Ratio Mathematica Volume 40, 2021, pp. 163-177

Generalized double Fibonomial numbers

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Abstract

From the beginning of 20th century, generalization of binomial coefficient has been deliberated broadly. One of the most famous generalized binomial coefficients are Fibonomial coefficients, obtained by substituting Fibonacci numbers in place of natural numbers in the binomial coefficients. In this paper, we further generalize the concept of Fibonomial coefficient and called it Generalized double Fibonomial number and obtain interesting properties of it. We also discuss its special case, double Fibonomial number along with the situation in which they give integer values. Other properties of it have also been discussed along with its upper and lower bounds.

Keywords: Fibonacci numbers, Lucas numbers, Fibonomial numbers, Binomial coefficient, Double factorial.
2020 AMS subject classifications: 11B39,05A10,11B65. [1]

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[1]Received on January 12th, 2021. Accepted on May 12th, 2021. Published on June 30th, 2021. doi: 10.23755/rm.v40i1.625. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

In combinatorics, the factorial of a positive integer n, denoted by $n!$, is defined by $n! = n \times n-1 \times \cdots \times 2 \times 1; n \geq 1$ and $0! = 1$. Whereas the double factorial of a positive integer n, usually denoted by $n!!$ is defined as

$$n!! = \begin{cases} n \times n-2 \times \cdots \times 3 \times 1; & n \, is \, odd \\ n \times n-2 \times \cdots \times 4 \times 2; & n \, is \, even \\ 1; & n = 0 \end{cases}$$

Note that $n!!$ is not the same as the iterated factorial $(n!)!$, which grows much faster. We do not know precisely when, where, or by whom, the double factorial notation was devised. It was used by Meserve [6] in 1948, and it is not mentioned by Cajori in his very detailed work in history of mathematical notations during 1928 – 1929 [2]. Thus, we summaries that the notation was introduced at some times during the period 1928 – 1948.

In this definition of $n!!$, if we replace the natural numbers by the terms of the generalized Fibonacci numbers w_n defined by the recurrence relation $w_n = pw_{n-1} + qw_{n-2}$, for $n \geq 2$; $w_0 = a$ and $w_1 = b$, where a, b, p and q are any integers, then what we get will be called Generalized double Fibonorial $n!!_w$ and is defined as

$$n!!_w \equiv \begin{cases} w_n \times w_{n-2} \times \cdots \times w_3 \times w_1 & n > 0 \, is \, odd \\ w_n \times w_{n-2} \times \cdots \times w_4 \times w_2 & n > 0 \, is \, even \\ 1 & n = 0 \end{cases} \tag{1}$$

Here note that when we substitute $p = q = b = 1$ and $a = 0$ in the definition of w_n, we get regular Fibonacci numbers F_n. The definition of $n!!_w$ helps to express the generalized double Fibonorial in terms of regular generalized Fibonorial as shown in the following lemma.

Lemma 1.1. $n!!_w = \frac{n!_w}{(n-1)!!_w} = \frac{(n+1)!_w}{(n+1)!!_w}; n \geq 1.$

In 1964 Fontene [3] generalized the notion of binomial coefficients and introduce the new concept of Fibonomial coefficients. In the definition of binomial coefficients $\binom{m}{k}$, he replaced the natural numbers by the terms of an arbitrary sequence $\{A_n\}$ of real or complex numbers. Since then there has been an accelerated interest in the study of Fibonomial coefficients. When the sequence $\{A_n\}$ is considered as the sequence $\{F_n\}$ of Fibonacci numbers, the Fibonomial coefficients $\begin{bmatrix} m \\ k \end{bmatrix}_F$, for $1 \geq k \geq m$, is defined as $\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{m!_F}{k!_F (m-k)!_F}$. The elaborated study on the generalized Fibonomial coefficients can be found in literature. (See [5])

This quantity will always be an integer, which can be shown by an induction argument by replacing F_m in the numerator with $F_k F_{m-k+1} + F_{k-1} F_{m-k}$, resulting in

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = F_{m-k+1} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_F + F_{k-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_F \tag{2}$$

We use the concept of generalized double Fibonorial to further generalize the concept of generalized Fibonomial coefficients. We define the generalized double Fibonomial numbers $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w$ as

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w = \frac{m!!_w}{k!!_w \, (m-k)!!_w} \tag{3}$$

It is easy to observe that

$$\left[\begin{bmatrix} m \\ 0 \end{bmatrix}\right]_w = 1, \left[\begin{bmatrix} m \\ 2 \end{bmatrix}\right]_w = w_m and \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w = \left[\begin{bmatrix} m \\ m-k \end{bmatrix}\right]_w \tag{4}$$

2 Generalized double Fibonomial numbers:

2.1 Some properties of generalized double Fibonomial numbers:

The following results are now easy consequences from this definition:

