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Classifications of Hyper Pseudo BCK-algebras of Order 3

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Abstract

In this paper by considering the notion of hyper pseudo BCK-algebra, we classify the set of all non-isomorphic hyper pseudo BCK-algebras of order 3. For this, we define the notion of simple and normal condition and we characterize the all of hyper pseudo BCK-algebras of order 3 that satisfies these conditions.

Key words: Hyper pseudo BCK-algebra, Simple condition, Normal condition.

2000 AMS subject classifications: 97U99.

1 Introduction

The study of BCK-algebras was initiated at 1966 by Y. Imai and K. Iséki in [5] as a generalization of the concept of set-theoretic difference and propositional calculi. In order to extend BCK-algebras in a noncommutative form, Georgescu and Iorgulescu [4] introduced the notion of pseudo BCK-algebras and studied their properties. The hyperstructure theory (called also multialgebra) was introduced in 1934 by F. Marty [10] at the 8th Congress of Scandinavian Mathematicians. Since then many researchers have worked on algebraic hyperstructures and developed it. A recent book [3] contains a wealth of applications. Via this book, Corsini and Leoreanu presented some of the numerous applications of algebraic hyperstructures, especially those from the last fifteen years, to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [1, 9], R. A. Borzooei et al. applied the hyperstructures to (pseudo) BCK-algebras and introduced the notion of hyper (pseudo) BCK-algebra which is a generalization of (pseudo) BCK-algebra and investigated some related properties. In [2], R. A. Borzooei et al. classified all hyper BCK-algebras of order 3. Now, in this paper we classify the set of all non-isomorphic hyper pseudo BCK-algebras of order 3.

2 Preliminaries

Definition 2.1. *[9] By a hyper BCK-algebra we mean a nonempty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfy the following axioms:*

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x \circ H \ll \{x\}$,

(HK4) $x \ll y$ *and* $y \ll x$ *imply* $x = y$.

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$. In such case, we call " $\ll$ " the hyperorder in H.

Definition 2.2. *[1] A hyper pseudo BCK-algebra is a structure $(H, \circ, *, 0)$ where "$*$" and "$\circ$" are hyper operations on H and "0" is a constant element, that satisfies the following: for all $x, y, z \in H$,*

(PHK1) $(x \circ z) \circ (y \circ z) \ll, x \circ y$, $(x * z) * (y * z) \ll x * y$,

(PHK2) $(x \circ y) * z = (x * z) \circ y$,

(PHK3) $x \circ H \ll \{x\}$, $x * H \ll \{x\}$,

(PHK4) $x \ll y$ *and* $y \ll x$ *imply* $x = y$.

where $x \ll y \Leftrightarrow 0 \in x \circ y \Leftrightarrow 0 \in x * y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$.

Theorem 2.1. *[2, 9] Any BCK-algebra and hyper BCK-algebra is a hyper pseudo BCK-algebra.*

Proposition 2.1. *[1] In any hyper pseudo BCK-algebra H, the following hold:*

(i) $0 \circ 0 = \{0\}$, $0 * 0 = \{0\}$, $x \circ 0 = \{x\}$, $x * 0 = \{x\}$,

(ii) $0 \ll x$, $x \ll x$, $A \ll A$,

(iii) $0 \circ x = \{0\}$, $0 * x = \{0\}$, $0 \circ A = \{0\}$, $0 * A = \{0\}$.

for all $x, y, z \in H$ and for all nonempty subsets A and B of H.

Theorem 2.2. *[2] There are 19 non-isomorphic hyper BCK-algebras of order 3.*

Note: From now on, in this paper $H = \{0, a, b\}$ is a hyper pseudo BCK-algebra of order 3, unless otherwise state.

3 Characterization of Hyper Pseudo BCK-algebras of Order 3

Definition 3.1. *[2] We say that H satisfies the normal condition if one of the conditions $a \ll b$ or $b \ll a$ holds. If no one of these conditions hold, then we say that H satisfies the simple condition.*

Definition 3.2. *Let $(H_1, \circ_1, *_1, 0_1)$ and $(H_2, \circ_2, *_2, 0_2)$ be two hyper pseudo BCK-algebras and $f : H_1 \to H_2$ be a function. Then f is said to be a homomorphism iff*

(i) $f(0_1) = 0_2$

(ii) $f(x \circ_1 y) = f(x) \circ_2 f(y), \quad \forall x, y \in H_1$

(iii) $f(x *_1 y) = f(x) *_2 f(y), \quad \forall x, y \in H_1$.

If f is one to one (onto) we say that f is a monomorphism (epimorphism) and if f is both one to one and onto, we say that f is an isomorphism.

Definition 3.3. *Let $I \subseteq H$. Then we say that I is a proper subset of H if $I \neq \{0\}$ and $I \neq H$.*

3.1 Characterization of hyper pseudo BCK-algebras of order 3 that satisfy the simple condition

Theorem 3.1. There are only 10 hyper pseudo BCK-algebras of order 3, that satisfy the simple condition.

