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On odd integers and their couples of divisors

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Abstract

A composite odd integer can be expressed as the product of two odd integers. Possibly, this decomposition is not unique. From $2n + 1 = (2i + 1)(2j + 1)$ it follows that $n = i + j + 2ij$. This form of n characterizes the composite odd integers. It allows the formulation of simple algorithms to compute all the couples of divisors of odd integers and to identify the odd integers with the same number of couples of divisors (including the primes, with the number of non trivial divisors equal to zero). The distributions of odd integers $\leq 2n+1$ vs. the number of their couples of divisors have been computed up to $n \simeq 5\ 10^7$, and the main features are illustrated.

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1 Introduction: characterization of composite odd and prime numbers

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{P}$ the set of prime numbers with the exception of 2. Composite odd integers $2n + 1$, $n \in \mathbb{N}$, may be expressed as product of two odd integers,

$$2n + 1 = (2i + 1)(2j + 1),\ i, j \in \mathbb{N}, \tag{1}$$

or of more than two odd integers, i.e. as product of two odd integers in different ways. The decomposition (1) implies that

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$$n = k_{ij} = i + j + 2ij, \tag{2}$$

which may also be rewritten in the form

$$n = k_{ij} = i(j + 1) + (i + 1)j. \tag{3}$$

Either equation (2) or (3) specifies the structure of a composite odd integer $2n+1$.

Let $\mathbb{K} \subset \mathbb{N}$ be the set of the integers k_{ij} $\forall i, j \in \mathbb{N}$. Since any odd integer $2n + 1$ greater than one is either a composite or a prime number, it follows that

$$n \in \mathbb{K} \quad \Longleftrightarrow \quad 2n + 1 \in \mathbb{N} \setminus \mathbb{P},$$

or, equivalently,

$$n \in \mathbb{N} \setminus \mathbb{K} \quad \Longleftrightarrow \quad 2n + 1 \in \mathbb{P}.$$

Remark. More involved characterizations of prime numbers can be formulated. They are obtained starting from the observation that all prime numbers greater than $c \in \mathbb{N}$, are of the form $c\#h + \iota$, where $c\#$ represents c primorial, $h, \iota \in \mathbb{N}$, and $\iota < c\#$ is coprime to $c\#$, i.e. $gcd(\iota, c\#) = 1$. As an example, let $c = 4$, $c\# = 6$; thus, all prime numbers > 4 may be expressed as $6h + \iota$ with $\iota = 1, 5$. Since $6h + 5 = 6(h + 1) - 1$, then all prime numbers may be expressed in the form $6h \pm 1$, with the exception of 2 and 3. Let the odd integer $2n + 1$ be written as $2n + 1 = 6h \pm 1$, so that either $n = 3h$ or $n = 3h - 1$. For composite integers $n = k_{ij}$, and consequently 3 should be a dvisor of either k_{ij} or $k_{ij} + 1$.

The paper is organized as follows. In section 2 varios formulations of the relationship between n and the pair (i, j) are viewed. An algorithm to compute the divisors of an odd integer is described; it can also be used as a primality test. In sections 3 and 4 it is shown how odd integers with the same number of couples of divisors can be identified. Moreover, the distributions of odd integers $\leq 2n+1$ vs. the number of their couples of divisors are computed up to $n = 5\ 10^7$ and illustrated. Some concluding remarks can be found in section 5. Details of calculations are reported in appendix.

2 The relationship between n and the pair (i, j)

The functional relationship between a composite integer $2n+1$ and the factors $2i + 1$, $2j + 1$ of its decompositions, or between n, i, j, can be written in different forms. The decomposition (1) is an inverse proportional relationship (hyperbolic relation) between $2i + 1$ and $2j + 1$. Here and in the following it is assumed that $i \leq j$, so that $2i + 1 \leq \sqrt{2n + 1} \leq 2j + 1$ (equality holds iff $i = j$), or equivalently

$$i \leq I_n = \frac{1}{2}(-1 + \sqrt{2n+1}) \leq j. \tag{4}$$

The relation (1) has been written in the forms (2) and (3). These equations define the entries of the matrix $K = \{k_{ij}\}$, used for the computation of the distribution of odd integers vs. the number of their couples of divisors. Properties of K can be found in appendix 1.

By making explicit the variable j, (2) can be written in the form of an homographic function

$$j = \phi_n(i) = \frac{n-i}{2i+1}, \quad 1 \leq i \leq I_n. \tag{5}$$

Thus, $2i + 1$ is a divisor of both $2n + 1$ and $n - i$. From (12) in appendix 1, it follows that $2i + 1$ is also a divisor of $n - k_{ii}$.

Equation (5) can be used to compute the couples of divisors of an integer $2n+1$ by means of the following algorithm:

given n, compute $\phi_n(i)$ for $i = 1, 2, ..., [I_n]$,
where $[\cdot]$ is the integer part of the real argument;
if for some $i = i_q$ we obtain that $j_q = \phi_n(i_q) \in \mathbb{N}$,
then $2i_q + 1 \leq \sqrt{2n+1} \leq 2j_q + 1$ is a couple of divisors of $2n + 1$.

The order of the number of operations is $\sqrt{n/2}$. The algorithm can also be used as a primality test: if the computed $\phi_n(i) \notin \mathbb{N}\ \forall i$, then $2n + 1$ is a prime.

The functions $y = \phi_n(x)$, $x+y$, xy, $y-x$, of the real variable x, are monotone for $0 \leq x \leq I_n$ (figure 1). I_n, defined in (4), is the unique positive solution to the equation $\phi_n(x) = x$, i.e. $2x^2 + 2x - n = 0$. The point $x = I_n$ corresponds to the minimum of $x + y$, to the maximum of xy, and, obviously, to $y - x = 0$.

By means of a change of variables, the relationship (1) can be put in the form

$$2n + 1 = (s+t)(s-t) = s^2 - t^2, \quad with\ s = i + j + 1,\ t = j - i, \tag{6}$$

while (2) and (3), representing partitions of the integer n in two sections, can be put in linear forms

$$n = s + 2t, \quad with\ s = i + j,\ t = ij, \tag{7}$$

$$n = s + t, \quad with\ s = i(j+1),\ t = (i+1)j. \tag{8}$$

Equation (6) shows the well known fact that composite odd integers can be written as a difference of two squares in different ways, while for a prime only holds the decomposition $2n + 1 = (n+1)^2 - n^2$.