Lemma 2.1. *For any positive integers k, m and n,*

1. $(Iterative\ rule) \left[\begin{bmatrix} n \\ k \end{bmatrix}\right]_w \left[\begin{bmatrix} k \\ m \end{bmatrix}\right]_w = \left[\begin{bmatrix} n \\ m \end{bmatrix}\right]_w \left[\begin{bmatrix} n-m \\ k-m \end{bmatrix}\right]_w.$

2. $w_{m-k} \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w = w_m \left[\begin{bmatrix} m-2 \\ k \end{bmatrix}\right]_w.$

3. $w_k \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w = w_{m-k+2} \left[\begin{bmatrix} m \\ k-2 \end{bmatrix}\right]_w.$

4. $w_k \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w = w_m \left[\begin{bmatrix} m-2 \\ k-2 \end{bmatrix}\right]_w.$

Lemma 2.2. $(m-1)!!_w \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w$ *will always give an integer value.*

This result is an easy derived from the definition of generalized Fibonorial and generalized double Fibonomial numbers. The basic recurrence relations for the generalized double Fibonomial numbers is as follows:

Lemma 2.3. $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w - \left[\begin{bmatrix} m-2 \\ k \end{bmatrix}\right]_w = \left[\begin{bmatrix} m-2 \\ k-2 \end{bmatrix}\right]_w \left\{\frac{w_m - w_{m-k}}{w_k}\right\}$.

By changing k to $m-k$ and using (4), we get

Lemma 2.4. $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w - \left[\begin{bmatrix} m-2 \\ k-2 \end{bmatrix}\right]_w = \left[\begin{bmatrix} m-2 \\ k \end{bmatrix}\right]_w \left\{\frac{w_m - w_k}{w_{m-k}}\right\}$.

The following result can be easily obtained when we apply the sum on both sides with respect to the upper index such that m and k have the same parity.

Lemma 2.5. $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_w = \sum_{j=k}^{m} \left\{\frac{w_j - w_{j-k}}{w_k}\right\} \left[\begin{bmatrix} j-2 \\ k-2 \end{bmatrix}\right]_w$ *; where the sum is taken over integers starting from* k *with spacing of* 2 *up to* m.

2.2 Star of David theorem:

In 1972, Gould gave a result related to one interesting arithmetic property of binomial coefficients which was named as the Star of David theorem, which was stated as “The greatest common divisors of the binomial coefficients forming each of the two triangles in the Star of David shape in Pascal’s triangle are equal:

$$\gcd\left\{\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right\}.$$

The two sets of three numbers, which the Star of David theorem says, have equal greatest common divisors and equal products. Interestingly, Gould’s result can be imitated for generalized double Fibonomial numbers too as shown in the following result.

Theorem 2.1. $\left[\begin{bmatrix} m-a \\ k-b \end{bmatrix}\right]_w \left[\begin{bmatrix} m \\ k+b \end{bmatrix}\right]_w \left[\begin{bmatrix} m+b \\ k \end{bmatrix}\right]_w$
$= \left[\begin{bmatrix} m-a \\ k \end{bmatrix}\right]_w \left[\begin{bmatrix} m+b \\ k+a \end{bmatrix}\right]_w \left[\begin{bmatrix} m \\ k-b \end{bmatrix}\right]_w$ *; where* a, b *are positive integers.*

Proof. Using the definition of generalized double Fibonomial numbers, the left side of the result becomes

$$\left[\begin{bmatrix} m-a \\ k-b \end{bmatrix}\right]_w \left[\begin{bmatrix} m \\ k+b \end{bmatrix}\right]_w \left[\begin{bmatrix} m+b \\ k \end{bmatrix}\right]_w$$
$$= \frac{(m-a)!!_w}{(k-b)!!_w(m-k-a+b)!!_w} \times \frac{(m)!!_w}{(k+a)!!_w(m-k-a)!!_w} \times \frac{(m+b)!!_w}{(k)!!_w(m-k+b)!!_w}$$

$= \frac{(m-a)!!_w}{(k)!!_w(m-k-a)!!_w} \times \frac{(m+b)!!_w}{(k+a)!!_w(m-k-a+b)!!_w} \times \frac{(m)!!_w}{(k-b)!!_w(m-k+b)!!_w}$

$= \left[\left[\begin{matrix} m-a \\ k \end{matrix}\right]\right]_w \left[\left[\begin{matrix} m+b \\ k+a \end{matrix}\right]\right]_w \left[\left[\begin{matrix} m \\ k-b \end{matrix}\right]\right]_w$, as required.□

Corolary 2.1. *If $a = b = 1$, we get the product of six generalized double Fibonomial numbers, which are equally spaced around* $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_w$.

2.3 Generalized Double Multinomial Numbers:

Let $m = k_1 + k_2 + \cdots + k_r$ then we can define generalized double multinomial number as

$$\left[\left[\begin{matrix} m \\ k_1, k_2, \cdots, k_r \end{matrix}\right]\right]_w = \frac{m!!_w}{k_1!!_w k_2!!_w \cdots k_r!!_w}$$

Following result expresses generalized double multinomial numbers as the multiplication of generalized double Fibonomial numbers.

