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Mixed and Non-mixed Normal Subgroups of Dihedral Groups Using Conjugacy classes

Vinod S*
Biju G.S†

Abstract

In this paper, we characterize and compute the mixed and non-mixed basis of Dihedral groups. Also, by computing the conjugacy classes, we describe all the mixed and non-mixed normal subgroups of Dihedral Groups.
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*Department of Mathematics, Government College for Women, Thiruvananthapuram, Kerala, India; wenod76@gmail.com.
†Department of Mathematics, College of Engineering, Thiruvananthapuram-695016, Kerala, India; gsbiju@cet.ac.in.
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1 Introduction

There are many interesting functions from the family of Dihedral groups to set of natural numbers. For the Dihedral group D_n of order $2n$, Cavior [1975] proved that the number of subgroups is $d(n) + \sigma(n)$ where $\sigma(n)$ is the sum of positive divisors of n and $d(n)$ denote number of positive divisors of n. For elementary facts about dihedral groups see Conrad [Retrieveda]. Conrad [Retrievedb] describes the subgroups of D_n, including the normal subgroups. using characterization of dihedral groups in terms of generators and relations. Calugareanu [2004] presents a formula for the total number of subgroups of a finite abelian group. In Tărnăuceanu [2010] an arithmetic method is developed to count the number of some types of subgroups of finite abelian groups.

Subgroups of groups of smaller sizes are widely studied because their group properties can be easily verified and larger groups are usually studied in terms of their subgroups (see Miller [1940]). In this paper we characterize and compute the different basis of Dihedral groups. Also we describe all mixed and non-mixed normal subgroups of Dihedral groups via conjugacy classes.

2 Notations and Basic Results

Most of the notations, definitions and results we mentioned here are standard and can be found in Gallian [1994] and Dummit and Foote [2003]. For any given natural number n denote:

$d(n) =$ the number of positive divisors of n.
$\sigma(n) =$ the sum of positive divisors of n.
$\varphi(n) =$ the number of non- negative integers less than n and relatively prime to n.

Also, the greatest common divisor of m and n is denoted by (m, n). Let G be a group and $a_1, a_2, \ldots, a_p \in G$. Then the subgroup generated by $a_1, a_2, \ldots, a_p$ is denoted by $< a_1, a_2, \ldots, a_p >$.

Definition 2.1. *A group generated by two elements r and s with orders n and 2 such that $srs^{-1} = r^{-1}$ is said to be the n^{th} dihedral group and is denoted by D_n.*

Theorem 2.1. *For each divisor d of n, the group $\mathbb{Z}_n$ has a unique subgroup of order d, namely $\langle \frac{n}{d} \rangle$.*

Theorem 2.2. *For each divisor d of n, the group $\mathbb{Z}_n$ has exactly $\varphi(d)$ elements of order d, namely $\{k\frac{n}{d} : 0 \leq k \leq d-1, (k, d) = 1\}$.*

Theorem 2.3. *The number of subgroups of $\mathbb{Z}_n$ is $d(n)$, namely $\langle \frac{n}{d} \rangle$ where d is a divisor of n.*

Theorem 2.4. *Let G be a group generated by a and b such that $a^n = e$, $b^2 = e$ and $bab^{-1} = a^{-1}$. If the size of G is $2n$ then G is isomorphic to D_n.*

By theorem 2.4, we make an abstract definition for dihedral groups.

Definition 2.2. *For $n \geq 3$, let $R_n = \{r_0, r_1, \ldots, r_{n-1}\}$ and $S_n = \{s_0, s_1, \ldots, s_{n-1}\}$. Define a binary operation on $G_n = R_n \cup S_n$ by the following relations:*

$$r_i \cdot r_j = r_{i+j\,mod(n)} \qquad r_i \cdot s_j = s_{i+j\,mod(n)}$$
$$s_i \cdot s_j = r_{i-j\,mod(n)} \qquad s_i \cdot r_j = s_{i-j\,mod(n)} \quad \text{for all} \quad 0 \leq i, j \leq n-1.$$

Then $(G_n, \cdot)$ is a group of order $2n$.