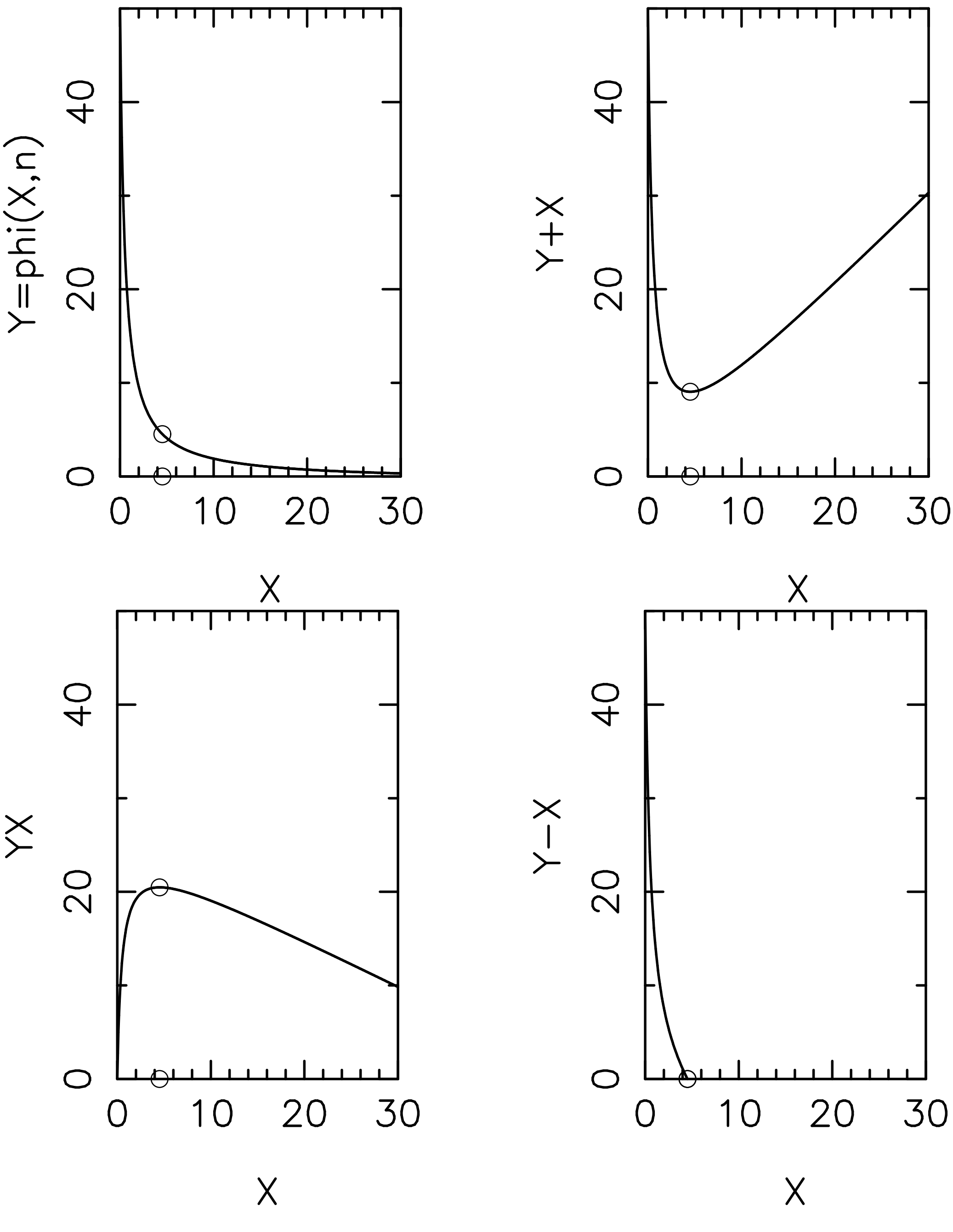


Figure 1: Top left $y = \phi_n(x)$, top right $x + y$, bottom left xy, bottom right $y - x$. Circle: point $x = I_n$ on the x axis, and corresponding points on the curves. $n = 50$, $I_n = 4.52$.

Given $n, s \in \mathbb{N}$, it is possible to prove when s and $t = n - s$ can be expressed as either in (7) or in (8). The details are reported in appendix 2: it is shown that i, j are solutions to second order equations, and they are integer satisfying either (7) or (8), iff the square root of a quadratic form in n and s is an integer,

3 Identification of odd integers $\leq 2n + 1$ with the same number of couples of divisors

Let $2m + 1$ be a composite integer and let

$\psi(m) =$ number of couples of divisors of $2m + 1$.

Obviously, $\psi(m)$ is also equal to the number of divisors of $2m + 1 \leq \sqrt{2m+1}$. If $\psi(m) = \nu$, then the entry $m = k_{ij}$, with $i \leq j$, appears ν times in the matrix $K = \{k_{ij}\}$.

Composite integers $2m + 1$ with $m \leq n$ are identified by the pairs (i, j) such that

$$4 \leq m = k_{ij} \leq n. \tag{9}$$

By assuming $i \leq j$, it follows that (9) holds for the pairs

$$(i, j) \in \Omega(4, n) = \{i, j \in \mathbb{N} : \ i = 1, 2, ..., [I_n]; \ j = i, i + 1, ..., [\phi_n(i)]\}.$$

An estimation of the number of these pairs as $n \longrightarrow +\infty$ is given by

$$\kappa_n^* \simeq n(\frac{1}{4}\ln(n) + c). \tag{10}$$

with $c = -0.4415$. The details can be found at the end of appendix 1. In doing so we do not consider the couple $(0, n)$, corresponding to the couple of trivial divisors $(1, 2n + 1)$.

The odd integers $2m+1, \ m \leq n$, with the same number of couples of divisors can be identified by means of the following algorithm:

let $\psi(m) = 0, \ m = 1, ..., n$;
compute $k_{ij}, \ \forall (i, j) \in \Omega(4, n)$;
for $k_{ij} = m$ let $\psi(m) = \psi(m) + 1$.

When $\psi(m) = 0$, then the integer $2m + 1$ is a prime. All the integers $2m + 1$, with ν couples of divisors, are identified by the values of m for which $\psi(m) = \nu$.

Furthermore, let

$\Pi_n(\nu) =$ number of odd integers $\leq 2n+1$ with ν couples of divisors.

$\Pi_n(0)$ is the number of primes $\leq 2n+1$, except 2. $\Pi_n(\nu)$ is estimated as follows:

for $\nu = 0: \ \Pi_n(0) =$ number of $\psi(m) = 0$,

for $\nu > 0: \ \Pi_n(\nu) = \frac{1}{\nu}\sum_{\psi(m)=\nu}\psi(m)$.

This approach, used to identify the prime numbers, is an equivalent formulation of the common implementation of the Eratostene's sieve (see for example the C program source in (2), section 6.3). In this case $\psi(m)$ could be a logical variable.

The algorithm may be easily applied to the integers in a generic set $[2a+1, 2n+1]$, with $4 < a < n$, to identify either the odd integers in this interval with the same number of couples of divisors or the primes. The inequalities identifying these integers,

$$a \leq k_{ij} \leq n, \quad with\ i \leq j,$$

hold for the pairs

$$(i,j) \in \Omega(a,n) = \{i,j \in \mathbb{N}: \ i = 1,2,...,[I_n]; \ j = J_a(i), J_a(i){+}1, ..., [\phi_n(i,)]\},$$

where:

$$when\ i \leq [I_a]: \ either\ J_a(i) = [\phi_a(i)] + 1, \ \phi_a(i) \notin \mathbb{N}, \ or\ J_a(i) = \phi_a(i) \in \mathbb{N};$$

$$when\ i > [I_a]: J_a(i) = i.$$

The set of the points $(i,j) \in \Omega(a,n)$, with integer coordinates, is contained in a closed and convex set $\Omega^*(a,n)$ of a plane. See figure 2, where the boundaries of this set are plain defined.

Some remarks on the case with large n and $n - a << n$ can be found in appendix 3.

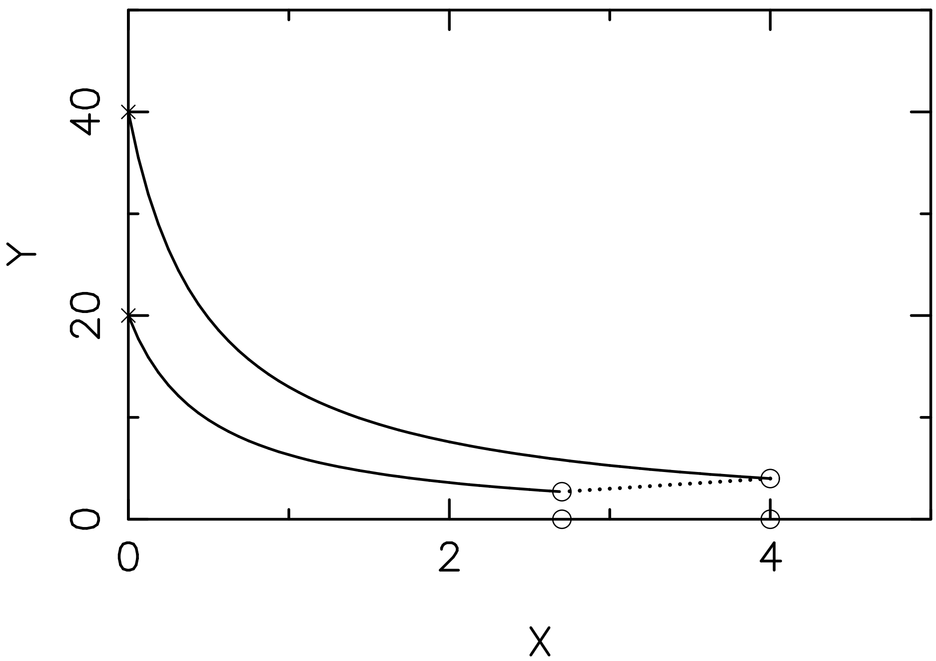


Figure 2: Set $\Omega^*(a,n)$ in the plane (x,y). Continuous lines: $y = \phi_a(x) < y = \phi_n(x)$; dotted line: $y = x$; asterisk: points $(0,a),(0,n)$; circle: points $(I_a,0),(I_n,0)$, and corresponding points on the curves. Different scales for x and y.