Lemma 2.6. *Generalized double multinomial numbers can be expressed as the multiplication of generalized double Fibonomial numbers.*

Proof. In the definition of generalized double multinomial numbers, consider $r = 2$, then we have $\left[\left[\begin{matrix} m \\ k_1, k_2 \end{matrix}\right]\right]_w = \left[\left[\begin{matrix} m \\ k_1 \end{matrix}\right]\right]_w$; where $k_1 + k_2 = m$.
For $r = 3$ and $m = k_1 + k_2 + k_3$, $\left[\left[\begin{matrix} m \\ k_1, k_2, k_3 \end{matrix}\right]\right]_w = \left[\left[\begin{matrix} m \\ k_1 \end{matrix}\right]\right]_w \left[\left[\begin{matrix} m-k_1 \\ k_2 \end{matrix}\right]\right]_w$.
Let us now consider $r = n$ and $m = k_1 + k_2 + \cdots + k_n$. Thus
$\left[\left[\begin{matrix} m \\ k_1, k_2, \cdots, k_r \end{matrix}\right]\right]_w = \frac{m!!_w}{k_1!!_w k_2!!_w \cdots k_n!!_w} = \frac{m!!_w}{k_1!!_w k_2!!_w \cdots k_{n-2}!!_w} \times \frac{1}{k_{n-1}!!_w k_n!!_w}$
$= \frac{m!!_w}{k_1!!_w k_2!!_w \cdots k_{n-2}!!_w \times (m-k_1-k_2-\cdots-k_{n-2})!!_w} \times \frac{(m-k_1-k_2-\cdots-k_{n-2})!!_w}{k_{n-1}!!_w (m-k_1-k_2-\cdots-k_{n-2}-k_{n-1})!!_w}$
$= \left[\left[\begin{matrix} m \\ k_1 \end{matrix}\right]\right]_w \left[\left[\begin{matrix} m-k_1 \\ k_2 \end{matrix}\right]\right]_w \cdots \left[\left[\begin{matrix} m-k_1-k_2-\cdots-k_{n-2} \\ k_{n-1} \end{matrix}\right]\right]_w$.
Hence, by the principle of Mathematical induction, we get the required result.□
It is obvious that all the above results related to generalized double Fibonorials and generalized double Fibonomial numbers are also true for double Fibonomials $n!!_F$ and double Fibonomial coefficients $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F$. But there are some additional results related to them, which are discussed in the following article.

3 Double Fibonomial numbers:

3.1 Definition and some properties of double Fibonomial numbers:

Using the definitions (1) and (3), Double Fibonorials and double Fibonomial numbers can be respectively expressed as

$$n!!_F \equiv \begin{cases} F_n \times F_{n-2} \times \cdots \times F_3 \times F_1 & n > 0 \, is \, odd \\ F_n \times F_{n-2} \times \cdots \times F_4 \times F_2 & n > 0 \, is \, even \\ 1 & n = 0 \end{cases}$$

and

$$\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = \frac{m!!_F}{k!!_F(m-k)!!_F}.$$

The following table shows first few terms of double Fibonorials for some initial values of n.

n	0	1	2	3	4	5	6	7	8	9	10
$n!!_F$	1	1	1	2	3	10	24	130	504	4420	27720

Table 1: Double Fibonorial numbers

Also by (4), double FIbonomial numbers have the symmetry property. Thus Table 2 shows the first few terms of double Fibonomial numbers of one side only.

								1
							1	
						1		1
					1		2	
				1		3/2		3
			1		10/3		5	
		1		24/10		8		6
	1		65/12		13		65/3	
1		252/65		21		126/5		56

Table 2: Double Fibonomial numbers

We further show how Double Fibonorial and Double Fibonomial numbers are connected with the sequence $\{L_n\}$ of Lucas numbers. This sequence is famously known as the twin sequence of Fibonacci sequence, which can be obtained by substituting $p = q = b = 1$ and $a = 2$ in the definition of w_n. That is $L_n = L_{n-1} + L_{n-2}$; $L_0 = 2$ and $L_1 = 1$. It is easy to observe that $F_{2n} = F_n L_n$. If we define $n!_L = L_n \times L_{n-1} \times \cdots \times L_2 \times L_1$, then the following lemma follows easily.

Lemma 3.1. *$n!!_F = k!_F \times k!_L$, for even positive integer $n = 2k$.*

If we consider $\begin{bmatrix} m \\ k \end{bmatrix}_L = \frac{m!_L}{k!_L(m-k)!_L}$,then the following is an easy consequence of lemma 3.1.

Lemma 3.2. $\left[\begin{bmatrix} 2m \\ 2k \end{bmatrix}\right]_F = \begin{bmatrix} m \\ k \end{bmatrix}_F \times \begin{bmatrix} m \\ k \end{bmatrix}_L$

From the Table 2 it is clear that double Fibonomial numbers are not always an integer. Obviously, for any integer m, $\left[\begin{bmatrix} m \\ 0 \end{bmatrix}\right]_F = \left[\begin{bmatrix} m \\ m \end{bmatrix}\right]_F = 1$, will always have an integer value. Also $\left[\begin{bmatrix} m \\ 2 \end{bmatrix}\right]_F = \left[\begin{bmatrix} m \\ m-2 \end{bmatrix}\right]_F = F_m$ will be integers. These two will serve as the trivial cases. Following theorem speaks about when double Fibonomial numbers attain integer values.