Note that in the group $(G_n, \cdot)$, the identity element is r_0, $r_i = r_j$ if and only if $i = j\,mod(n)$, $s_i = s_j$ if and only if $i = j\,mod(n)$, the inverse of r_i is r_{n-i} and the inverse of s_i is s_i for all $0 \leq i, j \leq n-1$. It is also clear that $r_1^i = r_i$ and $r_j \cdot s_0 = s_j$ for all $0 \leq i, j \leq n-1$. Since G_n is a group of order $2n$ and can be generated by r_1 and s_0 such that:

$$r_1^n = r_n = r_0,\ s_0^2 = r_0 \text{ and } s_0 r_1 s_0^{-1} = s_0 r_1 s_0 = s_{-1} s_0 = r_{-1} = r_{n-1} = r_1^{-1}.$$

Therefore the group G_n is isomorphic to $D_n =< r_1, s_0 >$. The elements of R_n are called rotations and that of S_n are called reflections. A subgroup of D_n which contain both rotations and reflections is called a mixed subgroup and subgroups contain rotations only is called non-mixed subgroup. From the group D_n, we have the following.

Theorem 2.5. *R_n is a subgroup of D_n and is isomorphic to $\mathbb{Z}_n$.*

Theorem 2.6. *If n is even, the number of elements of order 2 in D_n is $n+1$, namely $\{r_{n/2}, s_j :\ 0 \leq j \leq n-1\}$.*

Theorem 2.7. *If n is odd, the number of elements of order 2 in D_n is n, namely $\{s_j :\ 0 \leq j \leq n-1\}$.*

Theorem 2.8. *If d divide n and $d \neq 2$, the number of elements of order d in D_n is $\varphi(d)$ namely $\{r_{kn/d} :\ 0 \leq k \leq d-1,\ (k,d) = 1\}$.*

Theorem 2.9. *If a and b are two elements in D_n, then $< a, b >= \{a^k b^m\ : 0 \leq k, m \leq n-1\}$*

Definition 2.3. *Let G be a finite group. An element $y \in G$ is said to be a conjugate of $x \in G$ iff $y = gxg^{-1}$, for some g in G.*

This relation conjugacy in a group G is an equivalence relation on G. The equivalence class determined by the element x is denoted by $cl(x)$. Thus $cl(x) = \{gxg^{-1} \ : g \in G\}$. The summation, $\sum_{x \in G} |cl(x)|$, where summation runs over one element from each conjugacy class of x is called the class equation of G.

Definition 2.4. *A subgroup H of the group G is said to be a normal subgroup if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.*

A normal subgroup which contain rotations alone is called a non- mixed normal subgroup and normal subgroups which contains both reflections and rotations is called mixed normal subgroup.

Theorem 2.10. *Every normal subgroup is a union of conjugacy classes.*

Theorem 2.11. *Every subgroup of a cyclic normal subgroup of the group G is also normal in G.*

3 Subgroups of D_n

Theorem 3.1. *The number of non-mixed subgroups of D_n is $d(n)$, namely $\{< r_{n/d} >: \ d \ \text{is a divisor of} \ n\}$.*

Proof. The non-mixed subgroups of D_n are subgroups of R_n. Since R_n is isomorphic to $\mathbb{Z}_n$, for each divisor d of n, the group R_n has a unique subgroup of order d, namely $< r_{n/d} >$. Hence the number of non-mixed subgroups of D_n is $d(n)$, namely $\{< r_{n/d} >: \ d \ \text{is a divisor of} \ n\}$. □

Theorem 3.2. *Every mixed subgroup of D_n has even order of which half of them are rotation and half of them are reflection.*

Proof. Let H be a mixed subgroup of D_n containing a reflection s. Let A denote the set of rotations of H and B denote the set of all reflections of H. Define a map $\psi : A \to B$ by $\psi(r) = r \cdot s$ for all $r \in A$. If s_j is an element in B then $s_j \cdot s$ is an element of A and $\psi(s_j \cdot s) = s_j ss = s_j$. Hence ψ is onto. Also $\psi(r) = \psi(r') \implies rs = r's \implies r = r'$ and hence ψ is one-one. □

Theorem 3.3. *Every mixed subgroup of D_n is Dihedral.*

Proof. Let H be a mixed subgroup of D_n. By theorem 3.2 , $|H| = 2d$ for some d and $H \cap R_n =< r_{n/d} >$. Since order of H is $2d$ and $< r_{n/d} >$ is its subgroup of order d, we have $H =< r_{n/d} > \cup < r_{n/d} > s =< r_{n/d}, s >$, for some s in H. Since $(r_{n/d})^d = r_o, s^2 = r_0$ and $sr_{n/d}s^{-1} = (r_{n/d})^{-1}$, we have $H \equiv D_d$ and hence the proof. □

Corolary 3.1. *If H is a mixed subgroup of D_n then,*

1. $|H| = 2d$, *for some d which divides n.*
2. $H \equiv D_n =< r_{n/d}, s >$ *for some $s \in H$.*

Here we have a usual question: If d divides n, does there exist a subgroup of order $2d$? If it exists, how many?

