then $0<\frac{\epsilon-\delta\sqrt{D}}{kN}\leq\frac{A_{ij}-B_{ij}\sqrt{D}}{kN}$. This gives $0<\frac{\epsilon-\delta\sqrt{D}}{k}\leq\frac{A_{ij}-B_{ij}\sqrt{D}}{k}$. Subsequently we get

$$\frac{2\delta\sqrt{D}}{k}=\left(\frac{\epsilon+\delta\sqrt{D}}{k}\right)-\left(\frac{\epsilon-\delta\sqrt{D}}{k}\right)\geq\left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right)-\left(\frac{A_{ij}-B_{ij}\sqrt{D}}{k}\right)=\frac{2B_{ij}\sqrt{D}}{k}.$$

This now gives $\delta\geq B_{ij}>0$. Hence both $\epsilon>0,\delta>0$, which proves (ii).

Finally, we prove that $\frac{\epsilon+\delta\sqrt{D}}{k}$ is the smallest solution of (1.3) for the fixed classes i,j. On the contrary suppose $\frac{\epsilon+\delta\sqrt{D}}{k}$ is positive solution of (1.3) but not the smallest solution of any fixed classes i,j. In this case $\frac{\epsilon+\delta\sqrt{D}}{k}\left(x_1-y_1\sqrt{D}\right)$ will be the positive solution of (1.3). Then by (2.7) we have

$$\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\leq\frac{\epsilon+\delta\sqrt{D}}{k}<\left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right)\left(x_1+y_1\sqrt{D}\right),$$

which gives

$$\left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right)\left(x_1-y_1\sqrt{D}\right)\leq\frac{\epsilon+\delta\sqrt{D}}{k}\left(x_1-y_1\sqrt{D}\right)<\frac{A_{ij}+B_{ij}\sqrt{D}}{k}.$$

Here $\frac{A_{ij}+B_{ij}\sqrt{D}}{k}$ runs through every fundamental solution of (1.3). Thus, we can now say that $\frac{\epsilon+\delta\sqrt{D}}{k}\left(x_1 - y_1\sqrt{D}\right)$ is a positive solution of (1.3) smaller than all the fundamental solutions $\frac{A_{ij}+B_{ij}\sqrt{D}}{k}$ of (1.3), which is a contradiction. This contradiction finally assures that there cannot exist any solution of (1.3) which is not covered by (2.5), as required.

We illustrate this theorem by the following example:

Illustration. Once again we consider the Pellian equation $U^2 - 7V^2 = 18$ whose all the fundamental solutions are given by $\frac{A_{11}+B_{11}\sqrt{2}}{3} = \frac{19+7\sqrt{7}}{3}$; $\frac{A_{12}+B_{12}\sqrt{2}}{3} = \frac{5+\sqrt{7}}{3}$ and $\frac{A_{13}+B_{13}\sqrt{2}}{3} = \frac{9+3\sqrt{7}}{3}$ and $x_1 + y_1\sqrt{D} = 8 + 3\sqrt{7}$ is the fundamental solution of $x^2 - 7y^2 = 1$. Then the above theorem asserts that all the integer solutions $\frac{u_{1j,n}+v_{1j,n}\sqrt{7}}{3}$ of Pellian equation $U^2 - 7V^2 = 18$ are covered by $\frac{u_{1j,n}+v_{1j,n}\sqrt{7}}{3} = \left(\frac{A_{1j}+B_{1j}\sqrt{7}}{3}\right)\left(x_1 + y_1\sqrt{D}\right)^n; j = 1,2,3\ ;\ n \geq 0.$ This gives $\frac{u_{11,n}+v_{11,n}\sqrt{7}}{3} = \left(\frac{19+7\sqrt{7}}{3}\right)\left(8 + 3\sqrt{7}\right)^n$;
$\frac{u_{12,n}+v_{12,n}\sqrt{7}}{3} = \left(\frac{5+\sqrt{7}}{3}\right)\left(8 + 3\sqrt{7}\right)^n$ and $\frac{u_{13,n}+v_{13,n}\sqrt{7}}{3} = \left(\frac{9+3\sqrt{7}}{3}\right)\left(8 + 3\sqrt{7}\right)^n$.