Theorem 3.1. *The nontrivial double Fibonomial numbers $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F$ are integers only when either m and k both are even integers together or $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \left[\begin{bmatrix} 6 \\ 3 \end{bmatrix}\right]_F$.*

Proof. We prove the result in four cases depending on the parity of m and k. Case 1: When m and k both are even integers, we have

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \left[\begin{bmatrix} 2n \\ 2l \end{bmatrix}\right]_F = \frac{(2n)!!_F}{(2l)!!_F(2n-2l)!!_F} = \frac{F_{2n} \times F_{2n-2} \times \cdots \times F_{2n-2l+2}}{F_{2l} \times \cdots \times F_4 \times F_2}$$

Note that number of elements in numerator and denominator are same. Also, they are Fibonacci numbers with even subscripts, such that in the denominator we have first l even subscripted Fibonacci numbers. Since these numbers always divide multiplication of any l consecutive even subscripted Fibonacci numbers, it follows that $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F$ will always be an integer.

Case 2: When m and k both are odd integers, we have In this case, we have

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \left[\begin{bmatrix} 2n+1 \\ 2l+1 \end{bmatrix}\right]_F = \frac{(2n+1)!!_F}{(2l+1)!!_F(2n-2l)!!_F} = \frac{F_{2n+1} \times F_{2n-1} \times \cdots \times F_{2l+3}}{F_{2n-2l} \times \cdots \times F_4 \times F_2}$$

In the numerator, every Fibonacci number is with odd subscript. Consequently, none of them will be divisible by $F_4 = 3$. Thus $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F$ will not be an integer in this case.
Case 3: When m is odd integer and k is even integer, we have In this case, we have

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \left[\begin{bmatrix} 2n+1 \\ 2l \end{bmatrix}\right]_F = \frac{(2n+1)!!_F}{(2l)!!_F(2n-2l+1)!!_F} = \frac{F_{2n+1}\times F_{2n-1}\times\cdots\times F_{2n-2l+3}}{F_{2l}\times\cdots\times F_4\times F_2}$$

Here again in the numerator, every Fibonacci number is with odd subscript, so none of them will be divisible by $F_4 = 3$. And therefore, in this case $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F$ will not be an integer.
Case 4: When m is even integer and k is odd integer, we have

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \left[\begin{bmatrix} 2n \\ 2l+1 \end{bmatrix}\right]_F = \frac{(2n)!!_F}{(2l+1)!!_F(2n-2l-1)!!_F} = \frac{F_{2n}\times F_{2n-2}\times\cdots\times F_4\times F_2}{(F_{2l+1}\times\cdots\times F_3\times F_1)(F_{2n-2l-1}\times\cdots\times F_3\times F_1)}$$

Here number of terms in the numerator and denominator are same. Also, the Fibonacci numbers in the numerator are with only even subscripts and in the denominator with only odd subscripts. But, for any Fibonacci number F_n, there exists a prime p such that if $p \mid F_n$, then p will only divide F_{mn}; for every $m \geq 1$.

Since $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \left[\begin{bmatrix} m \\ m-k \end{bmatrix}\right]_F$, for convenience we take $k > m-k$, that is, $2k > m$. Then there will not be the same Fibonacci numbers in the numerator and denominator. Also, there will not be any multiple subscripts of k in the numerator. Thus, there will exist a prime p in the denominator such that $p \mid F_k$ which will not divide any of the Fibonacci number in the numerator.

Likewise, when $k = m-k$, then except for $k = 3$, there will be a prime p such that $p \mid F_k$ as well as $p \mid F_m$, but it will appear in the denominator only once where as in the numerator twice. Thus in this case, except for $\left[\begin{bmatrix} 6 \\ 3 \end{bmatrix}\right]_F = 6$, $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F$ will not be an integer.□

In the following theorem we obtain the recurrence relation for the double Fibonomial numbers.

Theorem 3.2. $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_{k-1}\left[\begin{bmatrix} m-2 \\ k \end{bmatrix}\right]_F + F_{m-k+1}\left[\begin{bmatrix} m-2 \\ k-2 \end{bmatrix}\right]_F.$

Proof. From [4], we observe that the Fibonomial coefficients $\begin{bmatrix} m \\ k \end{bmatrix}_F$ has the recurrence relation

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = F_{k-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_F + F_{m-k+1} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_F$$