Theorem 3.4. *If d divides n, the number of mixed subgroups of order $2d$ is $\frac{n}{d}$.*

Proof. By the corollary 3.1, it is clear that the mixed subgroups D_n of order $2d$ are $\{< r_{n/d}, s_j >: \ 0 \leq j \leq n-1\}$, all of them need not be distinct. Suppose $< r_{n/d}, s_i >=< r_{n/d}, s_j >$ for some $0 \leq i, j \leq n-1$.

$$\begin{aligned}
& < r_{n/d}, s_i > =< r_{n/d}, s_j > \\
\Longleftrightarrow & < r_{n/d} > \cup < r_{n/d} > s_i =< r_{n/d} > \cup < r_{n/d} > s_j \\
\Longleftrightarrow & < r_{n/d} > s_i =< r_{n/d} > s_j \\
\Longleftrightarrow & \ s_i s_j^{-1} \in < r_{n/d} > \\
\Longleftrightarrow & \ s_i s_j^{-1} = r_{kn/d} \ \text{ for some } 0 \leq k \leq d-1 \\
\Longleftrightarrow & \ s_i s_j = r_{kn/d} \\
\Longleftrightarrow & \ r_{i-j} = r_{kn/d} \\
\Longleftrightarrow & \ i-j \equiv \frac{kn}{d} \, mod(n) \ \text{ for some } 0 \leq k \leq d-1 \\
\Longleftrightarrow & \ d(i-j) \equiv 0 \, mod(n) \\
\Longleftrightarrow & \ i-j \equiv 0 \, mod(\frac{n}{d}) \\
\Longleftrightarrow & \ i \equiv j \, mod(\frac{n}{d})
\end{aligned}$$

Hence the number of distinct mixed subgroups of order $2d$ in D_n is $\frac{n}{d}$, namely $\{< r_{n/d}, s_i >: \ 0 \leq i < \frac{n}{d}\}$. □

Theorem 3.5. *The number of mixed subgroups of D_n is $\sigma(n)$.*

Proof. By theorem 3.4, the mixed subgroups of D_n is $\sum_{d/n} \frac{n}{d} = \sum_{d/n} d = \sigma(n)$. They are $\cup_{d/n}\{< r_{n/d}, s_i >: \ 0 \leq i \leq \frac{n}{d} - 1\}$. □

From theorem 3.1 and theorem3.5 we have,

Theorem 3.6. *The number of subgroups of D_n is $\sigma(n) + d(n)$.*

Theorem 3.7. *The number of abelian subgroups of D_n is $d(n)+n$ if n is odd and $d(n)+n+\frac{n}{2}$ if n is even.*

Proof. All non-mixed subgroups of D_n are cyclic and hence abelian. So by theorem 3.1, there are $d(n)$ non- mixed abelian subgroups for D_n. If n is odd, by theorem 3.3 and corollary 3.1, the mixed abelian subgroups of D_n are of order 2 and hence there are n such subgroups. Thus if n is odd, the number of abelian subgroups of D_n is $d(n)+n$. If n is even, by theorem 3.3 and corollary 3.1, the mixed abelian subgroups of D_n are of order 2 and 4, and hence there are $n+\frac{n}{2}$ such subgroups. Thus if n is even, the number of abelian subgroups of D_n is $d(n)+n+\frac{n}{2}$. □

Theorem 3.8. *The number of cyclic subgroups of D_n is $d(n)+n$.*

Proof. By theorem 3.1, the number of non-mixed cyclic subgroups of D_n is $d(n)$. Also by theorem 3.3 and corollary 3.1,the mixed cyclic subgroups of D_n is n. Hence the number of cyclic subgroups of D_n is $d(n)+n$. □

4 Basis of D_n

A basis of D_n which contain both rotation and reflection is called a mixed basis and other basis is called non-mixed basis. By the definition 2.2, it is obvious that two rotations cannot generate D_n. Hence non-mixed basis of D_n are basis consisting of two reflections.

Theorem 4.1. *For $n \geq 3$, the number of mixed basis of D_n is $n\varphi(n)$.*

Proof. Let $s_j (0 \leq j \leq n-1)$ be a reflection in D_n. Then for any $0 \leq i \leq n-1$,

$$\begin{aligned}
< r_i, s_j > &= \{r_i^m s_j^t \ : 0 \leq m, t \leq n-1\} \quad \text{; by theorem 2.9} \\
&= \{r_i^m s_j,\, r_i^m r_0 \ : 0 \leq m \leq n-1\} \quad \text{; since } s_j^t = s_j \text{ or } r_0 \\
&= \{r_i^m s_j,\, r_i^m \ : 0 \leq m \leq n-1\} \\
&= \{r_i^m s_j \ : 0 \leq m \leq n-1\} \cup \{r_i^m \ : 0 \leq m \leq n-1\} \\
&= < r_i > s_j \cup < r_i > = D_n \ \text{ if and only if } (i,n)=1
\end{aligned}$$