We now prove some recurrence relations for the values of $u_{ij,n}$ and $v_{ij,n}$. We assume that $x_1 + y_1\sqrt{D}$ is the fundamental solution of (1.2), $u_i + v_i\sqrt{D}$ runs through all the fundamental solutions of (1.1), $\frac{r_j+s_j\sqrt{D}}{k}$ runs through all the fundamental solutions of $R^2 - DS^2 = k^2$, $r \geq 1$ is a fixed integer and $n \geq 1$.

Theorem 2.5. a) $u_{ij,n+r} = \frac{x_r u_{ij,n}+Dy_r v_{ij,n}}{k}, v_{ij,n+r} = \frac{y_r u_{ij,n}+x_r v_{ij,n}}{k}$.

b) $u_{ij,n+r} = \frac{x_1 u_{ij,n+r-1}+Dy_1 v_{ij,n+r-1}}{k}, v_{ij,n+r} = \frac{y_1 u_{ij,n+r-1}+x_1 v_{ij,n+r-1}}{k}$.

c) $u_{ij,n+2r} = \frac{2kx_r u_{ij,n+r}-u_{ij,n}}{k^2}\ ,\ v_{ij,n+2r} = \frac{2kx_r v_{ij,n+r}-v_{ij,n}}{k^2}$.

Proof. (a) By (2.5), we have

$$\frac{u_{ij,n+r}+v_{ij,n+r}\sqrt{D}}{k} = \left(\frac{A_{ij}+B_{ij}\sqrt{D}}{k}\right)\left(x_1 + y_1\sqrt{D}\right)^{n+r} = \left(\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k}\right)\left(x_r + y_r\sqrt{D}\right)$$
$$= \frac{(x_r u_{ij,n}+Dy_r v_{ij,n})+(y_r u_{ij,n}+x_r v_{ij,n})\sqrt{D}}{k}.$$

Hence $u_{ij,n+r} = \frac{x_r u_{ij,n}+Dy_r v_{ij,n}}{k}, v_{ij,n+r} = \frac{y_r u_{ij,n}+x_r v_{ij,n}}{k}$.

(b) To prove the second result, we first consider $r = 1$ and replace n by $n - 1$ in the above result. We thus get

$$u_{ij,n} = \frac{x_1 u_{ij,n-1} + D y_1 v_{ij,n-1}}{k}, v_{ij,n} = \frac{y_1 u_{ij,n-1} + x_1 v_{ij,n-1}}{k}.$$

Then $u_{ij,n+r} = \frac{x_r u_{ij,n} + D y_r v_{ij,n}}{k}$

$$= \frac{1}{k}\left\{x_r\left(\frac{x_1 u_{ij,n-1} + D y_1 v_{ij,n-1}}{k}\right) + D y_r\left(\frac{y_1 u_{ij,n-1} + x_1 v_{ij,n-1}}{k}\right)\right\}$$
$$= \frac{1}{k}\left\{x_1\left(\frac{x_r u_{ij,n-1} + D y_r v_{ij,n-1}}{k}\right) + D y_1\left(\frac{x_r v_{ij,n-1} + y_r u_{ij,n-1}}{k}\right)\right\}$$
$$= \frac{x_1 u_{ij,n+r-1} + D y_1 v_{ij,n+r-1}}{k}$$

Value of $v_{ij,n+r}$ can be obtained accordingly.

(c) To prove the final part, we replace n by $n+r$ in the first result and using that in (a) above, we obtain

$$u_{ij,n+2r} = \frac{1}{k}\left\{x_r u_{ij,n+r} + D y_r\left(\frac{y_r u_{ij,n} + x_r v_{ij,n}}{k}\right)\right\}$$
$$= \frac{1}{k}\left\{x_r u_{ij,n+r} + \frac{D y_r^2 u_{ij,n}}{k} + x_r\left(\frac{D y_r v_{ij,n}}{k}\right)\right\}$$
$$= \frac{1}{k}\left\{x_r u_{ij,n+r} + \frac{D y_r^2 u_{ij,n}}{k} + x_r\left(u_{ij,n+r} - \frac{x_r u_{ij,n}}{k}\right)\right\}$$
$$= \frac{1}{k}\left\{2 x_r u_{ij,n+r} - \frac{u_{ij,n}}{k}(x_r^2 - D y_r^2)\right\}$$

Since $x_r + y_r\sqrt{D}$ is a solution of (1.2), we have $x_r^2 - D y_r^2 = 1$. Thus, we obtain $u_{ij,n+2r} = \frac{2k x_r u_{ij,n+r} - u_{ij,n}}{k^2}$.