Now, using this relation and lemma 3.2, we get

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \frac{m!_F}{(m-1)!!_F} \times \frac{(k-1)!!_F}{k!_F} \times \frac{(m-k-1)!!_F}{(m-k)!_F} = \begin{bmatrix} m \\ k \end{bmatrix}_F \times \frac{(k-1)!!_F \times (m-k-1)!!_F}{(m-1)!!_F}.$$

$$= \left\{ F_{k-1} \begin{bmatrix} m-1 \\ k \end{bmatrix}_F + F_{m-k+1} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_F \right\} \times \frac{(k-1)!!_F \times (m-k-1)!!_F}{(m-1)!!_F}.$$

$$= \left\{ \frac{F_{k-1}(m-1)!_F}{(m-1)!!_F} \times \frac{(k-1)!!_F}{k!_F} \times \frac{(n-k-1)!!_F}{(n-k-1)!_F} \right\} + \left\{ \frac{F_{n-k+1}(m-1)!_F}{(m-1)!!_F} \times \frac{(k-1)!!_F}{(k-1)!_F} \times \frac{(n-k-1)!!_F}{(n-k-1)!_F} \right\}$$

$$= \frac{F_{k-1}(n-2)!!_F}{k!!_F \times (n-k-2)!!_F} + \frac{F_{n-k+1}(n-2)!!_F}{(k-2)!!_F \times (n-k)!!_F}$$

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_{k-1} \left[\begin{bmatrix} m-2 \\ k \end{bmatrix}\right]_F + F_{m-k+1} \left[\begin{bmatrix} m-2 \\ k-2 \end{bmatrix}\right]_F$$, as required. □

Lemma 3.3. $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \sum_{j=1}^{\left|\frac{m-k}{2}\right|} F_{k-1}^{j-1} F_{m-k+1-2(j-1)} \left[\begin{bmatrix} m-2j \\ k-2 \end{bmatrix}\right]_F + F_{k-1}^{\left|\frac{m-k}{2}\right|} A;$

where $A = \begin{cases} 1; \textit{when } m \textit{ and } k \textit{ both are even or both are odd integers} \\ \left[\begin{bmatrix} k+1 \\ k \end{bmatrix}\right]_F; \textit{otherwise} \end{cases}$

Proof. From above theorem, we have

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_{m-k+1} \left[\begin{bmatrix} m-2 \\ k-2 \end{bmatrix}\right]_F + F_{k-1} \left[\begin{bmatrix} m-2 \\ k \end{bmatrix}\right]_F$$

$$= F_{m-k+1} \left[\begin{bmatrix} m-2 \\ k-2 \end{bmatrix}\right]_F + F_{k-1} \left\{ F_{m-k-1} \left[\begin{bmatrix} m-4 \\ k-2 \end{bmatrix}\right]_F + F_{k-1} \left[\begin{bmatrix} m-4 \\ k \end{bmatrix}\right]_F \right\}$$

$$= F_{m-k+1} \left[\begin{bmatrix} m-2 \\ k-2 \end{bmatrix}\right]_F + F_{k-1} F_{m-k-1} \left[\begin{bmatrix} m-4 \\ k-2 \end{bmatrix}\right]_F$$

$$. \quad + F_{k-1}^2 \left\{ F_{m-k-3} \left[\begin{bmatrix} m-6 \\ k-2 \end{bmatrix}\right]_F + F_{k-1} \left[\begin{bmatrix} m-6 \\ k \end{bmatrix}\right]_F \right\}$$

Continuing this process, we get

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \begin{cases} \sum_{j=1}^{\left|\frac{m-k}{2}\right|} F_{k-1}^{j-1} F_{m-k+1-2(j-1)} \left[\begin{bmatrix} m-2j \\ k-2 \end{bmatrix}\right]_F + F_{k-1}^{\left|\frac{m-k}{2}\right|} \left[\begin{bmatrix} k \\ k \end{bmatrix}\right]_F; \\ \textit{when } n \textit{ and } k \textit{ both are even or odd} \\ \sum_{j=1}^{\left|\frac{m-k}{2}\right|} F_{k-1}^{j-1} F_{m-k+1-2(j-1)} \left[\begin{bmatrix} m-2j \\ k-2 \end{bmatrix}\right]_F + F_{k-1}^{\left|\frac{m-k}{2}\right|} \left[\begin{bmatrix} k+1 \\ k \end{bmatrix}\right]_F; \\ \textit{otherwise} \end{cases}$$

,as required. □

To illustrate the result, we consider $m = 9$ and $k = 5$. Then

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \sum_{j=1}^{\left|\frac{m-k}{2}\right|} F_{k-1}^{j-1} F_{m-k+1-2(j-1)} \left[\begin{bmatrix} m-2j \\ k-2 \end{bmatrix}\right]_F + F_{k-1}^{\left|\frac{m-k}{2}\right|} A$$

$$= \sum_{j=1}^{2} F_4^{j-1} F_{5-2(j-1)} \left[\begin{bmatrix} 9-2j \\ 3 \end{bmatrix}\right]_F + F_4^2 \left[\begin{bmatrix} 5 \\ 5 \end{bmatrix}\right]_F$$

$= F_5 \left[\left[\begin{matrix}7\\3\end{matrix}\right]\right]_F + F_4F_3 \left[\left[\begin{matrix}5\\3\end{matrix}\right]\right]_F + F_4^2 = \left(5 \times \frac{65}{3}\right) + (3 \times 2 \times 5) + (3^2)$
$= \frac{442}{3} = \left[\left[\begin{matrix}9\\5\end{matrix}\right]\right]_F$, as expected.