4 Distributions of odd numbers vs. the number of their couples of divisors

The computation of the distributions $\Pi_n(\nu)$ has been performed, by means of the algorithm described in the previous section, for $n \leq 5\ 10^7$, i.e. for odd integers $2n+1 \leq 10^8+1$ (see the tables 1, 2 for some values of n).

	$n=5$		$n=50$		$n=5\ 10^2$		$n=5\ 10^3$		$n=5\ 10^4$	
ν										
0	4		25		167		1228		9591	
1		1		20		207		1964		18259
2			5		52		382		2824	
3						56		925		11380
4					5		50		308	
5						12		264		3200
6					0		4		32	
7						1		128		2785
8							22		265	
9								9		188
10							0		3	
11								24		826
12									1	
13										18
14									27	
15										195
16									0	
17										66
18									0	
19										13
20									0	
21										0
22									1	
23										18
tot.	4	1	30	20	224	276	1686	3314	13052	36948

Table 1: Distribution $\Pi_n(\nu)$ of odd integers $\leq 2n+1$, with ν couples of divisors, for $n = 5, 50, 5\ 10^2, 5\ 10^3, 5\ 10^4$.

Let

ν_n^* = maximum number of couples of divisors of odd integers $\leq 2n+1$.

	$n = 5\ 10^5$	$n = 5\ 10^6$		$n = 5\ 10^5$	$n = 5\ 10^6$
ν			ν		
0	78497	664578	1	168522	1555858
2	21711	174188	3	126518	1336044
4	2030	14919	5	32314	309137
6	236	1758	7	42022	542740
8	2228	17481	9	2171	21649
10	16	74	11	13521	172181
12	2	12	13	238	2343
14	295	2376	15	5733	105676
16	0	4	17	1403	17487
18	0	0	19	545	8847
20	24	268	21	0	15
22	11	60	23	1537	34648
24	6	57	25	0	0
26	50	705	27	17	566
28	0	0	29	67	1503
30	0	0	31	179	8098
32	0	0	33	0	0
34	0	7	35	88	3589
36	0	0	37	0	4
38	0	0	39	13	922
40	0	11	41	0	69
42	0	0	43	0	0
44	0	65	45	0	0
46	0	0	47	6	1693
48		0	49		7
50		0	51		0
52		0	53		118
54		0	55		16
56		0	57		0
58		0	59		86
60		0	61		0
62		1	63		91
64		0	65		0
66		0	67		0
68		0	69		0
70		0	71		46
72		0	73		0
..		..	..		..
78		0	79		3
tot.	105106	876564	tot.	394894	4123436

Table 2: Distribution $\Pi_n(\nu)$ of odd integers $\leq 2n+1$, with ν couples of divisors, for $n = 5\ 10^5, 5\ 10^6$. (Since $\nu_n^* = 143$ for $n = 5\ 10^7$, this case is not reported here).

Thus, $\Pi_n(\nu) = 0$ for $\nu > \nu_n^*$. It has been estimated (table 3, figure 3) that ν_n^* increases as a power of n. The following approximation has been found by a fitting procedure

$$\nu_n^* = \mu n^\lambda, \quad \mu = e^{0.3992 \pm 0.1050}, \ \lambda = 0.2586 \pm 0.0083, \quad 5\ 10^2 \leq n \leq 5\ 10^7.$$

n	$5\ 10^2$	$5\ 10^3$	$5\ 10^4$	$5\ 10^5$	$5\ 10^6$	$5\ 10^7$
ν_n^*	8	12	24	48	80	144
$\mu\ n^\lambda$	7.43	13.48	24.46	44.37	80.48	145.99

Table 3: Computed values of ν_n^* and those produced by $\nu_n^* = \mu n^\lambda$ for some values of n.

A visual inspection of the patterns of $\Pi_n(\nu)$ (the scattered plots of $\ln(\Pi_n(\nu))$ vs. ν are shown in the figures 4, 5) suggests that the odd integers with even and odd numbers of couples of divisors should belong to different populations. This view has to be considered only as a guess of the author, trying to interpret special features of $\Pi_n(\nu)$. Anyhow, to avoid repetitions, we nickname these integers as

$ravens$ the odd integers with 2ν couples of divisors,
$cods$ the odd integers with $2\nu + 1$ couples of divisors.

The primes, identified by $\nu = 0$, are included in the ravens. We have that

$\Pi_n(2\nu) =$ number of $ravens \leq 2n + 1$,
$\Pi_n(2\nu + 1) =$ number of $cods \leq 2n + 1$.

For $n > 150$, in general

$$\Pi_n(2\nu) < \Pi_n(2\nu + 1). \tag{11}$$

Only for few values of ν this inequality is not satisfied in the computed distributions (table 4). The number of $ravens$ is less large than that of $cods$ (see the last row in the tables 1, 2).

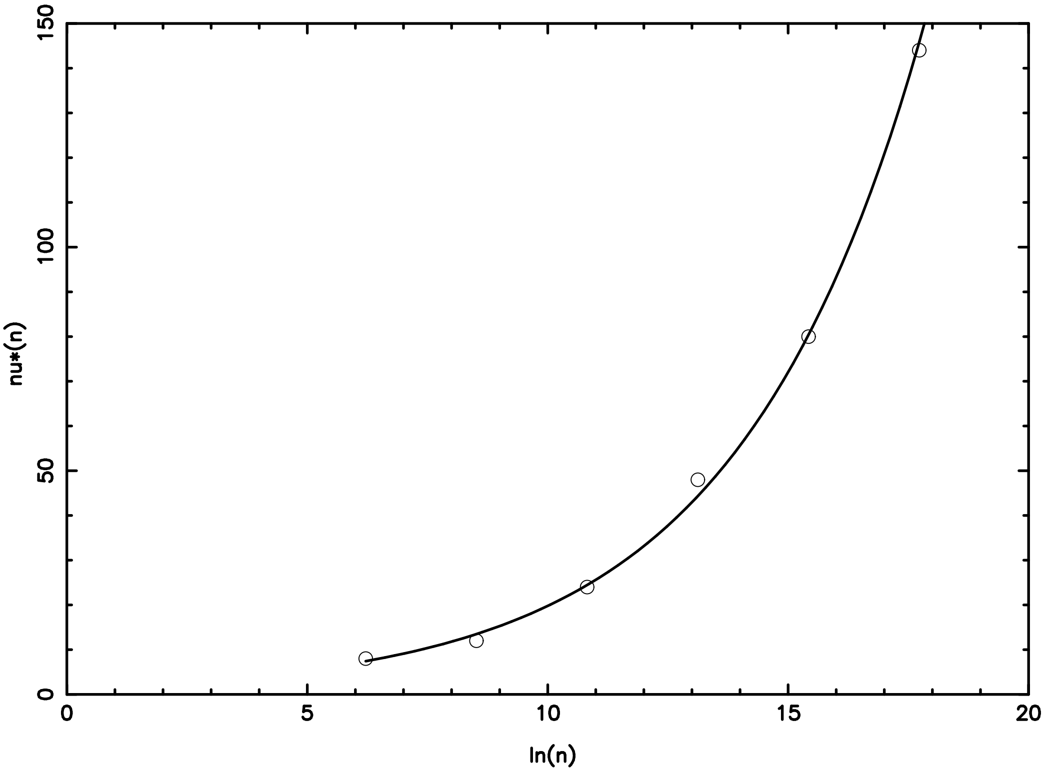


Figure 3: ν_n^* vs. $ln(n)$. Circles: computed values, continuous line: approximation $\nu_n^* = \mu\ exp(\lambda\ ln(n))$ for $2\ 10^2 \leq n \leq 5\ 10^7$.