Proof. Let H satisfy the simple condition. Now, we prove the following statements:

(i) For all $x, y \in H$ which $x \neq y$, then $x \notin y \circ x$ and $x \notin y * x$.

(ii) $a \circ b = a * b = \{a\}$ and $b \circ a = b * a = \{b\}$.

(iii) $a \circ a$ and $a * a$ are equal to $\{0\}$ or $\{0, a\}$ and $b \circ b$ and $b * b$ are equal to $\{0\}$ or $\{0, b\}$.

For the proof of (i), let $x \neq y$ and $x \in y \circ x$, by the contrary. Clearly $x \neq 0$. Because if $x = 0$, then $y \neq 0$ and $0 \in y \circ 0 = \{y\}$, which is impossible. Moreover, since $y \circ x \leq y$, then $x \leq y$ which is impossible by the simplicity of H. By the similar way, we can prove that $x \notin y * x$.

(ii) Since $a \not\leq b$, then $0 \notin a \circ b$ and $0 \notin a * b$. Hence $a \circ b$ and $a * b$ can not be equal to $\{0\}, \{0, a\}, \{0, b\}$ or $\{0, a, b\}$. Since by (i), we have $b \notin a \circ b$ and $b \notin a * b$, we conclude that $a \circ b$ and $a * b$ can not be equal to $\{b\}$ or $\{a, b\}$. Thus $a \circ b = a * b = \{a\}$. By the similar way, we can prove that $b \circ a = b * a = \{b\}$.

(iii) Since $a \ll a$, the only cases for $a \circ a$ and $a * a$ are $\{0\}, \{0, a\}, \{0, b\}$ or $\{0, a, b\}$. Also we have $a \circ a \leq a$ and $a * a \leq a$. Thus $b \notin a \circ a$ and $b \notin a * a$. Hence the only cases for $a \circ a$ and $a * a$ are $\{0\}$ or $\{0, a\}$. By the similar way, we can prove that $b \circ b$ and $b * b$ are equal to $\{0\}$ or $\{0, b\}$.

Therefore, by (i), (ii) and (iii) we conclude that there are 16 hyper pseudo BCK-algebras of order 3, which satisfy the simple condition. But some of them are isomorphic under the map $f : H \to H$ which is defined by $f(0) = 0$, $f(a) = b$ and $f(b) = a$. Hence there are 10 hyper pseudo BCK-algebras of order 3, that satisfy the simple condition. Now, we give these hyper pseudo BCK-algebras:

$\circ_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0}

$*_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0}

$\circ_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0}

$*_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$\circ_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0}

$*_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0,b}

$\circ_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$*_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0}

$\circ_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$*_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$\circ_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$*_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0,b}

$\circ_7$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$*_7$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0}

$\circ_8$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0,b}

$*_8$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0}

$\circ_9$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0,b}

$*_9$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{a}
b	{b}	{b}	{0,b}

$\circ_{10}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0,b}

$*_{10}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{a}
b	{b}	{b}	{0,b}

3.2 Characterization of hyper pseudo BCK-algebras of order 3 that satisfy the normal condition

Note: From now on, in this section we let $H = \{0, a, b\}$ satisfies the normal condition. Since in this condition, $a \leq b$ or $b \leq a$, so without loss of generality we let $a \leq b$ and $b \not\leq a$ i.e., $0 \in a \circ b \cap a * b$, $0 \notin b \circ a$ and $0 \notin b * a$.
Lemma 3.1. Only one of the following cases hold for $b * a$ and $b \circ a$.

(NPHB1) $b \circ a = b * a = \{a\}$,

(NPHB2) $b \circ a = \{a\}, b * a = \{a, b\}$,

(NPHB3) $b \circ a = b * a = \{b\}$,

(NPHB4) $b \circ a = \{a, b\}, b * a = \{a\}$,

(NPHB5) $b \circ a = b * a = \{a, b\}$.

Proof. Since $0 \notin b \circ a$ and $0 \notin b * a$, then $b \circ a$ and $b * a$ are equal to one of the sets $\{a\}, \{b\}$ or $\{a, b\}$.

If $b \circ a = \{a\}$, then $b * a \neq \{b\}$. Since if $b * a = \{b\}$, then $(b \circ a) * a \neq (b * a) \circ a$. Hence $b * a = \{a\}$ or $\{a, b\}$.

If $b \circ a = \{b\}$, then $b * a \neq \{a, b\}$ and $\{a\}$. Since if $b * a = \{a, b\}$ or $\{a\}$, then $(b \circ a) * a \neq (b * a) \circ a$. Hence $b * a = \{b\}$.

If $b \circ a = \{a, b\}$, then $b * a \neq \{b\}$. Since if $b * a = \{b\}$, then $(b \circ a) * a \neq (b * a) \circ a$. Hence $b * a = \{a\}$ or $\{a, b\}$. Therefore, we have the above cases.
Lemma 3.2. Only one of the following cases hold for $a * b$ and $a \circ b$.

(NPHA1) $a \circ b = a * b = \{0\}$.

(NPHA2) $a \circ b = a * b = \{0, a\}$.