Value of $v_{ij,n+2r}$ can also be obtained accordingly.

We further derive some more interesting properties related with the value of $u_{ij,n}$ and $v_{ij,n}$. The following interesting recursive formula connects three $u_{ij,n}$'s as well as $v_{ij,n}$'s when the suffixes are in arithmetic progression.

Corollary 2.6. (a) $u_{ij,n} u_{ij,n+2r} - u_{ij,n+r}^2 = k^2 D N y_r^2$.

(b) $v_{ij,n} v_{ij,n+2r} - v_{ij,n+r}^2 = -k^2 N y_r^2$.

Proof. By (2.5) we have $\frac{u_{ij,n} + v_{ij,n}\sqrt{D}}{k} = \left(\frac{A_{ij} + B_{ij}\sqrt{D}}{k}\right)(x_1 + y_1\sqrt{D})^n$.

Taking its surd conjugate, we get

$$\frac{u_{ij,n} - v_{ij,n}\sqrt{D}}{k} = \left(\frac{A_{ij} - B_{ij}\sqrt{D}}{k}\right)(x_1 - y_1\sqrt{D})^n.$$

For convenience we write $\gamma = x_1 + y_1\sqrt{D}, \bar{\gamma} = x_1 - y_1\sqrt{D}$ and $\mu_{ij} = \frac{A_{ij} + B_{ij}\sqrt{D}}{k}, \overline{\mu_{ij}} = \frac{A_{ij} - B_{ij}\sqrt{D}}{k}$. Then we have $\frac{u_{ij,n} + v_{ij,n}\sqrt{D}}{k} = \mu_{ij}\gamma^n$ and $\frac{u_{ij,n} - v_{ij,n}\sqrt{D}}{k} = \overline{\mu_{ij}}\bar{\gamma}^n$.

Adding and subtracting these two relations, we get

$$u_{ij,n} = \frac{k}{2}\{\mu_{ij}\gamma^n + \overline{\mu_{ij}}\bar{\gamma}^n\} \text{ and } v_{ij,n} = \frac{k}{2\sqrt{D}}\{\mu_{ij}\gamma^n - \overline{\mu_{ij}}\bar{\gamma}^n\}.$$

It can be easily observed that $\gamma\bar{\gamma} = x_1^2 - Dy_1^2 = 1$ and $\mu_{ij}\overline{\mu_{ij}} = \frac{A_{ij}^2 - DB_{ij}^2}{k^2} = N$. Then

$$
\begin{aligned}
&u_{ij,n}u_{ij,n+2r} - u_{ij,n+r}^2 \\
&\quad = \frac{k^2}{4}\left\{\begin{matrix}(\mu_{ij}\gamma^n + \overline{\mu_{ij}}\bar{\gamma}^n)(\mu_{ij}\gamma^{n+2r} + \overline{\mu_{ij}}\bar{\gamma}^{(n+2r)}) \\ -(\mu_{ij}\gamma^{n+r} + \overline{\mu_{ij}}\bar{\gamma}^{(n+r)})^2\end{matrix}\right\} \\
&\quad = \frac{k^2}{4}\left\{\begin{matrix}\mu_{ij}\overline{\mu_{ij}}(\gamma^{n+2r}\bar{\gamma}^{\,n} + \gamma^n\bar{\gamma}^{\,(n+2r)}) \\ -2\mu_{ij}\overline{\mu_{ij}}\gamma^{n+r}\bar{\gamma}^{(n+r)}\end{matrix}\right\} = \frac{k^2}{4}\{N(\gamma^{2r} + \bar{\gamma}^{\,2r}) - 2N\} \\
&\quad = \frac{k^2N}{4}(\gamma^r - \bar{\gamma}^{\,r})^2.
\end{aligned}
$$

Since $x_r + y_r\sqrt{D} = (x_1 + y_1\sqrt{D})^r = \gamma^r$ and $x_r - y_r\sqrt{D} = (x_1 - y_1\sqrt{D})^r = \bar{\gamma}^{\,r}$, we get $u_{ij,n}u_{ij,n+2r} - u_{ij,n+r}^2 = k^2DNy_r^2$.
Second result can be proved accordingly.