n	$5\ 10^2$	$5\ 10^3$	$5\ 10^4$	$5\ 10^5$	$5\ 10^6$	$5\ 10^7$
		8-9	8-9	8-9 20-21 24-25 26-27	20-21 24-25 44-45	20-21 24-25 32-33 44-45 74-75 80-81
N. couples	0	1	1	4	3	6

Table 4: Couples of ν for which inequality (11) is not satisfied.

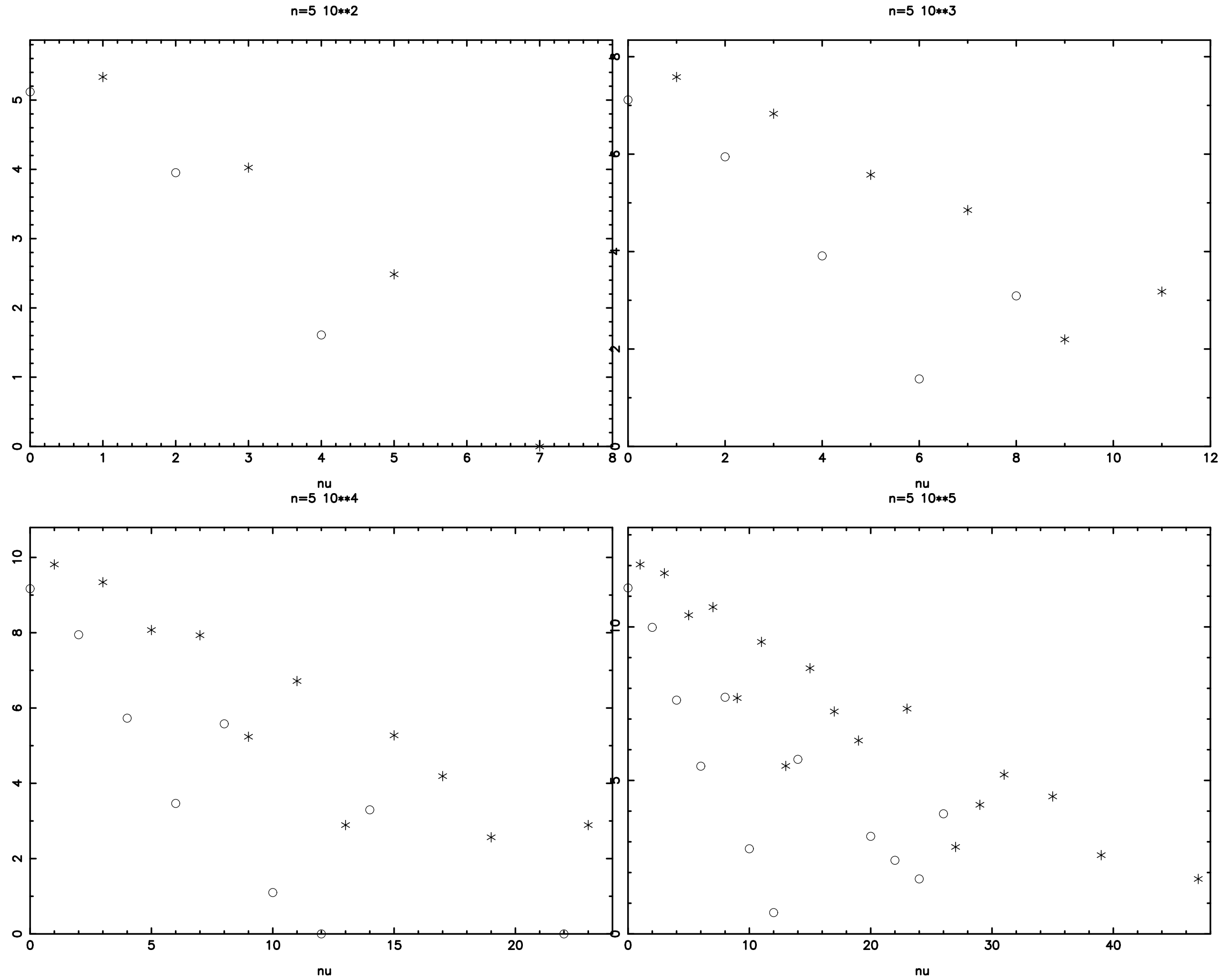


Figure 4: Distributions $ln(\Pi_n(\nu)$ vs. ν for $n = 5\ 10^2,\ 5\ 10^3\ 5\ 10^4,\ 5\ 10^5$. Circle: $ravens$, asterisk: $cods$.

