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About a definition of metric over an abelian linearly ordered group

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Abstract

A $\mathcal{G}$-metric over an abelian linearly ordered group $\mathcal{G} = (G, \odot, \leq)$ is a binary operation, $d_{\mathcal{G}}$, verifying suitable properties. We consider a particular $\mathcal{G}$ metric derived by the group operation $\odot$ and the total weak order $\leq$, and show that it provides a base for the order topology associated to $\mathcal{G}$.

Key words: $\mathcal{G}$-metric, abelian linearly ordered group, multi-criteria decision making.

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1 Introduction

The object of the investigation in our previous papers have been the pairwise comparison matrices that, in a Multicriteria Decision Making context, are a helpful tool to determine a weighted ranking on a set X of alternatives or criteria [1], [2], [3]. The pairwise comparison matrices play a basic role in the Analytic Hierarchy Process (AHP), a procedure developed by T.L. Saaty [17], [18], [19]. In [14], the authors propose an application of the AHP for reaching consensus in Multiagent Decision Making problems; other consensus models are proposed in [6], [11], [15], [16].

The entry a_{ij} of a pairwise comparison matrix $A = (a_{ij})$ can assume different meanings: a_{ij} can be a preference ratio (multiplicative case) or a preference difference (additive case) or a_{ij} is a preference degree in $[0, 1]$ (fuzzy case). In order to unify the different approaches and remove some drawbacks linked to the measure scale and a lack of an algebraic structure, in [7] we consider pairwise comparison matrices over abelian linearly ordered groups (*alo-groups*). Furthermore, we introduce a more general notion of metric over an alo-group $\mathcal{G} = (G, \odot, \leq)$, that we call *$\mathcal{G}$-metric*; it is a binary operation on G

$$d : (a, b) \in G^2 \to d(a, b) \in G,$$

verifying suitable conditions, in particular: $a = b$ if and only if the value of $d(a, b)$ coincides with the identity of $\mathcal{G}$. In [7], [8], [9], [10] we consider a particular $\mathcal{G}$-metric, based upon the group operation $\odot$ and the total order $\leq$. This metric allows us to provide, for pairwise comparison matrices over a divisible alo-group, a consistency index that has a natural meaning and it is easy to compute in the additive and multiplicative cases.

In this paper, we focus on a particular $\mathcal{G}$-metric introduced in [7] looking for a topology over the alo-group in which the $\mathcal{G}$-metric is defined. By introducing the notion of *$d_{\mathcal{G}}$-neighborhood* of an element in an alo-group $\mathcal{G} = (G, \odot, \leq)$, we show that the above $\mathcal{G}$-metric generates the order topology that is naturally induced in $\mathcal{G}$ by the total weak order $\leq$.

2 Abelian linearly ordered groups

Let G be a non empty set, $\odot : G \times G \to G$ a binary operation on G, $\leq$ a total weak order on G. Then $\mathcal{G} = (G, \odot, \leq)$ is an *alo-group*, if and only if $(G, \odot)$ is an abelian group and

$$a \leq b \Rightarrow a \odot c \leq b \odot c. \tag{1}$$

As an abelian group satisfies the cancellative law, that is $a \odot c = b \odot c \Leftrightarrow a = b$, (1) is equivalent to the strict monotonicity of $\odot$ in each variable:

$$a < b \Leftrightarrow a \odot c < b \odot c. \tag{2}$$

Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, we will denote with:

- e the identity of $\mathcal{G}$;
- $x^{(-1)}$ the symmetric of $x \in G$ with respect to $\odot$;
- $\div$ the inverse operation of $\odot$ defined by $a \div b = a \odot b^{(-1)}$,
- $x^{(n)}$, with $n \in \mathbb{N}_0$, the (n)-*power* of $x \in G$:

$$x^{(n)} = \begin{cases} e, & if \quad n = 0 \\ x^{(n-1)} \odot x, & if \quad n \geq 1; \end{cases}$$

- $<$ the strict simple order defined by $x < y \Leftrightarrow x \leq y$ *and* $x \neq y$;
- $\geq$ and $>$ the opposite relations of $\leq$ and $<$ respectively.