(NPHA3) $a \circ b = \{0\}, a * b = \{0, a\}$.

(NPHA4) $a \circ b = \{0, a\}, a * b = \{0\}$.

Proof. Since $0 \in a \circ b \cap a * b$, then $a \circ b$ and $a * b$ are equal to one of the sets $\{0\}, \{0, a\}, \{0, b\}$ or $\{0, a, b\}$. Moreover, since $a \circ b \leq a$ and $a * b \leq a$, then $b \notin a \circ b$ and $b \notin a * b$. Hence the only cases for $a \circ b$ and $a * b$ are $\{0\}$ or $\{0, a\}$. Therefore, we have the above cases.
Lemma 3.3. Only one of the following cases hold for $a * a$ and $a \circ a$.

(i) $a \circ a = a * a = \{0\}$,

(ii) $a \circ a = \{0\}, a * a = \{0, a\}$,

(iii) $a \circ a = \{0, a\}, a * a = \{0\}$,

(iv) $a \circ a = a * a = \{0, a\}$.

Proof. Since $0 \in a \circ a \cap a * a$, then $a \circ a$ and $a * a$ can be equal to the one of cases $\{0\}, \{0, a\}, \{0, b\}$ or $\{0, a, b\}$. Moreover, since $a \circ a \leq a$ and $a * a \leq a$, then $b \notin a \circ a$ and $b \notin a * a$. Hence the only cases for $a \circ a$ and $a * a$ are $\{0\}$ or $\{0, a\}$. Therefore, we have the above cases.

Theorem 3.4. There are only 5 non-isomorphic hyper pseudo BCK-algebras of order 3, that satisfy the normal condition and condition (NPHB1).

Proof. Since H satisfies the condition (NPHB1), then $b \circ a = b * a = \{a\}$.

Case (NPHA1): We have $a * b = a \circ b = \{0\}$. If $a \circ a = \{0, a\}$ or $a * a = \{0, a\}$, then $(a \circ a) \circ (b \circ a) \not\ll a \circ b$ and $(a * a) * (b * a) \not\ll a * b$. Therefore, $a \circ a = a * a = \{0\}$. Moreover, if $b \circ b$ and $b * b$ are equal to the one of sets $\{0, b\}$ or $\{0, a, b\}$, then $(b \circ b) \circ (a \circ b) \not\ll b \circ a$ and $(b * b) * (a * b) \not\ll b * a$. Therefore, in this case $b \circ b$ and $b * b$ are equal to the one of sets $\{0\}$ or $\{0, a\}$. Now, we consider the following cases:

(1) $b \circ b = b * b = \{0\}$. Thus in this case, we have the following tables:

$*_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0}

$\circ_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0}

(2) $b \circ b = \{0\}$ and $b * b = \{0, a\}$. Thus in this case, we have the following tables:

$*_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0}

$\circ_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0,a}

(3) $b \circ b = \{0, a\}$ and $b * b = \{0\}$. Thus similar to (2), we have one hyper pseudo BCK-algebra in this case.

(4) $b \circ b = b * b = \{0, a\}$. Thus in this case, we have the following tables:

$*_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0,a}

$\circ_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0,a}

Case (NPHA2): We have $a * b = a \circ b = \{0, a\}$. If $a \circ a = \{0\}$ or $a * a = \{0\}$, then $(a \circ b) \circ (a \circ b) \not\ll a \circ a$ and $(a * b) * (a * b) \not\ll a * a$. Therefore, $a \circ a = a * a = \{0, a\}$. Moreover, if $b \circ b$ and $b * b$ are equal to the one of sets $\{0, b\}$

or $\{0,a,b\}$, then $(b \circ b) \circ (a \circ b) \not\ll b \circ a$ and $(b * b) * (a * b) \not\ll b * a$ and if $b \circ b = \{0\}$ or $b * b = \{0\}$, then $(b \circ a) \circ (b \circ a) \not\ll b \circ b$ and $(b * a) * (b * a) \not\ll b * b$. Therefore, $b \circ b = b * b = \{0,a\}$. Thus we have the following case:

$*_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a}	{0,a}

$\circ_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a}	{0,a}

Case (NPHA3): We have $a * b = \{0,a\}$ and $a \circ b = \{0\}$. Since $(a * b) * (a * b) \ll a * a$, then $a * a = \{0,a\}$. Moreover, since $a \circ b = \{0\}$, then $a \circ a \neq \{0,a\}$, because $(a \circ a) \circ (b \circ a) \not\ll a \circ b$. Thus $a \circ a = \{0\}$ and in this case $(b * a) \circ a \neq (b \circ a) * a$.

Case (NPHA4): We have $a * b = \{0\}$ and $a \circ b = \{0,a\}$. In this case we can show that $(b * a) \circ a \neq (b \circ a) * a$. Therefore, we have not any hyper pseudo BCK-algebras.

We can check that all of the these 5 cases are hyper pseudo BCK-algebras and each of them are not isomorphic together.

Theorem 3.5. There are only 6 non-isomorphic hyper pseudo BCK-algebras of order 3, that satisfy the normal condition and condition (NPHB2).