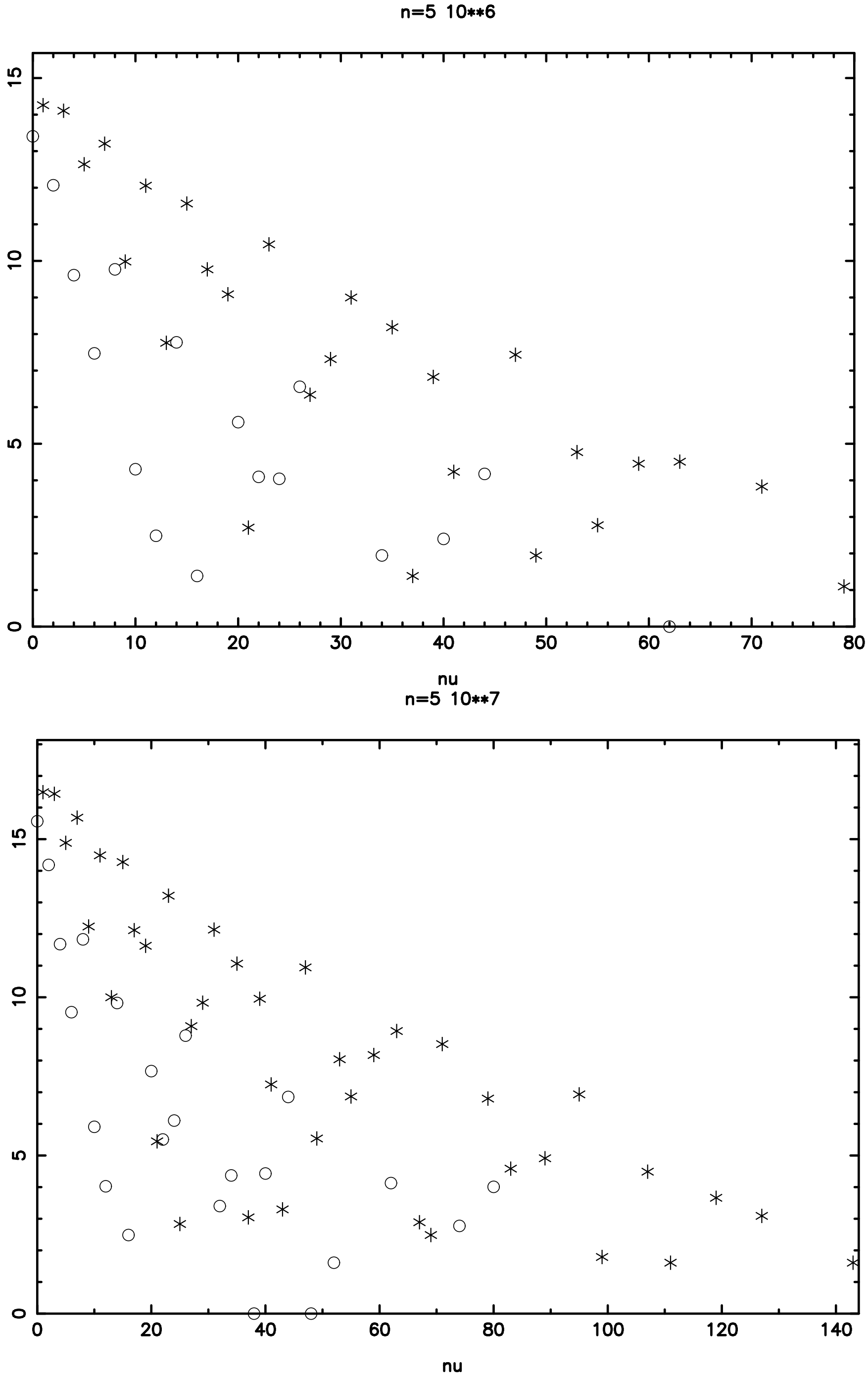


Figure 5: Distribution $ln(\Pi_n(\nu))$ vs. ν for $n = 5\ 10^6$, $n = 5\ 10^7$. Circle: $ravens$, asterisk: $cods$.

For $\simeq 5\ 10^2 \leq n \leq\simeq 5\ 10^4$ both the points $\Pi_n(2\nu)$ and $\Pi_n(2\nu+1)$ show well distinct decreasing trends with ν (figure 4). However, points not belonging to the initial trends begin to appear for $n \simeq 5\ 10^3$. Indeed, new branches (generally decreasing with ν) grow for increasing n, beginning at ν values not detected in the previous branches (figure 5).

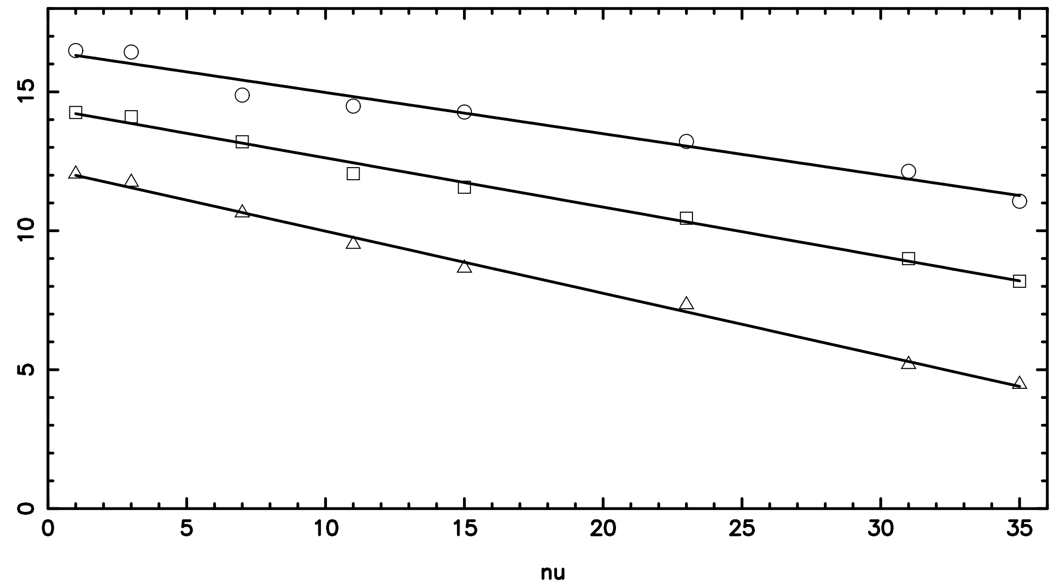
A branch may be roughly defined as a sequence of points in the plane $(\nu, \ln \Pi)$ which approximately lay on a straight line. For example, in the plot for $n = 5\ 10^5$ in figure 4, we can recognize two $raven$ branches: the initial at the points $\nu = 0, 2, 4, 6, 8$ and a second branch at $\nu = 8, 14, 20, 22, 24$ (the point at $\nu = 8$ is already present in the distribution for $n = 5\ 10^3$), and a single point at $\nu = 26$. Moreover, four cod branches: the initial at the points $\nu = 1, 3, 7, 11, 15, 23, 31, 35$, and then at $\nu = 5, 9, 13$, at $\nu = 17, 19, 27$, and at $\nu = 29, 39$. The attribution of a point to a branch is sometimes uncertain. Indeed, the interpretation of the evolution of the distributions $\Pi_n(\nu)$ with n in terms of growing branches is arbitrary. The straight lines in the plane $(\nu, \ln \Pi)$ approximating the initial trends of both $ravens$ and $cods$ are estimated by a fitting procedure (table 5, figure 6).

	n	$\alpha \pm \sigma$	$\beta \pm \sigma$	σ
$ravens$				
	$5\ 10^5$	11.3131 ± 0.2663	-0.8846 ± 0.0377	0.3901
	$5\ 10^6$	13.4700 ± 0.2292	-0.9257 ± 0.0324	0.3358
	$5\ 10^7$	15.7156 ± 0.1925	-0.9823 ± 0.0272	0.2820
$cods$				
	$5\ 10^5$	12.2196 ± 0.1155	-0.2234 ± 0.0059	0.1971
	$5\ 10^6$	14.3920 ± 0.1244	-0.1770 ± 0.0063	0.2122
	$5\ 10^7$	16.4599 ± 0.2084	-0.1484 ± 0.0106	0.3557

Table 5: Initial branches of $\Pi_n(\nu)$: coefficients of the linear relationship $\ln(\Pi_n(\nu)) = \alpha + \beta\nu$ and their standard deviations σ. Last column: σ of $\ln(\Pi_n(\nu))$.

All the points $(\nu, \ln(\Pi_n(\nu))$ are contained in a bounded region of the plane $(\nu, \ln \Pi)$ (figures 4, 5). This region is bounded from the bottom by the initial branch of $ravens$, starting from the number of primes $\ln(\Pi_n(0))$ and ending in $\nu \simeq 20$, and then by the axis $\ln \Pi = 0$. From the top by the initial branch of $cods$, starting from $\ln(\Pi_n(1))$ and ending in $\nu \simeq 40$, and then by sparse decreasing cod points, belonging to different branches. The upper boundary can be approximated by a straight line with $\simeq$ the same slope of the initial cod trend.

A guess about the description of the evolution of $\Pi_n(\nu)$ with n has been sug-



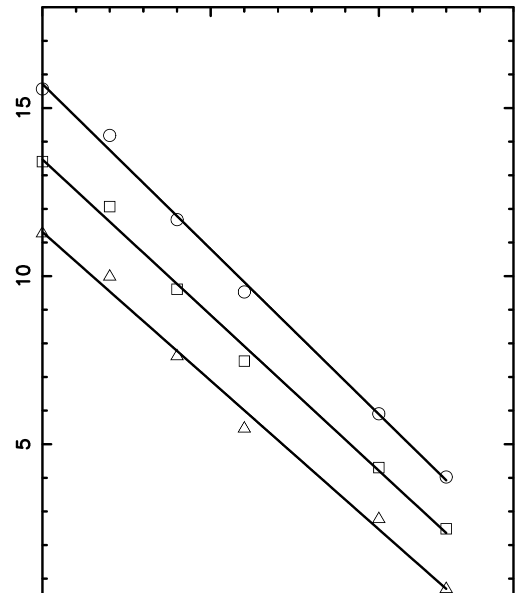


Figure 6: Top: initial branch of cods at $\nu = 1, 3, 7, 11, 15, 23, 31, 35$. Bottom: initial branch of ravens at $\nu = 0, 2, 4, 6, 10, 12$. Triangle: $n = 5\ 10^5$; square: $n = 5\ 10^6$; circle: $n = 5\ 10^7$. Continuous line: linear approximation.