Following result gives some more recurrence relations in the form of a determinant.

Theorem 2.7. a) $\begin{vmatrix} u_{ij,n} & u_{ij,n+r} \\ v_{ij,n} & v_{ij,n+r}\end{vmatrix} = ky_rN$

b) $\begin{vmatrix} u_{ij,n+r-1} & u_{ij,n+r} \\ v_{ij,n+r-1} & v_{ij,n+r}\end{vmatrix} = ky_1N$

c) $\begin{vmatrix} 1 & 1 & 1 \\ u_{ij,n-r} & u_{ij,n} & u_{ij,n+r} \\ v_{ij,n-r} & v_{ij,n} & v_{ij,n+r}\end{vmatrix} = -2kNy_r(x_r - 1)$.

Proof. We only prove (c), since first two results follow easily through theorem 2.5. Now

$$
\begin{aligned}
&\begin{vmatrix} 1 & 1 & 1 \\ u_{ij,n-r} & u_{ij,n} & u_{ij,n+r} \\ v_{ij,n-r} & v_{ij,n} & v_{ij,n+r}\end{vmatrix} \\
&\qquad = \begin{vmatrix} u_{ij,n} & u_{ij,n+r} \\ v_{ij,n} & v_{ij,n+r}\end{vmatrix} - \begin{vmatrix} u_{ij,n-r} & u_{ij,n+r} \\ v_{ij,n-r} & v_{ij,n+r}\end{vmatrix} + \begin{vmatrix} u_{ij,n-r} & u_{ij,n} \\ v_{ij,n-r} & v_{ij,n}\end{vmatrix} \\
&\qquad = ky_rN - ky_{2r}N + ky_rN = kN(2y_r - y_{2r}).
\end{aligned}
$$

Now $(x_1 + y_1\sqrt{D})^{2r} = x_{2r} + y_{2r}\sqrt{D} = (x_r + y_r\sqrt{D})^2$. This gives $y_{2r} = 2x_ry_r$. Thus

$$
\begin{vmatrix} 1 & 1 & 1 \\ u_{ij,n-r} & u_{ij,n} & u_{ij,n+r} \\ v_{ij,n-r} & v_{ij,n} & v_{ij,n+r}\end{vmatrix} = kN(2y_r - 2x_ry_r) = -2kNy_r(x_r - 1).
$$

3. Number of solutions up to a desired limit

In this final section, we define and obtain the values of the sums $R(Z) = \sum_{\substack{\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k} \leq Z \\ U^2-DV^2=k^2N}} 1$, $S(Z) = \sum_{\substack{u_{ij,n} \leq Z \\ U^2-DV^2=k^2N}} 1$ and $T(Z) = \sum_{\substack{v_{ij,n} \leq Z \\ U^2-DV^2=k^2N}} 1$, the total number of positive solutions $\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k}, u_{ij,n}$ and $v_{ij,n}$ respectively of (1.3) that do not exceed any given large positive real number Z. For convenience, we denote $\delta = \frac{1}{\log \gamma}, \gamma = x_1 + y_1\sqrt{D}$ and let $\mathcal{A}_{ij} = \frac{A_{ij}+B_{ij}\sqrt{D}}{k}$ runs through all the fundamental solutions of (1.3) for any fixed class i. We also assume that (1.1) has β fundamental solutions and the equation $R^2 - DS^2 = k^2$ has η fundamental solutions.

Thus, throughout we have $1 \leq i \leq \beta$ and $1 \leq j \leq \eta$. We first obtain the value of $R(Z)$ which gives the number of all the solutions of (1.3) not exceeding any fixed given positive real number Z.

Theorem 3.1. $R(Z) = \delta\left\{\beta\eta \log(Z) - \log\left(\prod_{i=1}^{\beta}\prod_{j=1}^{\eta}\mathcal{A}_{ij}\right)\right\} + C$, where C is the effective constant such that $0 \leq C < \beta\eta$.