Proof. Since H satisfies the condition (NPHB2), then $b \circ a = \{a\}$ and $b * a = \{a,b\}$.

Case (NPHA1): We have $a * b = a \circ b = \{0\}$. If one of the $a \circ a$ or $a * a$ are equal to $\{0,a\}$, then $(a \circ a) \circ (b \circ a) \not\ll a \circ b$ and $(a * a) * (b * a) \not\ll a * a$. Hence we have $a * a = a \circ a = \{0\}$. But in this case $(b * a) \circ a \neq (b \circ a) * a$. Therefore, we have not any hyper pseudo BCK-algebras in this case.

Case (NPHA2): We have $a * b = a \circ b = \{0,a\}$. If $a \circ a = \{0\}$ or $a * a = \{0\}$, then $(a \circ b) \circ (a \circ b) \not\ll a \circ a$ and $(a * b) * (a * b) \not\ll a * a$. Therefore, $a \circ a = a * a = \{0,a\}$. Moreover, if $b * b$ is equal to the one of sets $\{0\}$ or $\{0,a\}$, then $(b * a) * (b * a) \not\ll b * b$. Therefore, in this case $b * b$ is equal to the one of sets $\{0,b\}$ or $\{0,a,b\}$ and if $b \circ b$ is equal to the one of sets $\{0,b\}$ or $\{0,a,b\}$, then $(b \circ b) \circ (a \circ b) \not\ll b \circ a$. Hence in this case $b \circ b$ is equal to the one of sets $\{0\}$ or $\{0,a\}$. Moreover, since $a \circ a = \{0,a\}$, if $b \circ b = \{0\}$, then $(b \circ a) \circ (b \circ a) \not\ll b \circ b$. Thus $b \circ b = \{0,a\}$. Now, we consider the following cases:

(1) $b \circ b = \{0,a\}$ and $b * b = \{0,b\}$. Thus in this case, we have the following tables:

$\circ_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a}	{0,a}

$*_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

(2) $b \circ b = \{0, a\}$ and $b * b = \{0, a, b\}$. Thus in this case, we have the following tables:

$\circ_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a}	{0,a}

$*_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

Case (NPHA3): We have $a * b = \{0, a\}$ and $a \circ b = \{0\}$. If $a * a = \{0\}$ and $a \circ a$ is equal to $\{0\}$ or $\{0, a\}$, then $(a * b) * (a * b) \not\ll a * a$. Thus $a * a = \{0, a\}$. Moreover, if $b * b$ is equal to the one of sets $\{0\}$ or $\{0, a\}$, then $(b * a) * (b * a) \not\ll b * b$. Therefore, in this case $b * b$ is equal to the one of sets $\{0, b\}$ or $\{0, a, b\}$ and if $b \circ b$ is equal to the one of sets $\{0, b\}$ or $\{0, a, b\}$, then $(b \circ b) \circ (a \circ b) \not\ll b \circ a$. Thus in this case $b \circ b$ is equal to the one of sets $\{0\}$ or $\{0, a\}$. Now, we consider the following cases:

(1) $b \circ b = \{0\}$ and $b * b = \{0, b\}$. If $a \circ a = \{0, a\}$, then $(b \circ a) \circ (b \circ a) \not\ll b \circ b$. Hence $a \circ a = \{0\}$. Thus in this case, we have the following tables:

$\circ_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0}

$*_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

(2) $b \circ b = \{0\}$ and $b * b = \{0, a, b\}$. If $a \circ a = \{0, a\}$, then $(b \circ a) \circ (b \circ a) \not\ll b \circ b$. Hence $a \circ a = \{0\}$. Thus in this case, we have the following tables:

$\circ_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0}

$*_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

(3) $b \circ b = \{0, a\}$ and $b * b = \{0, b\}$. If $a \circ a = \{0, a\}$, then $(a \circ a) \circ (b \circ a) \not\ll a \circ b$. Hence $a \circ a = \{0\}$. Thus in this case, we have the following tables:

$\circ_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0,a}

$*_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

(4) $b \circ b = \{0, a\}$ and $b * b = \{0, a, b\}$. If $a \circ a = \{0, a\}$, then $(a \circ a) \circ (b \circ a) \not\ll a \circ b$. Hence $a \circ a = \{0\}$. Thus in this case, we have the following tables:

$\circ_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a}	{0,a}

$*_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

Case (NPHA4): We have $a * b = \{0\}$ and $a \circ b = \{0, a\}$. If $a \circ a = \{0\}$ and $a * a$ is equal to $\{0\}$ or $\{0, a\}$, then $(a \circ b) \circ (a \circ b) \not\ll a \circ a$. Thus $a \circ a = \{0, a\}$. Moreover, if $b * b$ is equal to the one of sets $\{0\}$ or $\{0, a\}$, then $(b * a) * (b * a) \not\ll b * b$. Therefore, in this case $b * b$ is equal to the one of sets $\{0, b\}$ or $\{0, a, b\}$. Moreover, if $a * a = \{0, a\}$, then $(a * a) * (b * a) \not\ll a * b$. Hence $a * a = \{0\}$. But in this case $(b \circ a) * b \neq (b * b) \circ a$. Therefore, we have not any hyper pseudo BCK-algebras.