gested by the most simple formula ((1), p. 8) approximating the number of primes $\leq 2n+1$:

$$\Pi_n(0) = \frac{2n+1}{\ln(2n+1)}.$$

Taking into account that $\ln(\ln(n)) \simeq 1.2334 + 0.0992\ln(n),\ 10^2 \leq n \leq 10^7$, this relationship can be approximated by $\ln(\Pi_n(0)) \simeq -0.5403 + 0.9008\ln(n)$. Fitting of $\ln(\Pi_n(\nu))$ to the linear expression $\alpha + \beta\ln(n)$ has been carried out for $\nu = 0,1,2,3$ (table 6, figure 7). The straight lines are $\simeq$ parallel for the $ravens$ $\nu = 0,2$, while the lines for the $codes$ $\nu = 1,3$ show different slopes (the line for $\nu = 3$ is not shown in figure 7 for clearness of the figure). It is worth here to remind that a logarithmic approximation of a quantity may lead to a rough estimation of the quantity.

ν	$\alpha \pm \sigma$	$\beta \pm \sigma$	σ
0	-0.4902 ± 0.0755	0.8993 ± 0.0068	0.0847
1	-0.7130 ± 0.0282	0.9714 ± 0.0025	0.0317
2	-1.7076 ± 0.0550	0.8944 ± 0.0049	0.0617
3	-2.3972 ± 0.1632	1.0721 ± 0.0140	0.1528

Table 6: $\ln(\Pi_n(\nu))$ vs. $\ln(n)$: coefficients of the linear relationship $\ln(\Pi_n(\nu)) = \alpha + \beta\ln(n)$ and their standard deviations σ. Last column: σ of $\ln(\Pi_n(\nu))$. Ten $n-points$, from $n = 100$ to $n = 5\ 10^7$ are used in fitting $\ln(\Pi_n(\nu))$ for $\nu = 0,1,2$. Since $\Pi_n(3)$ is very small for $n = 100$, this point is not included for $\nu = 3$.

The "regularity" of some relationships between $\Pi_n(\nu)$ (figure 8) may arouse some surprise. We have performed a survey on the ratios between $\Pi_n(\nu)$ with $\nu = 0,1,2,3$. The trends of the ratios $\Pi_n(1)/\Pi_n(0)$, $\Pi_n(3)/\Pi_n(0)$, $\Pi_n(3)/\Pi_n(1)$ (figure 8 top), increasing with n, seem to be reasonable. On the other hand, the trends of the ratios $\Pi_n(2)/\Pi_n(\nu),\ \nu = 0,1,3$, (figure 8 bottom), are disturbing. This might be due to the shortage of $ravens$ with $\nu = 2$ detected. Obviously, computations with n greater than $n = 5\ 10^7$, the maximum value here considered, should be carried out to confirm the results, and to try to explain the trends. A careful analysis to produce a thorough knowledge has to be hoped for.

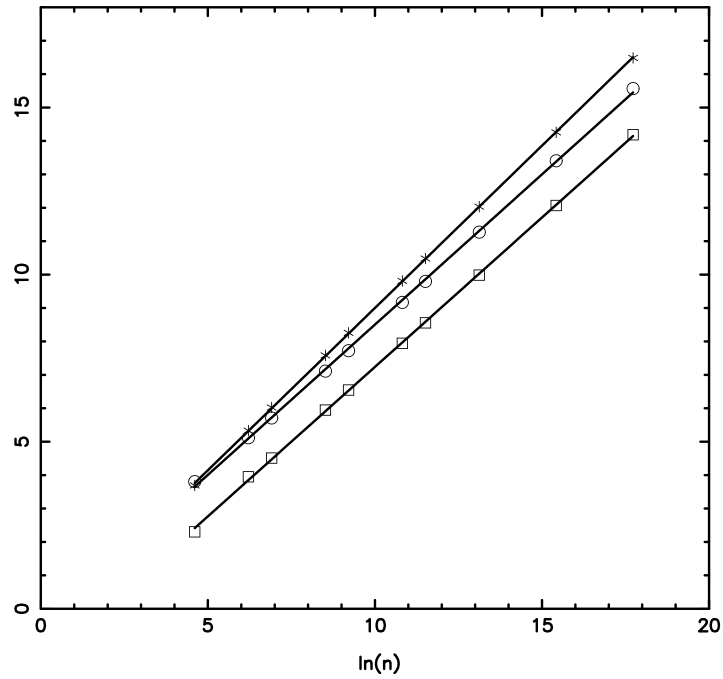


Figure 7: Distributions $\ln(\Pi_n(\nu))$ vs. $\ln(n)$. Circle: $\Pi_n(0)$, asterisk: $\Pi_n(1)$, square: $\Pi_n(2)$. Continuous line: linear approximation.

5 Concluding remarks

We have focused our attention on the computation of the couples of divisors of odd integers. Indeed, any even integer can be written in the form

$$2^m\ (2n+1), \quad with\ m \geq 1,\ n \geq 0.$$

Thus, it is characterized by the power of two, and possibly by an odd integer with its divisors. The following considerations hold for the computed distributions for n up to $5\ 10^7$.

For small n the following inequalities hold:

$$\Pi_n(0) > \Pi_n(1) > \Pi_n(2) > \Pi_n(\nu), \quad \nu > 2, \quad n < 149.$$

$\Pi_n(0)$ is the number of primes, $\Pi_n(1)$ is the number of $cods$ either products of two primes or primes cubed, while $\Pi_n(2)$ is the number of $ravens$ either products of primes by primes squared or primes to the fourth. The previous inequalities can be explained by the following reasonings: for small n, (1) the density of primes is high, and (2) the prime factors in the divisors of both $cods$ and $ravens$ should be small. As an example, the possible decompositions of $cods \leq 101$ with $\nu = 1$ are reported here:

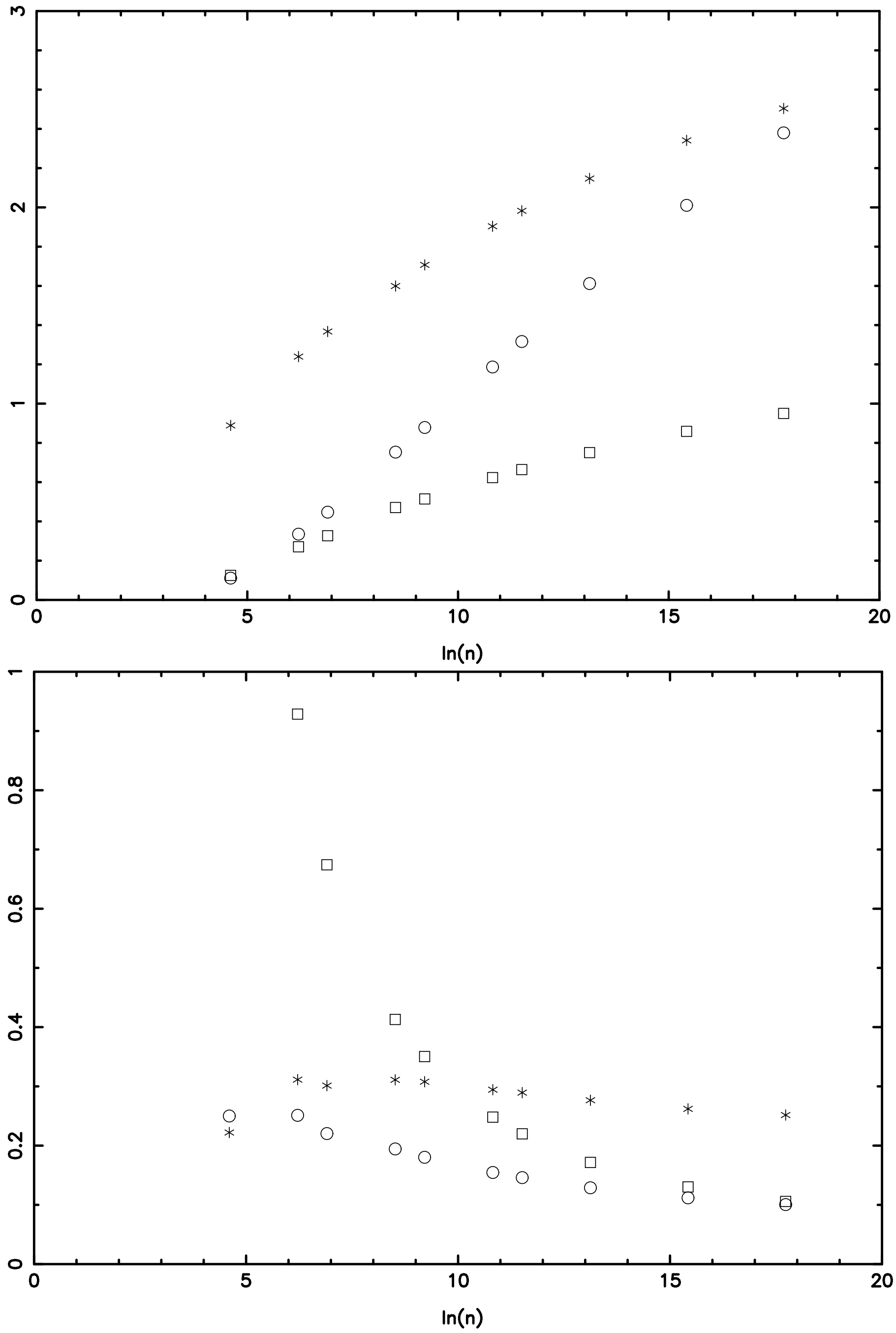


Figure 8: Ratios of distributions $\Pi_n(\nu)$ vs. $\ln(n)$. Top: $\Pi_n(1)/\Pi_n(0)$ asterisk, $\Pi_n(3)/\Pi_n(0)$ circle, $\Pi_n(3)/\Pi_n(1)$ square. Bottom: $\Pi_n(2)/\Pi_n(0)$ asterisk, $\Pi_n(2)/\Pi_n(1)$ circle, $\Pi_n(2)/\Pi_n(3)$ square.

$$p_1p_i,\ i=1,...,10;\ p_2p_i,\ i=2,3,...,7;\ p_3p_i,\ i=3,4,5;\ p_1p_1^2;$$

where $p_1=3,\ p_2=5,...$ are the primes. Thus, for small n, few factors produce *cod* integers $\leq 2n+1$. For increasing n, the inequality $\Pi_n(1)>\Pi_n(\nu),\ \nu\neq 1,$ hold. For $n=149$ we have $\Pi_n(0)=\Pi_n(1)$ (table 7.

n	$\Pi_n(0)$	$\Pi_n(1)$	$\Pi_n(2)$
50	25	20	5
100	45	40	10
148	61	60	16
149	61	61	16
150	61	62	16
200	78	83	21
250	94	104	25

Table 7: The transition from $\Pi_n(0)>\Pi_n(1)$ to $\Pi_n(0)<\Pi_n(1)$.

The distributions $\Pi_n(\nu)$ have been obtained by identifying all the couples $(2i+1,2j+1)$ of divisors of the integers $2m+1$ with $m=k_{ij}\leq n$. For large n the number κ_n^* of $k_{ij}\leq n$ is $n(\ln(n)/4+c)$ (10). The number k_{a+1n}^* of k_{ij} such that $a+1\leq k_{ij}\leq n$ can be estimated by

$$k_{a+1n}^*=k_n^*-k_a^*=\frac{1}{4}(n\ln(n)-a\ln(a))+c(n-a).$$

Under the assumption $n-a<<a<n$ it follows that

$$1<\frac{n}{a}=1+(\frac{n}{a}-1)<<2,\quad \textit{so that}\ 0<\frac{n}{a}-1<<1.$$

Thus,

$$k_{a+1n}^*=k_n^*-k_a^*=(n-a)(\frac{1}{4}\ln(n)+c)+\frac{1}{4}a\ln(\frac{n}{a})=(n-a)(\frac{1}{4}\ln(n)+c+\frac{1}{4}).$$

Some explanations on the numerical computations are given. The algorithms described in sections 2 and 3 can be easily implemented in Fortran language.

The algorithm (in section 2) for the computation of the couples of divisors of a given integer $2n+1$ does not require the storage of large dimension vectors. It has been successfully used to determine the couples of divisors of odd composite integers (and whether a number is prime or composite), up to input numbers of order $2^{60} \simeq 10^{18}$. Note that quadruple precision for floating point operations is necessary for numbers of order 2^{60}. We do not have recourse to computer algebra systems, with numbers of variable length (3; 4).

The algorithm (in section 3) for the computaion of the distributions $\Pi_n(\nu)$ requires the storage of an INTEGER*8 vector; the computation has been carried out up to the limit of the storage carrying capacity of the available computer (about a vector of $6\ 10^7$ of INTEGERS*8 entries). The computation of the primes in a given interval $[2a+1, 2n+1]$ has been performed either with small $n-a \in [5, 50]$ and a up to 10^{18}, or with large $n-a \in [10^2, 4\ 10^7]$ and a up to 10^9.

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Appendix 1. The matrix $K = \{k_{ij}\}$

Obviously, the symmetry property holds for the elements k_{ij} of K: $k_{ij} = k_{ji}, \forall\ i, j \in \mathbb{N}$. Thus, they can be represented by means of a symmetric matrix. (see the table 8).

Since some composite odd numbers $2n + 1$ may be expressed as product of two odd numbers in different ways, it follows that

$$2n + 1 = (2i_1 + 1)(2j_1 + 1) = (2i_2 + 1)(2j_2 + 1) \Longrightarrow k_{i_1 j_1} = k_{i_2 j_2},$$

as it can be observed in the matrix K (table 8). The number of couples (i, j) such that $n = k_{ij}$, if they exist, is the number of decompositions of $2n + 1$ in two factors.

$- \downarrow i\ j \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	4	7	10	13	16	19	22	25	28	31	34	37	40	43	46
2		12	17	22	27	32	37	42	47	52	57	62	67	72	77
3			24	31	38	45	52	59	66	73	80	87	94	101	108
4				40	49	58	67	76	85	94	103	112	121	130	139
5					60	71	82	93	104	115	126	137	148	159	170
6						84	97	110	123	136	149	162	175	188	201
7							112	127	142	157	172	187	202	217	232
8								144	161	178	195	212	229	246	263
9									180	199	218	237	256	275	294
10										220	241	262	283	304	325
11											264	287	310	333	356
12												312	337	362	387
13													364	391	418
14														420	449
15															480

Table 8: Matrix $K = \{k_{ij}\}$ for $1 \le i \le j \le 15$. i=row and j=column index.

Besides the symmetry identity, the elements k_{ij} satisfy other combinatorial properties, obtained from the equation

$$(2n + 1)(2k_{ij} + 1) = 2k_{nk_{ij}} + 1.$$

The identities

$$k_{pk_{qr}} = k_{qk_{rp}} = k_{rk_{pq}}, \quad p, q, r \in N$$

follow from the product of three odd numbers $(2p+1)(2q+1)(2r+1)$, while the identities

$$k_{k_{pq}k_{rs}} = k_{k_{pr}k_{qs}} = ... = k_{pk_{qk_{rs}}} = k_{qk_{pk_{rs}}} = ..., \quad p, q, r, s \in N$$

follow from the product of four odd numbers $(2p + 1)(2q + 1)(2r + 1)(2s + 1)$. Obviously, more involved identities are obtained from products of more than four odd numbers.