Then

$$b^{(-1)} = e \div b, \quad (a \odot b)^{(-1)} = a^{(-1)} \odot b^{(-1)}, \quad (a \div b)^{(-1)} = b \div a; \tag{3}$$

moreover, assuming that G is no trivial, that is $G \neq \{e\}$, by (2) we get

$$a < e \Leftrightarrow a^{(-1)} > e, \quad a < b \Leftrightarrow a^{(-1)} > b^{(-1)} \tag{4}$$

$$a \odot a > a \quad \forall a > e, \qquad a \odot a < a \quad \forall a < e. \tag{5}$$

By definition, an alo-group $\mathcal{G}$ is a *lattice ordered group* [4], that is there exists $a \vee b = max\{a, b\}$, for each pair $(a, b) \in G^2$. Nevertheless, by (5), we get the following proposition.

Proposition 2.1. *A no trivial alo-group* $\mathcal{G} = (G, \odot, \leq)$ *has neither the greatest element nor the least element.*

Order topology. If $\mathcal{G} = (G, \odot, \leq)$ is an alo-group, then G is naturally equipped with the order topology induced by $\leq$ that we will denote with $\tau_{\mathcal{G}}$. An open set in $\tau_{\mathcal{G}}$ is union of the following open intervals:

- $]a, b[= \{x \in G : a < x < b\}$;
- $] \leftarrow, a[= \{x \in G : x < a\}$;
- $]b, \rightarrow [= \{x \in G : x > b\}$;

and a neighborhood of $c \in G$ is an open set to which c belongs. Then $G \times G$ is equipped with the related product topology. We say that $\mathcal{G}$ is a *continuous* alo-group if and only if $\odot$ is continuous.

Isomorphisms between alo- groups An *isomorphism* between two alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{G}' = (G', \circ, \leq)$ is a bijection $h : G \to G'$ that is both a lattice isomorphism and a group isomorphism, that is:

$$x < y \Leftrightarrow h(x) < h(y) \quad and \quad h(x \odot y) = h(x) \circ h(y). \tag{6}$$

Thus, $h(e) = e'$, where e' is the identity in $\mathcal{G}'$, and

$$h(x^{(-1)}) = (h(x))^{(-1)}. \tag{7}$$

By applying the inverse isomorphism $h^{-1} : G' \to G$, we get:

$$h^{-1}(x' \circ y') = h^{-1}(x') \odot h^{-1}(y'), \quad h^{-1}(x'^{(-1)}) = (h^{-1}(x'))^{(-1)}. \tag{8}$$

By the associativity of the operations $\odot$ and $\circ$, the equality in (6) can be extended by induction to the n-operation $\bigodot_{i=1}^{n} x_i$, so that

$$h(\bigodot_{i=1}^{n} x_i) = \bigcirc_{i=1}^{n} h(x_i), \quad h(x^{(n)}) = h(x)^{(n)}. \tag{9}$$

3 $\mathcal{G}$-metric

Following [5], we give the following definition of norm:

Definition 3.1. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, the function:*

$$|| \cdot || : a \in G \to ||a|| = a \vee a^{(-1)} \in G \tag{10}$$

*is a $\mathcal{G}$-*norm*, or a* norm *on $\mathcal{G}$.*

The norm $||a||$ of $a \in G$ is also called *absolute value* of a in [4].

Proposition 3.1. *[7] The $\mathcal{G}$-norm satisfies the properties:*

1. $||a|| = ||a^{(-1)}||$;
2. $a \leq ||a||$;
3. $||a|| \geq e$;
4. $||a|| = e \Leftrightarrow a = e$;
5. $||a^{(n)}|| = ||a||^{(n)}$;
6. $||a \odot b|| \leq ||a|| \odot ||b||$. *(triangle inequality)*

Definition 3.2. *Let $\mathcal{G} = (G, \odot, \leq)$ be an alo-group. Then, the operation*

$$d : (a, b) \in G^2 \to d(a, b) \in G$$

*is a $\mathcal{G}$-*metric *or $\mathcal{G}$-*distance *if and only if:*

1. $d(a, b) \geq e$;
2. $d(a, b) = e \Leftrightarrow a = b$;

3. $d(a,b) = d(b,a)$*;*

4. $d(a,b) \leq d(a,c) \odot d(b,c)$*.*

Proposition 3.2. *[7] Let* $\mathcal{G} = (G, \odot, \leq)$ *be an alo-group. Then, the operation*

$$d_{\mathcal{G}} : (a,b) \in G^2 \to d_{\mathcal{G}}(a,b) = ||a \div b|| \in G \tag{11}$$

is a $\mathcal{G}$*-distance.*

Proposition 3.3. *[7] Let* $\mathcal{G} = (G, \odot, \leq)$ *and* $\mathcal{G}' = (G', \circ, \leq)$ *be alo-groups,* $h : G \to G'$ *an isomorphism between* $\mathcal{G}$ *and* $\mathcal{G}'$*. Then, for each choice of* $a, b \in G$ *:*