We can check that all of the these 6 cases are hyper pseudo BCK-algebras and each of them are not isomorphic together.

Theorem 3.6. There are only 70 non-isomorphic hyper pseudo BCK-algebras of order 3, that satisfy the normal condition and condition (NPHB3).

Proof. Since H satisfies the condition (NPHB3), then $b \circ a = b * a = \{b\}$.

Case (NPHA1): We have $a * b = a \circ b = \{0\}$. Now, we consider the following cases:

(1) $b \circ b = b * b = \{0\}$. In this case, we have the following tables:

$\circ_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$*_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$\circ_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$*_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$\circ_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$\circ_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_4$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

(2) $b \circ b = \{0\}$ and $b * b = \{0, b\}$. In this case, we have the following tables:

$\circ_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$*_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$\circ_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$*_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

$\circ_7$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_7$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$\circ_8$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_8$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

(3) $b * b = \{0\}$ and $b \circ b = \{0, b\}$. Similar to (2), we have four hyper pseudo BCK-algebras in this case.

(4) $b \circ b = \{0\}$ and $b * b = \{0, a\}$. If $a \circ a = \{0\}$, then $(b \circ a) * b \neq (b * b) \circ a$. Thus in this case, we have the following tables:

$\circ_{13}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{13}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,a}

$\circ_{14}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{14}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

(5) $b * b = \{0\}$ and $b \circ b = \{0, a\}$. If $a * a = \{0\}$, then $(b * a) \circ b \neq (b \circ b) * a$. Thus similar to (4), we have two hyper pseudo BCK-algebras in this case.

(6) $b \circ b = \{0\}$ and $b * b = \{0, a, b\}$. If $a \circ a = \{0\}$, then $(b \circ a) * b \neq (b * b) \circ a$. Thus in this case, we have the following tables:

$\circ_{17}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{17}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,a,b}

$\circ_{18}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{18}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

(7) $b*b = \{0\}$ and $b \circ b = \{0, a, b\}$. If $a*a = \{0\}$, then $(b*a) \circ b \neq (b \circ b)*a$. Thus similar to (6), we have two hyper pseudo BCK-algebras in this case.

(8) $b \circ b = b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{21}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$*_{21}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$\circ_{22}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$*_{22}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

$\circ_{23}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

$*_{23}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$\circ_{24}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

$*_{24}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

(9) $b \circ b = \{0, b\}$ and $b * b = \{0, a\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(10) $b * b = \{0, b\}$ and $b \circ b = \{0, a\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(11) $b \circ b = \{0, b\}$ and $b * b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(12) $b * b = \{0, b\}$ and $b \circ b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(13) $b \circ b = b * b = \{0, a\}$. If $a \circ a = \{0\}$, then $(b \circ a) * b \neq (b * b) \circ a$ and if $a * a = \{0\}$, then $(b * a) \circ b \neq (b \circ b) * a$. Therefore, $a \circ a = a * a = \{0, a\}$. Thus in this case, we have the following tables:

$\circ_{25}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

$*_{25}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

(14) $b \circ b = \{0, a\}$ and $b * b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(15) $b * b = \{0, a\}$ and $b \circ b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(16) $b \circ b = b * b = \{0, a, b\}$. If $a \circ a = \{0\}$, then $(b \circ a) * b \neq (b * b) \circ a$ and if $a * a = \{0\}$, then $(b * a) \circ b \neq (b \circ b) * a$. Therefore, $a \circ a = a * a = \{0, a\}$. Thus in this case, we have the following tables:

$\circ_{26}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

$*_{26}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

Case (NPHA2): We have $a * b = a \circ b = \{0, a\}$. If one of the $a * a$ or $a \circ a$ are equal to $\{0\}$, then $(a * b) * (a * b) \not\ll a * a$ or $(a \circ b) \circ (a \circ b) \not\ll a \circ a$. Hence $a * a = a \circ a = \{0, a\}$.

(1) $b \circ b = b * b = \{0\}$. Thus in this case, we have the following tables:

$\circ_{27}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$*_{27}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

(2) $b \circ b = \{0\}$ and $b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{28}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$*_{28}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

(3) $b * b = \{0\}$ and $b \circ b = \{0, b\}$. Thus similar to (2), we have one hyper pseudo BCK-algebra in this case.

(4) $b \circ b = \{0\}$ and $b * b = \{0, a\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(5) $b * b = \{0\}$ and $b \circ b = \{0, a\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(6) $b \circ b = \{0\}$ and $b * b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(7) $b * b = \{0\}$ and $b \circ b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(8) $b \circ b = b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{30}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{30}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

(9) $b \circ b = \{0, b\}$ and $b * b = \{0, a\}$. Thus in this case, we have the following tables:

$\circ_{31}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{31}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

(10) Let $b*b=\{0,b\}$ and $b\circ b=\{0,a\}$. Thus similar to (9), we have one hyper pseudo BCK-algebra in this case.