The quantities $k_{ij} - i = (2i + 1)j$ and $k_{ij} - j = i(2j + 1)$ are divisible by $2i + 1,\ j$ and by $i,\ 2j + 1$, respectively. Moreover, k_{ij} can be written in the form

$$k_{ij} = k_{ii}+(2i+1)(j-i), \quad i = 1, 2, ..., \ j = i, i+1, ... \ with \ k_{ii} = 2i^2+2i, \ (12)$$

which leads to the following recurrence formula for the computation (by means of additions) of the entries of the $i-th$ row, with $i \leq j$, of the matrix K:

$$k_{ij} = k_{i\ (j-1)} + 2i + 1, \quad j = i + 1, i + 2, ...$$

Now we estimate
$\kappa_n^* =$ number of pairs (i, j) with $1 \leq i \leq j$ such that $4 \leq k_{ij} \leq n$.
It is given by

$$\kappa_n^* = \sum_{i=1}^{I_n} [q_n(i)].$$

where

$$q_n(i) = \phi_n(i) - i + 1 = \frac{1}{2}(\frac{2n+1}{2i+1} - (2i-1)),$$

and here I_n denotes the integer part of the quantity defined in (4). $q_n(i)$ are decreasing with i, and

$$q_n(I_n) = 1 \leq q_n(i) \leq q_n(1) = \frac{n-1}{3}.$$

By direct calculation we have that

$$Q_n = \sum_{i=1}^{I_n} q_n(i) =$$

$$= (n+\frac{1}{2})\sum_{i=1}^{I_n}\frac{1}{2i+1} - \frac{1}{2}I_n^2 = (n+\frac{1}{2})\sum_{i=1}^{I_n}\frac{1}{2i+1} - \frac{1}{4}(1+n-\sqrt{2n+1}).$$

Taking into account the logarthmic growth of the harmonic series, we have for n large enough

$$\sum_{i=1}^{I_n}\frac{1}{2i+1} \simeq \ln(\frac{2I_n+1}{\sqrt{I_n}}) + \frac{\gamma}{2} - 1,$$

where $\gamma \simeq 0.5772$ is the Euler-Mascheroni constant. It follows that

$$\frac{Q_n}{n} \longrightarrow \ln(2\sqrt{I_n} + \frac{1}{\sqrt{I_n}}) + \frac{\gamma}{2} - \frac{5}{4} \simeq \frac{1}{4}\ln(n) + c \quad as\ n \longrightarrow +\infty,$$

with $c = 0.5(1.5\ln(2) + \gamma - 2.5) = -0.4415$.

Since the following inequalities

$$Q_n - I_n \leq k_n^* \leq Q_n.$$

hold, and $I_n/n \longrightarrow 0$ as $n \longrightarrow +\infty$, for the increasing function κ_n^*/n we have that

$$\frac{\kappa_n^*}{n} \longrightarrow \frac{1}{4}\ln(n) + c \quad as\ n \longrightarrow +\infty.$$

A linear fit of κ_n^*/n vs. $\ln(n)$, for $10 \leq n \leq 10^{17.5}$, produces the line $\kappa_n^*/n \simeq (-0.4017 \pm 0.0102) + (0.2486 \pm 0.0004)\ln(n)$.

Appendix 2. The partitions $n = s + 2t$ and $n = s + t$

Equation (2) represents a partition of the integer n in two sections

$$s = i + j \quad and \quad n - s = 2ij, \quad with \quad 2s \leq n\ (2s = n\ iff\ i = j = 1). \quad (13)$$

Since $i \in [1, I_n]$ and $j = \phi_n(i)$, the bounds for s in the partition (13) are given by $I_n + \phi_n(I_n) = 2I_n$ and $1 + \phi_n(1)$ (see figure 1, plot of $x + y$). Therefore, the set of admissible values for s is

$$\Omega_1 = [2I_n, \frac{n+2}{3}]. \quad (14)$$

From (13) it follows that i and j are the positive integer solutions,if they exist, to the equation

$$x^2 - sx + \frac{n-s}{2} = 0. \quad (15)$$

The solutions to (15) are

$$x^{\pm} = \frac{1}{2}(s \pm \sqrt{\Delta}, \quad with \quad \Delta = s^2 + 2s - 2n. \quad (16)$$

Since Δ and s have the same parity, positive integer solutions exist iff

$$\exists s \in \Omega_1 : \sqrt{\Delta} \in \mathbb{N}.$$

Equation (3) represents another partition of the integer n in two sections

$$s = i(j+1) \quad and \quad n-s = (i+1)j, \quad with \quad 2s \leq n\ (2s = n\ iff\ i = j). \quad (17)$$

The set of admissible values for s is

$$\Omega_2 = [\frac{n+2}{3}, \frac{n}{2}]. \quad (18)$$

From (17) it follows that i is solution to the following equation

$$i^2 + (n - 2s + 1)i - s = 0,$$

and

$$j = i + n - 2s.$$

The results are given by

$$i = \frac{1}{2}[-1 - (n - 2s) + \sqrt{\Delta}],$$

$$j = \frac{1}{2}[-1 + (n - 2s) + \sqrt{\Delta}],$$

where

$$\Delta = 2n + 1 + (n - 2s)^2.$$

Since Δ and s have the same parity, positive integer solutions to the system (17) exist iff

$$\exists s \in \Omega_2 : \sqrt{\Delta} \in \mathbb{N}.$$

We can summarize the reasonings on the partitions of n in the following implications:

$$n \in \mathbb{K},\ \Delta = s^2 + 2s - 2n \Longleftrightarrow \exists s \in \Omega_1 : \sqrt{\Delta} \in \mathbb{N},$$

$$n \in \mathbb{K}, \Delta = 4s^2 - 4ns + 2n + 1 \Longleftrightarrow \exists s \in \Omega_2 : \sqrt{\Delta} \in \mathbb{N}.$$

Appendix 3. Remarks on the sets $\Omega(a,n)$ and $\Omega^*(a,n)$

Here we consider the case

$$n - a << I_a = min(a, n, I_a, I_n). \quad (19)$$

For example, this situation happens when we are looking for very few primes in $[2a+1, 2n+1]$ with large a. In virtue of the prime distribution ((1), p. 8) we should choose

$$n - a \simeq \frac{\iota}{2} \ln(2a+1),$$

with $1 \le \iota \le 10$.

The difference between the top and bottom boundary lines of $\Omega^*(a,n)$ (figure 2) is

$$\phi_n(i) - \phi_a(i) = \frac{n-a}{2i+1}.$$

It is decreasing with i, and

$$\phi_n(i) - \phi_a(i) < 1 \quad for\ i \ge [I_0],\ I_0 = \frac{n-a-1}{2} + 1 < I_a. \quad (20)$$

When (20) holds, at most only one integer is in $[\phi_a(i), \phi_n(i)]$. It follows that for $i \ge [I_0]$ the points of $\Omega^*(a,n)$ with integer coordinates are the points (i,j) with $j = [\phi_a(i)+1] = [\phi_n(i)]$.

Furthermore, since $I_a < 2\sqrt{n+a}$, from (19) we have that also $n - a < 2\sqrt{n+a}$, which implies $I_n - I_a < 1$. Thus, the boundary of $\Omega^*(a,n)$ between the points (I_a, I_a) and (I_n, I_n) (figure 2) does not contain points with integer coordinates.

In the limit case $a = n$, the set $\Omega^*(a,n)$ (figure 2) reduces to the curve $y = \phi_n(x)$ for $0 \le x \le I_n$, and $\Omega(n,n)$ is then defined by

$$(i,j) \in \Omega(n,n) = \{i = 1, 2, ..., [I_n] : \phi_n(i) \in \mathbb{N},\ j = \phi_n(i)\}.$$

The algorithm described in section 2 identifies the points of the set $\Omega(n,n)$.