Proof. To find the value of $R(Z) = \sum_{\substack{\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k} \leq Z \\ U^2-DV^2=k^2N}} 1$, we first find the number of positive solutions $\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k}$ of (1.3) that do not exceed Z for some fixed class $i = \alpha$ $(1 \leq j \leq \eta)$. Since γ and $\mathcal{A}_{ij}$ are solutions of (1.2) and (1.3) respectively, (2.5) can be written as $\frac{u_{\alpha j,n}+v_{\alpha jn}\sqrt{D}}{k} = \mathcal{A}_{\alpha j}\gamma^n$. Now, for any given Z, it is clear that for some fixed class $i = \alpha$, there exists some n such that $\frac{u_{\alpha j,n}+v_{\alpha j,n}\sqrt{D}}{k} \leq Z < \frac{u_{\alpha j,n+1}+v_{\alpha j,n+1}\sqrt{D}}{k}$. Then we get $\mathcal{A}_{\alpha j}\gamma^n \leq Z < \mathcal{A}_{\alpha j}\gamma^{n+1}$. This implies $n < \frac{\log Z - \log \mathcal{A}_{\alpha j}}{\log \gamma} < n + 1$. Since n is an integer, we get $n = \left[\frac{\log Z - \log \mathcal{A}_{\alpha j}}{\log \gamma}\right]$, where $[x]$ denotes the integer part of x.

Now since $[x] = x - \{x\}$, where $\{x\}$ is the fractional part of x and as $0 \leq \{x\} < 1$, we have

$$R(Z) = \sum_{i=1}^{\beta}\sum_{j=1}^{\eta}\left[\frac{\log Z - \log \mathcal{A}_{ij}}{\log \gamma}\right] = \sum_{i=1}^{\beta}\sum_{j=1}^{\eta}\left(\frac{\log Z - \log \mathcal{A}_{ij}}{\log \gamma} + c'\right),$$

where $0 \leq c' < 1$. Thus $R(Z) = \frac{1}{\log \gamma}\sum_{i=1}^{\beta}\sum_{j=1}^{\eta}\left(\log Z - \log \mathcal{A}_{ij}\right) + \beta\eta c'$. If we write $C = \beta\eta c'$, then we get $0 \leq C < \beta\eta$ and

$$R(Z) = \delta\left\{\beta\eta \log(Z) - \log\left(\prod_{i=1}^{\beta}\prod_{j=1}^{\eta}\mathcal{A}_{ij}\right)\right\} + C.$$

We next find the value of $S(Z)$.

Theorem 3.2. $S(Z) = \delta\left\{\beta\eta\log(2Z/k) - \log\left(\prod_{i=1}^{\beta}\prod_{j=1}^{\eta}\mathcal{A}_{ij}\right)\right\} + C$, where C is the effective constant such that $-\beta\eta \le C < \beta\eta$.

Proof. To find the value of $S(Z)$, we first find the number of positive solutions of (1.3) where the values $u_{ij,n}$ of U do not exceed Z for some fixed class $i = \alpha$ $(1 \le j \le \eta)$. Now (2.5) can be written as $\frac{u_{\alpha j,n}+v_{\alpha jn}\sqrt{D}}{k} = \mathcal{A}_{\alpha j}\gamma^n$. Then for any given Z, it is clear that for some fixed class $i = \alpha$, there exists some *n* such that $u_{\alpha j,n} \le Z < u_{\alpha j,n+1}$. Since $\mathcal{A}_{ij} = \frac{A_{ij}+B_{ij}\sqrt{D}}{k}$, we write $\overline{\mathcal{A}_{ij}} = \frac{A_{ij}-B_{ij}\sqrt{D}}{k}$. We also have $\gamma^{-1} = x_1 - y_1\sqrt{D}$. Since γ and $\mathcal{A}_{ij}, \overline{\mathcal{A}_{ij}}$ are the solutions of (1.2) and (1.3) respectively, we have $\gamma\gamma^{-1} = 1$ and $\mathcal{A}_{\alpha j}\,\overline{\mathcal{A}_{\alpha j}} = N$. Then (2.5) can be written as

$$\frac{u_{\alpha j,n}+v_{\alpha j,n}\sqrt{D}}{k} = \mathcal{A}_{\alpha j}\gamma^n. \tag{3.1}$$