The following result is an easy consequence from the definition of double Fibonomial numbers and the basic identity $F_mL_n + F_nL_m = 2F_{m+n}$ relating both Fibonacci numbers and Lucas numbers.

Lemma 3.4. $\left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F = \frac{1}{2}\left(L_k \left[\left[\begin{matrix}m-2\\k\end{matrix}\right]\right]_F + L_{m-k} \left[\left[\begin{matrix}m-2\\k-2\end{matrix}\right]\right]_F\right).$

Proof. Since $2F_m = F_kL_{m-k} + F_{m-k}L_k$, we have $2\left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F F_m =$
$\left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F F_kL_{m-k} + \left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F F_{m-k}L_k$
$= \left[\left[\begin{matrix}m-2\\k-2\end{matrix}\right]\right]_F F_mL_{m-k} + \left[\left[\begin{matrix}m-2\\k\end{matrix}\right]\right]_F F_mL_k$. Thus
$2\left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F = L_k \left[\left[\begin{matrix}m-2\\k\end{matrix}\right]\right]_F + L_{m-k} \left[\left[\begin{matrix}m-2\\k-2\end{matrix}\right]\right]_F$, as required. □

Using lemma 3.4 and applying the same logic of lemma 3.3, the following result can be proved easily.

Lemma 3.5. $\left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F = \sum_{j=1}^{\left\lfloor\frac{m-k}{2}\right\rfloor} \frac{L_k^{j-1}L_{m-k-2(j-1)}}{2^j} \left[\left[\begin{matrix}m-2j\\k-2\end{matrix}\right]\right]_F + \left(\frac{L_k}{2}\right)^{\left\lfloor\frac{m-k}{2}\right\rfloor} A;$

where $A = \begin{cases} 1; when\ m\ and\ k\ both\ are\ even\ or\ odd\ integers \\ \left[\left[\begin{matrix}k+1\\k\end{matrix}\right]\right]_F; otherwise \end{cases}$

To illustrate the result, we consider $m = 10$ and $k = 3$. Then $\left[\left[\begin{matrix}m\\k\end{matrix}\right]\right]_F =$
$\sum_{j=1}^{\left\lfloor\frac{m-k}{2}\right\rfloor} \frac{L_k^{j-1}L_{m-k-2(j-1)}}{2^j} \left[\left[\begin{matrix}m-2j\\k-2\end{matrix}\right]\right]_F + \left(\frac{L_k}{2}\right)^{\left\lfloor\frac{m-k}{2}\right\rfloor} A$
$= \sum_{j=1}^{3} \frac{L_3^{j-1}L_{7-2(j-1)}}{2^j} \left[\left[\begin{matrix}10-2j\\3\end{matrix}\right]\right]_F + \left(\frac{L_3}{2}\right)^3 \left[\left[\begin{matrix}4\\3\end{matrix}\right]\right]_F$
$= \frac{L_7}{2} \left[\left[\begin{matrix}8\\1\end{matrix}\right]\right]_F + \frac{L_3L_5}{2^2} \left[\left[\begin{matrix}6\\1\end{matrix}\right]\right]_F + \frac{L_3^2L_3}{2^3} \left[\left[\begin{matrix}4\\1\end{matrix}\right]\right]_F + \frac{L_3^3}{2^3} \left[\left[\begin{matrix}4\\3\end{matrix}\right]\right]_F$
$= \left(\frac{29}{2} \times \frac{252}{65}\right) + \left(\frac{4\times 11}{2^2} \times \frac{24}{10}\right) + \left(\frac{4^3}{2^3}\right)\left(\frac{3}{2} + \frac{3}{2}\right)$
$= \frac{1386}{13} = \left[\left[\begin{matrix}10\\3\end{matrix}\right]\right]_F$, as expected.

In the following section we find the bounds of these numbers.

3.2 Bounds of double Fibonomial numbers:

The Binet formula for the Fibonacci number is given by $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$; where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The following theorem gives us the bounds of double Fibonomial numbers in terms of α.

Theorem 3.3. *For* $\chi(n) = \begin{cases} 0; when\ n\ is\ even \\ 1; when\ n\ is\ odd \end{cases}$,
$\alpha^{\frac{(k-\chi(k))(m-k-\chi(m(m-k-1)-1))}{2}} \leq \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \leq \alpha^{\frac{(k+\chi(k))(m-k+\chi(m(m-k-1)-1))}{2}}$.