$$d_{\mathcal{G}'}(h(a), h(b)) = h(d_{\mathcal{G}}(a,b)). \tag{12}$$

Corolary 3.1. *Let* $h : G \to G'$ *be an isomorphism between the alo-groups* $\mathcal{G} = (G, \odot, \leq)$ *and* $\mathcal{G}' = (G', \circ, \leq)$*. If* $a' = h(a), b' = h(b), r' = h(r) \in G'$*, then* $r > e \Leftrightarrow r' > e'$ *and*

$$d_{\mathcal{G}'}(a', b') < r' \Leftrightarrow d_{\mathcal{G}}(a,b) < r.$$

4 Examples of continuous alo-groups over a real interval

An alo-group $\mathcal{G} = (G, \odot, \leq)$ is a *real* alo-group if and only if G is a subset of the real line R and $\leq$ is the total order on G inherited from the usual order on R. If G is a proper interval of R then, by Proposition 2.1, it is an open interval.

Examples of real divisible continuous alo-groups are the following (see [8] [9]):

Additive alo-group $\mathcal{R} = (\mathbb{R}, +, \leq)$, where $+$ is the usual addition on $\mathbb{R}$. Then, $e = 0$ and for $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$a^{(-1)} = -a, \quad a \div b = a - b, \quad a^{(n)} = na.$$

The norm $||a|| = |a| = a \vee (-a)$ generates the usual distance over R:

$$d_{\mathcal{R}}(a,b) = |a - b| = (a - b) \vee (b - a).$$

Multiplicative alo-group $]\mathbf{0,+\infty}[$ $= (]0,+\infty[, \cdot, \leq)$, where $\cdot$ is the usual multiplication on $\mathbb{R}$. Then, $e = 1$ and for $a, b \in]0,+\infty[$ and $n \in \mathbb{N}$:

$$a^{(-1)} = 1/a, \quad a \div b = \frac{a}{b}, \quad a^{(n)} = a^n.$$

The norm $||a|| = |a| = a \vee a^{-1}$ generates the following $]\mathbf{0,+\infty}[$ - distance

$$d_{]\mathbf{0,+\infty}[}(a,b) = \frac{a}{b} \vee \frac{b}{a}.$$

Fuzzy alo-group $]\mathbf{0,1}[$ $= (]0,1[, \otimes, \leq)$, where $\otimes$ is the binary operation in $]0,1[$:

$$\otimes : (a,b) \in]0,1[\times]0,1[\mapsto \frac{ab}{ab + (1-a)(1-b)} \in]0,1[, \tag{13}$$

Then, 0.5 is the identity element, $1-a$ is the inverse of $a \in]0,1[$, $a \div b = \frac{a(1-b)}{a(1-b)+(1-a)b}$, $a^{(0)} = 0.5$,

$$a^{(n)} = \frac{a^n}{a^n + (1-a)^n} \quad \forall n \in \mathbb{N} \tag{14}$$

and

$$d_{]\mathbf{0,1}[}(a,b) = \frac{a(1-b)}{a(1-b)+(1-a)b} \vee \frac{b(1-a)}{b(1-a)+(1-b)a} = \frac{a(1-b) \vee b(1-a)}{a(1-b)+b(1-a)}. \tag{15}$$

Remark 4.1. *By Proposition 2.1, the closed unit interval $[0,1]$ can not be structured as an alo-group; thus, in [7], the authors propose $\otimes$ as a suitable binary operation on $]0,1[$, satisfying the following requirements: 0.5 is the identity element with respect to $\otimes$; $1-a$ is the inverse of $a \in]0,1[$ with respect to $\otimes$; $(]0,1[, \otimes, \leq)$ is an alo-group. The operation $\otimes$ is the restriction to $]0,1[\times]0,1[$ of the uninorm:*

$$U(a,b) = \begin{cases} 0, & (a,b) \in \{(0,1),(1,0)\}; \\ \frac{ab}{ab+(1-a)(1-b)}, & \text{otherwise.} \end{cases}$$

The uninorms have been introduced in [12] as a generalization of t-norm and t-conorm [13] and are commutative and associative operations on $[0,1]$, verifying the monotonicity property (1).

5 $d_{\mathcal{G}}$- neighborhoods and order topology

In this section $\mathcal{G} = (G, \odot, \leq)$ is an alo-group and $d_{\mathcal{G}}$ the $\mathcal{G}$-distance in (11).