(11) $b\circ b=\{0,b\}$ and $b*b=\{0,a,b\}$. Thus in this case, we have the following tables:

$\circ_{33}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{33}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

(12) $b*b=\{0,b\}$ and $b\circ b=\{0,a,b\}$. Thus similar to (11), we have one hyper pseudo BCK-algebra in this case.

(13) $b\circ b=b*b=\{0,a\}$. Thus in this case, we have the following tables:

$\circ_{35}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

$*_{35}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

(14) $b\circ b=\{0,a\}$ and $b*b=\{0,a,b\}$. Thus in this case, we have the following tables:

$\circ_{36}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

$*_{36}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

(15) $b*b=\{0,a\}$ and $b\circ b=\{0,a,b\}$. Thus similar to (14), we have one hyper pseudo BCK-algebra in this case.

(16) $b\circ b=b*b=\{0,a,b\}$. Thus in this case, we have the following tables:

$\circ_{38}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

$*_{38}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

Case (NPHA3): We have $a*b=\{0,a\}$ and $a\circ b=\{0\}$. If $a*a=\{0\}$ and $a\circ a$ is equal to $\{0\}$ or $\{0,a\}$, then $(a*b)*(a*b)\not\ll a*a$. Thus $a*a=\{0,a\}$.

(1) $b\circ b=b*b=\{0\}$. Thus in this case, we have the following tables:

$\circ_{39}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$*_{39}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$\circ_{40}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{40}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

(2) $b \circ b = \{0\}$ and $b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{41}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$*_{41}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$\circ_{42}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{42}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

(3) $b * b = \{0\}$ and $b \circ b = \{0, b\}$. Thus similar to (2), we have two hyper pseudo BCK-algebras in this case.

(4) $b \circ b = \{0\}$ and $b * b = \{0, a\}$. If $a \circ a = \{0\}$ then $(b \circ a) * b \neq (b * b) \circ a$. Thus in this case, we have the following tables:

$\circ_{45}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{45}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

(5) $b * b = \{0\}$ and $b \circ b = \{0, a\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(6) $b \circ b = \{0\}$ and $b * b = \{0, a, b\}$. If $a \circ a = \{0\}$, then $(b \circ a) * b \neq (b * b) \circ a$. Thus in this case, we have the following tables:

$\circ_{46}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

$*_{46}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

(7) $b * b = \{0\}$ and $b \circ b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(8) $b \circ b = b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{47}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$*_{47}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$\circ_{48}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

$*_{48}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

(9) $b \circ b = \{0, b\}$ and $b * b = \{0, a\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.

(10) $b * b = \{0, b\}$ and $b \circ b = \{0, a\}$. Thus in this case, we have the following tables:

$\circ_{49}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,a}

$*_{49}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$\circ_{50}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

$*_{50}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

(11) $b \circ b = \{0, b\}$ and $b * b = \{0, a, b\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.

(12) $b * b = \{0, b\}$ and $b \circ b = \{0, a, b\}$. Thus in this case, we have the following tables:

$\circ_{51}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,a,b}

$*_{51}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$\circ_{52}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

$*_{52}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

(13) $b \circ b = b * b = \{0, a\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.

(14) $b \circ b = \{0, a\}$ and $b * b = \{0, a, b\}$. If $a \circ a = \{0\}$, then $(b \circ a) * b \neq (b * b) \circ a$. Thus in this case, we have the following tables:

$\circ_{53}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

$*_{53}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

(15) $b * b = \{0, a\}$ and $b \circ b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(16) $b \circ b = b * b = \{0, a, b\}$. If $a \circ a = \{0\}$, then $(b \circ a) * b \neq (b * b) \circ a$. Thus in this case, we have the following tables:

$\circ_{54}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

$*_{54}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

Case (NPHA4): We have $a \circ b = \{0, a\}$ and $a * b = \{0\}$. If $a \circ a = \{0\}$ and $a * a$ is equal to $\{0\}$ or $\{0, a\}$, then $(a \circ b) \circ (a \circ b) \not\ll a \circ a$. Thus $a \circ a = \{0, a\}$.

(1) $b \circ b = b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{55}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$*_{55}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0}

$\circ_{56}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$*_{56}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

(2) $b \circ b = \{0\}$ and $b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{57}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$*_{57}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$\circ_{58}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0}

$*_{58}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

(3) $b * b = \{0\}$ and $b \circ b = \{0, b\}$. Thus similar to (2), we have two hyper pseudo BCK-algebras in this case.

(4) $b \circ b = \{0\}$ and $b * b = \{0, a\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(5) $b * b = \{0\}$ and $b \circ b = \{0, a\}$. If $a * a = \{0\}$, then $(b * a) \circ b \neq (b \circ b) * a$. Thus in this case, we have the following tables:

$\circ_{61}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a}

$*_{61}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

(6) $b \circ b = \{0\}$ and $b * b = \{0, a, b\}$. Then $(b \circ b) * b \neq (b * b) \circ b$.