Proof. It is well-known that $\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$; for all $n \geq 1$.
Then it is easy to observe that

$$\frac{F_{m-2t}}{F_{2t+2}} \leq \alpha^{m-4t-1} \tag{5}$$

and

$$\frac{F_{m-2t}}{F_{2t+2}} \geq \alpha^{m-4t-3} \tag{6}$$

Here we consider the four cases depending on the parity of m and k. When both m and k are even, using the definition of double Fibonomial numbers and (5), we have
$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = \frac{m!!_F}{k!!_F \times (m-k)!!_F} = \frac{F_m \times F_{m-2} \times \cdots \times F_{m-k+2}}{F_2 \times F_4 \times \cdots \times F_k}$
$\leq \alpha^{m-1} \times \alpha^{m-5} \times \cdots \times \alpha^{(m-2k+3)} = \alpha^{\frac{k(m-k+1)}{2}}$
Thus $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \leq \alpha^{\frac{k(m-k+1)}{2}}$.
Again using (6) in the definition of double Fibonomial number, we get

$$\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \geq \alpha^{m-3} \times \alpha^{m-7} \times \cdots \times \alpha^{m-2k+1} = \alpha^{\frac{k(m-k-1)}{2}}.$$

This shows that $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \geq \alpha^{\frac{k(m-k-1)}{2}}$. Thus when m and n both are even, we have

$$\alpha^{\frac{k(m-k-1)}{2}} \leq \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \leq \alpha^{\frac{k(m-k+1)}{2}}.$$

Considering $\chi(n) = \begin{cases} 0; when\ n\ is\ even \\ 1; when\ n\ is\ odd \end{cases}$, this result can be written as

$$\alpha^{\frac{(k-\chi(k))(m-k-\chi(m(m-k-1)-1))}{2}} \leq \left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \leq \alpha^{\frac{(k+\chi(k))(m-k+\chi(m(m-k-1)-1))}{2}}.$$

The required result can be proved using the similar technique for all the remaining cases.

To illustrate it, we consider $m = 9$ and $k = 4$. Then $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = \frac{442}{3}$.
Also, $\alpha^{\frac{(k-\chi(k))(m-k-\chi(m(m-k-1)-1))}{2}} = \alpha^{\frac{k(m-k-1)}{2}} = \alpha^8 = 46.97$
and $\alpha^{\frac{(k+\chi(k))(m-k+\chi(m(m-k-1)-1))}{2}} = \alpha^{\frac{k(m-k+1)}{2}} = \alpha^12 = 321$, which shows that

$$\alpha^{\frac{(k-\chi(k))(m-k-\chi(m(m-k-1)-1))}{2}} \leq \left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F \leq \alpha^{\frac{(k+\chi(k))(m-k+\chi(m(m-k-1)-1))}{2}}.$$

4 Double Fibonomial numbers and Fibonacci numbers:

By [1], it is known that a primitive divisor of a Fibonacci number F_n is any prime integer p such that $p \mid F_n$ but $p \nmid F_m$; where $m < n$. Also, primitive divisor theorem says that for $n \geq 13$, every F_n has a primitive divisor. We use this result to prove many interesting relations between generalized double Fibonomial numbers and Fibonacci numbers.

4.1 Double Fibonomial number as a power of Fibonacci number:

In literature, there are many results involving Fibonomial numbers and Fibonacci numbers. From (4), it is clear that $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_m$ for $k = 2$. Thus, the Diophantine equation $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_n$ will always have a trivial solution $(m, k, n) = (m, 2, m)$. Following result claims that there is no other solution for the considered Diophantine equation.

Lemma 4.1. *The Diophantine equation* $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_n$ *has no solution for* $k > 2$.

Proof. We know that except for $\left[\left[\begin{matrix} 6 \\ 3 \end{matrix}\right]\right]_F = 6$, which is not a Fibonacci number, and trivial cases, $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F$ is an integer only when both m and k are even integers. Thus, $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_n$ implies

$$\frac{F_m \times F_{m-2} \times \cdots \times F_{m-k+2}}{F_k \times F_{k-2} \times \cdots \times F_2} = F_n \tag{7}$$

If we consider $n \geq 13$ and $n > m$, then by the primitive divisor theorem, there exists a prime p such that $p \mid F_n$ but $p \nmid F_m$. That is, (7) has no solution possible. Similarly, for $m \geq 13$ and $m > n$, primitive divisor theorem implies that (7) has no solution.
Thus, we can narrow down the range of m and n as $max\,(m, n)$. A quick look at the Table 2 reveals that for $k > 2$, the Diophantine equation $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_n$ has no solution. □

The following result can be proved through the similar arguments.

Theorem 4.1. *For any positive integer* t*, the Diophantine equation* $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F = F_n^t$ *has no solution for* $k > 2$.

Though the double Fibonomial numbers do not possess the value of a Fibonacci number except for the trivial cases, they do stand in the neighborhood of Fibonacci number. We present this fact in the following final result.