Now taking surd-conjugate of (3.1) we get $\frac{u_{\alpha j,n}-v_{\alpha j,n}\sqrt{D}}{k} = \overline{\mathcal{A}_{\alpha j}}\gamma^{-n}$. Adding this with (3.1) we now have

$$u_{\alpha j,n} = \frac{k}{2}\left\{\mathcal{A}_{\alpha j}\gamma^n + \overline{\mathcal{A}_{\alpha j}}\gamma^{-n}\right\} = \frac{k}{2}\left\{\mathcal{A}_{\alpha j}\gamma^n + \frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n}\right\}.$$

Since $u_{\alpha j,n} \le Z < u_{\alpha j,n+1}$, for some *n*, we get

$$\mathcal{A}_{\alpha j}\gamma^n + \frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n} \le \frac{2Z}{k} < \mathcal{A}_{\alpha j}\gamma^{n+1} + \frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n-1}.$$

Also, since $\frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n} > 0$, $\overline{\mathcal{A}_{\alpha j}} = \frac{N}{\mathcal{A}_{\alpha j}} < \mathcal{A}_{\alpha j}$ and $\gamma^{-1} < \gamma$, we have $\mathcal{A}_{\alpha j}\gamma^n < \frac{2Z}{k} < 2\mathcal{A}_{\alpha j}\gamma^{n+1}$. Now $\gamma = x_1 + y_1\sqrt{D} > 2$. Then we have $\mathcal{A}_{\alpha j}\gamma^n < \frac{2Z}{k} < \mathcal{A}_{\alpha j}\gamma^{n+2}$. This implies $n < \frac{\log 2Z - \log\mathcal{A}_{\alpha j} - \log k}{\log\gamma} < n+2$. Since *n* is an integer, we get $n \le \left[\frac{\log 2Z - \log\mathcal{A}_{\alpha j} - \log k}{\log\gamma}\right] \le n+1$. This implies

$$n = \left[\frac{\log 2Z - \log\mathcal{A}_{\alpha j} - \log k}{\log\gamma}\right] \text{ or } n = \left[\frac{\log 2Z - \log\mathcal{A}_{\alpha j} - \log k}{\log\gamma}\right] - 1.$$

Thus $S(Z) = \sum_{i=1}^{\beta}\sum_{j=1}^{\eta}\left[\frac{\log 2Z - \log\mathcal{A}_{\alpha j} - \log k}{\log\gamma}\right] + c$, where $c = 0$ or -1.

Then $S(Z) = \frac{1}{\log\gamma}\sum_{i=1}^{\beta}\sum_{j=1}^{\eta}\left(\log 2Z - \log\mathcal{A}_{ij} - \log k + c + c'\right)$, where $0 \le c' < 1$. Now considering $c + c' = C'$, we have $-1 \le C' < 1$. We can now write $S(Z) = \frac{\beta\eta}{\log\gamma}\log(2Z/k) - \frac{1}{\log\gamma}\sum_{i=1}^{\beta}\sum_{j=1}^{\eta}\log\mathcal{A}_{ij} + \beta\eta C'$.

If we write $C = \beta\eta C'$, then we get $-\beta\eta \le C < \beta\eta$ and

$$S(Z) = \delta\left\{\beta\eta\log(2Z/k) - \log\left(\prod_{i=1}^{\beta}\prod_{j=1}^{\eta}\mathcal{A}_{ij}\right)\right\} + C.$$

Finally, we find the value of $T(Z)$.

Theorem 3.3. $T(Z)=\delta\left\{\beta\eta\log\left(2\sqrt{D}Z/k\right)-\log\left(\prod_{i=1}^{\beta}\prod_{j=1}^{\eta}\mathcal{A}_{ij}\right)\right\}+C$, where C is the effective constant such that $0\le C<2\beta\eta$.