(7) $b*b = \{0\}$ and $b\circ b = \{0, a, b\}$. If $a*a = \{0\}$, then $(b*a)\circ b \neq (b\circ b)*a$. Thus in this case, we have the following tables:

$\circ_{62}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

$*_{62}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0}

(8) $b \circ b = b * b = \{0, b\}$. Thus in this case, we have the following tables:

$\circ_{63}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{63}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,b}

$\circ_{64}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{64}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,b}

(9) $b \circ b = \{0, b\}$ and $b * b = \{0, a\}$. Thus in this case, we have the following tables:

$\circ_{65}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{65}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,a}

$\circ_{66}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{66}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

(10) $b * b = \{0, b\}$ and $b \circ b = \{0, a\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.

(11) $b \circ b = \{0, b\}$ and $b * b = \{0, a, b\}$. Thus in this case, we have the following tables:

$\circ_{67}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{67}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{b}	{0,a,b}

$\circ_{68}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,b}

$*_{68}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

(12) $b * b = \{0, b\}$ and $b \circ b = \{0, a, b\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.
(13) $b \circ b = b * b = \{0, a\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.
(14) $b \circ b = \{0, a\}$ and $b * b = \{0, a, b\}$. Then $(b * b) \circ b \neq (b \circ b) * b$.
(15) $b * b = \{0, a\}$ and $b \circ b = \{0, a, b\}$. If $a * a = \{0\}$, then $(b * a) \circ b \neq (b \circ b) * a$. Thus in this case, we have the following tables:

$\circ_{69}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

$*_{69}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a}

(16) $b \circ b = b * b = \{0, a, b\}$. If $a * a = \{0\}$, then $(b * a) \circ b \neq (b \circ b) * a$. Thus in this case, we have the following tables:

$\circ_{70}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{b}	{0,a,b}

$*_{70}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0}
b	{b}	{b}	{0,a,b}

We can check that all of the these 70 cases are hyper pseudo BCK-algebras and each of them are not isomorphic together.

Theorem 3.7. There are only 6 non-isomorphic hyper pseudo BCK-algebras of order 3, that satisfy the normal condition and condition (NPHB4).

Proof. The proof is the similar to the proof of Theorem 3.5, by the some modification.

Theorem 3.8. There are only 9 non-isomorphic hyper pseudo BCK-algebras of order 3, that satisfy the normal condition and condition (NPHB5).

Proof. Since H satisfies the condition (NPHB5), then $b \circ a = b * a = \{a, b\}$.

Case (NPHA1): We have $a * b = a \circ b = \{0\}$. If one of the $b \circ b$ or $b * b$ are equal to $\{0\}$, $\{0, a\}$ or $\{0, b\}$, then $(b \circ a) \circ (b \circ a) \not\ll b \circ b$ or $(b * a) * (b * a) \not\ll b * b$. Therefore, we have only the following case:

(1) $b \circ b = b * b = \{0, a, b\}$. If $a \circ a = \{0, a\}$, then $(a \circ a) \circ (b \circ a) \not\ll a \circ b$ and if $a * a = \{0, a\}$, then $(a * a) * (b * a) \not\ll a * b$. Therefore, $a \circ a = a * a = \{0\}$. Thus in this case, we have the following tables:

$\circ_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

$*_1$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

Case (NPHA2): We have $a * b = a \circ b = \{0, a\}$. If one of the $a * a$ or $a \circ a$ are equal to $\{0\}$, then $(a * b) * (a * b) \not\ll a * a$ or $(a \circ b) \circ (a \circ b) \not\ll a \circ a$. Hence $a * a = a \circ a = \{0, a\}$.

If $b \circ b = \{0\}$, but $b * b = \{0\}$, {0,b}, {0,a} or {0,a,b}, then $(b \circ a) \circ (b \circ a) \not\ll b \circ b$.

If $b \circ b = \{0, b\}$, but $b * b = \{0\}$ or $\{0, a\}$, then $(b * a) * (b * a) \not\ll b * b$. Hence $b * b = \{0, b\}$ or $\{0, a, b\}$.

If $b \circ b = \{0, a\}$, but $b * b = \{0\}$, $\{0, b\}$, $\{0, a\}$ or $\{0, a, b\}$, then $(b \circ a) \circ (b \circ a) \not\ll b \circ b$.

If $b \circ b = \{0, a, b\}$, but $b * b = \{0\}$ or $\{0, a\}$, then $(b * a) * (b * a) \not\ll b * b$. Hence $b * b = \{0, b\}$ or $\{0, a, b\}$. Therefore, we have the following cases:

(1) $b \circ b = b * b = \{0, b\}$. Thus we have the following tables:

$\circ_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

$*_2$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

(2) $b \circ b = \{0, b\}$ and $b * b = \{0, a, b\}$. Thus we have the following tables:

$\circ_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

$*_3$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

(3) $b \circ b = \{0, a, b\}$ and $b * b = \{0, b\}$. Thus similar to (2), we have one hyper pseudo BCK-algebra in this case.