Theorem 4.2. *The only solutions of the Diophantine equation* $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \pm 1 = F_n$ *are* $(m, k, n) = (3, 1, 2), (3, 2, 2), (4, 2, 3), (6, 3, 5), (8, 4, 10)$ *for* $'+'$ *case and* $(3, 1, 4), (3, 2, 4)$ *for* $'-'$ *case.*

Poof. From the Table 2, it is easy to observe that the Diophantine equation $\left[\begin{bmatrix} m \\ k \end{bmatrix}\right]_F \pm 1 = F_n$ has solution $(m, k, n) = (3, 1, 2), (3, 2, 2), (4, 2, 3)$ for $'+'$ case and $(m, k, n) = (3, 1, 4), (3, 2, 4)$ for the $'-'$ case for $m \leq 5$. Now for $m > 5$, when m is an odd integer, double Fibonomial number will not be an integer. And when m is an even integer such that k is an odd integer, $\left[\begin{bmatrix} 6 \\ 3 \end{bmatrix}\right]_F = 6$ is the only possibility integer value of double Fibonomial. Thus $(m, k, n) = (6, 3, 5)$ will be a solution of the given Diophantine equation for $'+'$ case.

Now, we can narrow down our possible solution to the even integers for both m and k. Since $F_a L_b = F_{a+b} + (-1)^b F_{a-b}$, the different factorizations for $F_n \pm 1$ depending on the class of $n modulo 4$ can be written as:

$$F_{4l} + 1 = F_{2l-1} L_{2l+1} \quad F_{4l} - 1 = F_{2l+1} L_{2l-1}$$

$$F_{4l+1} + 1 = F_{2l+1} L_{2l} \quad F_{4l+1} - 1 = F_{2l} L_{2l+1}$$

$$F_{4l+2} + 1 = F_{2l+2} L_{2l} \quad F_{4l+2} - 1 = F_{2l} L_{2l+2}$$

$$F_{4l+3} + 1 = F_{2l+1} L_{2l+2} \quad F_{4l+3} - 1 = F_{2l+2} L_{2l+1}$$

Therefore, the considered Diophantine equation, which can also be written as $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_n \mp 1$, can be factorized for the $'+'$ case as
$\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_{2l+1}L_{2l-1}$ $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_{2l}L_{2l+1}$ $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_{2l}L_{2l+2}$
$\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_{2l+2}L_{2l+1}$;
and for the $'-'$ case as
$\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_{2l-1}L_{2l+1}$ $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_{2l+1}L_{2l}$ $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_{2l+2}L_{2l}$
$\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_{2l+1}L_{2l+2}$;

It is obvious that all these eight cases can be handled in the similar manner. Thus, we shall only focus on the proof of the first case. Now, $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F = F_{2l+1}L_{2l-1}$ implies $\frac{F_m \times F_{m-2} \times \cdots \times F_{m-k+2}}{F_k \times F_{k-2} \times \cdots \times F_2} = F_{2l+1}L_{2l-1}$. Thus, we have

$$F_m \times F_{m-2} \times \cdots \times F_{m-k+2} = F_{2l+1} \times L_{2l-1} \times F_k \times F_{k-2} \times \cdots \times F_2$$

Since $F_2n = F_nL_n$, we write $L_{2l-1} = \frac{F_{4l-2}}{F_{2l-1}}$. Thus

$$F_m \times F_{m-2} \times \cdots \times F_{m-k+2} \times F_{2l-1} = F_{2l+1} \times F_{4l-2} \times F_k \times F_{k-2} \times \cdots \times F_2.$$

Since $l = \left\lfloor \frac{n}{4} \right\rfloor > 2$, we have$4l-2 > 2l+1$. Therefore, from primitive divisor theorem, we can write $m = 4l-2$. Thus,

$$F_{m-2} \times \cdots \times F_{m-k+2} \times F_{2l-1} = F_{2l+1} \times F_k \times F_{k-2} \times \cdots \times F_2 \tag{8}$$

If we assume that $m \geq max\{14, k+1\}$, we have $m-2 \geq 12$. So, again by primitive divisor theorem, we get $m-2 = max\{2l+1, k\}$. But $m-2 = 4l-4 > 2l+1$, which implies $m-2 = k$ and from (8), we get $F_{2l-1} = F_{2l+1}$, which is not possible. Thus, we only need to consider the range $4 \leq k \leq 10$ and $k+2 \leq m \leq 12$.

Again, from Table 2, we can easily claim that the only solution of the Diophantine equation $\left[\left[\begin{matrix} m \\ k \end{matrix}\right]\right]_F \pm 1 = F_n$ are $(m,k,n) = (3,1,2), (3,2,2), (4,2,3)$, $(6,3,5), (8,4,10)$ for $'+'$ case and $(3,1,4), (3,2,4)$ for $'-'$ case. □

5 Conclusion:

In this paper, we have defined double Fibonorial numbers and double Fibonomial numbers. We have proved many properties for these numbers including recursive equations in terms of Fibonacci numbers and Lucas numbers. We have extended the star of David theorem for double Fibonomial numbers and also discussed various Diophantine equations related to double Fibonomial numbers and Fibonacci numbers.

6 Acknowledgement:

The authors are thankful to the Department of Science and Technology for providing financial support under WOS – A fellowship.

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