Proof. To find the value of $T(Z)$, we first find the number of positive solutions of (1.3) where the values $v_{ij,n}$ of V do not exceed Z for some fixed class $i=\alpha$ $(1\le j\le\eta)$. Now by (2.5) since $\frac{u_{\alpha j,n}+v_{\alpha j,n}\sqrt{D}}{k}=\mathcal{A}_{\alpha j}\gamma^{n}$ and $\frac{u_{\alpha j,n}-v_{\alpha j,n}\sqrt{D}}{k}=\overline{\mathcal{A}_{\alpha J}}\gamma^{-n}$, on subtraction, we get

$$v_{\alpha j,n}=\frac{k}{2\sqrt{D}}\left\{\mathcal{A}_{\alpha j}\gamma^{n}-\overline{\mathcal{A}_{\alpha J}}\gamma^{-n}\right\}=\frac{k}{2\sqrt{D}}\left\{\mathcal{A}_{\alpha j}\gamma^{n}-\frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n}\right\}.$$

Since $v_{\alpha j,n}\le Z<v_{\alpha j,n+1}$, for some n, we have

$$\mathcal{A}_{\alpha j}\gamma^{n}-\frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n}\le\frac{2\sqrt{D}Z}{k}<\mathcal{A}_{\alpha j}\gamma^{n+1}-\frac{N}{\mathcal{A}_{\alpha j}}\gamma^{-n-1}.$$

Thus, we have $\mathcal{A}_{\alpha j}\gamma^{n-1}\le\frac{2\sqrt{D}Z}{k}<\mathcal{A}_{\alpha j}\gamma^{n+1}$. This implies

$$n-1<\frac{\log 2\sqrt{D}Z-\log\mathcal{A}_{\alpha j}-\log k}{\log\gamma}<n+1.$$

Since n is an integer, we have

$$n=\left[\frac{\log 2\sqrt{D}Z-\log\mathcal{A}_{\alpha j}-\log k}{\log\gamma}\right]\ \text{or}\ n=\left[\frac{\log 2\sqrt{D}Z-\log\mathcal{A}_{\alpha j}-\log k}{\log\gamma}\right]+1.$$

Thus $T(Z)=\sum_{i=1}^{\beta}\sum_{j=1}^{\eta}\left[\frac{\log 2\sqrt{D}Z-\log\mathcal{A}_{\alpha j}-\log k}{\log\gamma}\right]+c$, where $c=0$ or 1. Then

$$T(Z)=\frac{1}{\log\gamma}\sum_{i=1}^{\beta}\sum_{j=1}^{\eta}\left(\log 2\sqrt{D}Z-\log\mathcal{A}_{ij}-\log k+c+c'\right),$$

where $0\le c'<1$. Considering $c+c'=C'$, we have $0\le C'<2$. Thus, we now write $T(Z)=\frac{\beta\eta}{\log\gamma}\log\left(2\sqrt{D}Z/k\right)-\frac{1}{\log\gamma}\sum_{i=1}^{\beta}\sum_{j=1}^{\eta}\log\mathcal{A}_{ij}+\beta\eta C'$.

If we write $C=\beta\eta C'$, then we get $0\le C<2\beta\eta$ and

$$T(Z)=\delta\left\{\beta\eta\log\left(2\sqrt{D}Z/k\right)-\log\left(\prod_{i=1}^{\beta}\prod_{j=1}^{\eta}\mathcal{A}_{ij}\right)\right\}+C.$$

The Following interesting conclusions are now an easy consequence from these theorems.

Corollary 3.4. $T(Z)-S(Z)\approx\delta\beta\eta\log\sqrt{D}$.

Corollary 3.5. If the solutions $\frac{u_{ij,n}+v_{ij,n}\sqrt{D}}{k}$ of $U^2-DV^2=k^2N$ are considered as lattice points within the square $[0,Z]\times[0,Z]$, then density of these lattice points is zero.

This follows from the fact that $\lim_{n\to\infty}\frac{\log Z}{Z}=0$.

4. Conclusions

In this paper, we derived the necessary and sufficient condition for any two solutions of $U^2 - DV^2 = k^2N$ to belong to the same class and the bounds for the values of u, v occurring in the fundamental solution. We also derived an explicit formula which gives all its positive solutions. We further obtained some interesting recurrence relations connecting the values of u, v. Finally, we obtained the results for total number of its positive solutions not exceeding any given positive real number Z.

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