(4) Let $b \circ b = b * b = \{0, a, b\}$. Thus we have the following tables:

$\circ_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

$*_5$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

<u>*Case (NPHA3)*</u>: We have $a*b=\{0,a\}$ and $a\circ b=\{0\}$. If $a*a=\{0\}$ and $a\circ a$ is equal to $\{0\}$ or $\{0,a\}$, then $(a*b)*(a*b)\not\ll a*a$. Thus $a*a=\{0,a\}$.

If $b\circ b=\{0\}$, but $b*b=\{0\}$, {0,b}, {0,a} or {0,a,b}, then $(b\circ a)\circ(b\circ a)\not\ll b\circ b$.

If $b\circ b=\{0,b\}$, but $b*b=\{0\}$ or $\{0,a\}$, then $(b*a)*(b*a)\not\ll b*b$. If $b\circ b=b*b=\{0,b\}$, then $(b*a)\circ b\neq(b\circ b)*a$. If $b\circ b=\{0,b\}$ and $b*b=\{0,a,b\}$, then $(b*b)\circ b\neq(b\circ b)*b$.

If $b\circ b=\{0,a\}$, but $b*b=\{0\}$, $\{0,b\}$ or $\{0,a,b\}$, then $(b\circ a)\circ(b\circ a)\not\ll b\circ b$. If $b\circ b=b*b=\{0,a\}$, then $(b*b)\circ b\neq(b\circ b)*b$.

If $b\circ b=\{0,a,b\}$, but $b*b=\{0\}$ or $\{0,a\}$, then $(b*a)*(b*a)\not\ll b*b$. Hence $b*b=\{0,b\}$ or $\{0,a,b\}$. Therefore, we have the following cases:

(1) $b\circ b=\{0,a,b\}$ and $b*b=\{0,b\}$. If $a\circ a=\{0,a\}$, then $(a\circ a)\circ(b\circ a)\not\ll a\circ b$. Thus in this case, we have the following tables:

$\circ_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

$*_6$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

(2) $b\circ b=b*b=\{0,a,b\}$. If $a\circ a=\{0,a\}$, then $(a\circ a)\circ(b\circ a)\not\ll a\circ b$. Thus in this case, we have the following tables:

$\circ_7$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

$*_7$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

<u>*Case (NPHA4)*</u>: We have $a\circ b=\{0,a\}$ and $a*b=\{0\}$. If $a\circ a=\{0\}$ and $a*a$ is equal to $\{0\}$ or $\{0,a\}$, then $(a\circ b)\circ(a\circ b)\not\ll a\circ a$. Thus $a\circ a=\{0,a\}$.

If $b\circ b=\{0\}$, but $b*b=\{0\}$, {0,b}, {0,a} or {0,a,b}, then $(b\circ a)\circ(b\circ a)\not\ll b\circ b$.

If $b\circ b=\{0,b\}$, but $b*b=\{0\}$ or $\{0,a\}$, then $(b*a)*(b*a)\not\ll b*b$. If $b\circ b=b*b=\{0,b\}$, then $(b\circ a)*b\neq(b*b)\circ a$. Hence $b*b=\{0,a,b\}$.

If $b\circ b=\{0,a\}$, but $b*b=\{0\}$, $\{0,b\}$ or $\{0,a,b\}$, then $(b\circ a)\circ(b\circ a)\not\ll b\circ b$. If $b\circ b=b*b=\{0,a\}$, then $(b*b)\circ b\neq(b\circ b)*b$.

If $b\circ b=\{0,a,b\}$, but $b*b=\{0\}$ or $\{0,a\}$, then $(b*a)*(b*a)\not\ll b*b$. If $b\circ b=\{0,a,b\}$, but $b*b=\{0,b\}$, then $(b*b)\circ b\neq(b\circ b)*b$. Hence $b*b=\{0,a,b\}$. Therefore, we have the following cases:

(1) $b\circ b=\{0,b\}$ and $b*b=\{0,a,b\}$. If $a*a=\{0,a\}$, then $(a*a)*(b*a)\not\ll a*b$. Thus in this step, we have the following case:

$\circ_8$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

$*_8$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

(2) $b \circ b = b * b = \{0, a, b\}$. If $a * a = \{0, a\}$, then $(a * a) * (b * a) \not\ll a * b$. Thus in this step, we have the following case:

$\circ_9$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

$*_9$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

We can check that all of the these 9 cases are hyper pseudo BCK-algebras and each of them are not isomorphic together.

Theorem 3.9. There are 96 hyper pseudo BCK-algebras of order 3, that satisfies the normal condition.

Proof. By Theorems 3.4, 3.5, 3.6, 3.7 and 3.8, the proof is clear.

4 Conclusion

Theorem 4.1. *There are 106 hyper pseudo BCK-algebras of order 3 up to isomorphism.*

Proof. The proof follows by Theorems 3.1 and 3.9.

Definition 4.1. *We say that H is a proper hyper pseudo BCK-algebra if H is not a hyper BCK-algebra.*

Theorem 4.2. *There are 87 proper hyper pseudo BCK-algebras of order 3 up to isomorphism.*

Proof. The proof follows by Theorems 2.5 and 4.1.

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