Definition 5.1. *Let $c, r \in G$ and $r > e$; then the $d_{\mathcal{G}}$-*neighborhood *of c with radius r is the set:*

$$N_{d_{\mathcal{G}}}(c; r) = \{x \in G : d_{\mathcal{G}}(x, c) < r\}. \qquad (16)$$

Of course $c \in N_{d_{\mathcal{G}}}(c; r)$ for each $r > e$. Then, $N_{d_{\mathcal{G}}}(c)$ will denote a $d_{\mathcal{G}}$-neighborhood of c and $N_{d_{\mathcal{G}}}$ the set of the all $d_{\mathcal{G}}$-neighborhoods of the elements of $\mathcal{G}$.

Proposition 5.1. *Let $c, r \in G$ and $r > e$; then:*

$$N_{d_{\mathcal{G}}}(c; r) =]c \div r, c \odot r[$$

Proof. By properties (2), (3), (4) we get $c \div r = c \odot r^{(-1)} < c < c \odot r$ and:

$$x \in N_{d_{\mathcal{G}}}(c; r)$$
$$\Updownarrow$$
$$\begin{cases} e \leq x \div c < r \\ or \\ e < c \div x < r \end{cases}$$
$$\Updownarrow$$
$$\begin{cases} e \leq x \div c < r \\ or \\ r^{(-1)} < x \div c < e \end{cases}$$
$$\Updownarrow$$
$$\begin{cases} c \leq x < c \odot r \\ or \\ c \div r < x < c \end{cases}$$
$$\Updownarrow$$
$$x \in]c \div r, c \odot r[.$$

□

Proposition 5.2. *Let $h : G \to G'$ be an isomorphism between the alo-group $\mathcal{G} = (G, \odot, \leq)$ and the alo-group $\mathcal{G}' = (G', \circ, \leq)$. Then, for each choice of $c, r \in G$ and $c', r' \in G$ such that $c' = h(c)$, $r > e$ and $r' = h(r)$, the following equality holds:*

$$N_{d_{\mathcal{G}'}}(c'; r') = h(N_{d_{\mathcal{G}}}(c; r)). \tag{17}$$

Proof. By Proposition 3.3 and Corollary 3.1. □

Example 5.1. *The neighborhoods related to the examples in Section 4 are the following:*

- *in the additive alo-group $\mathcal{R} = (\mathbb{R}, +, \leq)$, the neighborhood of c with radius r is the open interval $]c - r, c + r[$;*
- *in the multiplicative alo-group **$]0,+\infty[$** $= (]0, +\infty[, \cdot, \leq)$, the neighborhood of c with radius r is the interval $]\frac{c}{r}, c \cdot r[$;*
- *in the fuzzy alo-group **$]0,1[$**$= (]0, 1[, \otimes, \leq)$, the neighborhood of c with radius r is the open interval $]\frac{c(1-r)}{c(1-r)+(1-c)r}, \frac{cr}{cr+(1-c)(1-r)}[$.*

By Proposition 5.1, $N_{d_{\mathcal{G}}}(c; r)$ is a particular neighborhood of c in the order topology $\tau_{\mathcal{G}}$. We show by means of the following results that the set $N_{d_{\mathcal{G}}}$ generates the order topology associated to $\mathcal{G}$.

Proposition 5.3. *Let A be an open set in the order topology $\tau_{\mathcal{G}}$. Then for each $c \in A$ there exists a $d_{\mathcal{G}}$-neighborhood of c included in A.*

Proof. It is enough to prove the assertion in the case that A is an open interval $]a, b[$. Let $c \in]a, b[$ and $r = d_{\mathcal{G}}(a, c) \wedge d_{\mathcal{G}}(b, c) = (c \div a) \wedge (b \div c)$. Let us consider the cases:

1. $r = c \div a \leq b \div c$;
2. $r = b \div c < c \div a$.

In the first case, $a = c \div r$, $c \odot r \leq b$ and so $]c \div r, c \odot r[\subseteq A =]a, b[$; thus, by Proposition 5.1, $N_{d_{\mathcal{G}}}(c; r) \subseteq A$. In the second case, the inclusion $N_{d_{\mathcal{G}}}(c; r) \subseteq A$ can be proved by similar arguments. □

Corolary 5.1. *The set $N_{d_{\mathcal{G}}}$ of the all $d_{\mathcal{G}}$-neighborhoods of the elements of $\mathcal{G}$ is a base for the order topology $\tau_{\mathcal{G}}$.*

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