Hence corresponding to each reflection $s_j (0 \leq j \leq n-1)$ there are $\varphi(n)$ mixed bases, namely $\{\{s_j, r_i\} : 0 \leq i \leq n-1 \text{ and } (i,n)=1\}$. So the number of mixed basis for $D_n (n \geq 3)$ is $n\varphi(n)$. □

Theorem 4.2. *For $n \geq 3$, the number of non-mixed basis of D_n is $\frac{n\varphi(n)}{2}$.*

Proof. Since the dimension of D_n is 2, any basis of D_n contain exactly two elements. The subgroup generated by two rotations always lies in R_n and hence cannot form a basis. Therefore any non- mixed basis of D_n contain exactly two reflections. : Let $s_j (0 \leq j \leq n-1)$ be a reflection in D_n. Then for any $0 \leq i \leq n-1$,

$$\begin{aligned} < s_i, s_j > &=< r_{i-j} s_j, s_j >=< r_{i-j}, s_j > \\ &\cong D_n \text{ if and only if } i - j \equiv k \ mod(n) \text{ and } (k, n) = 1 \end{aligned}$$

Hence corresponding to each reflection $s_j (0 \leq j \leq n-1)$ there are $\varphi(n)$ non-mixed basis for D_n namely $\{\{s_{i+j}, s_j\} : 0 \leq i \leq n-1 \text{ and } (i, n) = 1\}$. If $\{s_i, s_j\}$ is a mixed basis corresponding to the reflection s_i, then it is also a basis corresponding to the reflection s_j. Hence the number of non-mixed basis for $D_n (n \geq 3)$ is $\dfrac{n\varphi(n)}{2}$. □

Theorem 4.3. *For $n \geq 3$, the number of different basis for D_n is $\dfrac{3n}{2}\varphi(n)$.*

Proof. The collection of all different bases of $D_n (n \geq 3)$ is the union of all mixed and non-mixed bases. Hence the different bases of $D_n (n \geq 3)$ is $\dfrac{n\varphi(n)}{2} + n\varphi(n) = \dfrac{3n}{2}\varphi(n)$. □

5 Congugacy classes of D_n

In this section we will compute all conjugacy classes and class equation of Dihedral groups.

Theorem 5.1. *If n is odd, the number of conjugacy classes in D_n is $\dfrac{n+3}{2}$.*

Proof. Let $r_i (0 \leq i \leq n-1)$ be a rotation in D_n. Then

$$\begin{aligned} cl(r_i) &= \{r_j r_i r_j^{-1}, s_j r_i s_j^{-1} \ : \ 0 \leq j \leq n-1\} \\ &= \{r_j r_i r_{-j}, s_j r_i s_j \ : \ 0 \leq j \leq n-1\} \\ &= \{r_i, s_j r_i s_j \ : \ 0 \leq j \leq n-1\} \\ &= \{r_i, s_{j-i} s_j \ : \ 0 \leq j \leq n-1\} \\ &= \{r_i, r_{-i}\} \end{aligned}$$

Since n is odd, $r_i = r_{-i}$ if and only if $i = 0$. Therefore

$$cl(r_0) = \{r_0\} \text{ and } cl(r_i) = \{r_i, r_{-i}\}, \text{ a two element set, for all } 1 \leq i \leq n-1.$$

Also,

$$\begin{aligned} cl(s_0) &= \{r_j s_0 r_j^{-1}, s_j s_0 s_j^{-1} \ : \ 0 \leq j \leq n-1\} \\ &= \{r_j s_0 r_j^{-1}, s_j s_0 s_j \ : \ 0 \leq j \leq n-1\} \\ &= \{r_j s_0 r_{-j}, s_j s_0 s_j \ : \ 0 \leq j \leq n-1\} \\ &= \{s_{2j} \ : \ 0 \leq j \leq n-1\} \\ &= \{s_j \ : \ 0 \leq j \leq n-1\}, \text{ since } n \text{ odd.} \end{aligned}$$

Hence, if n is odd, $\{\{s_j \ : \ 0 \leq j \leq n-1\}, \{r_0\}, \{r_i, r_{-i} \ \} : \ 1 \leq i \leq (n-1)/2\}$ are the conjugacy classes of D_n. Thus if n is odd, the number of conjugacy class of D_n is $\dfrac{(n-1)}{2} + 2 = \dfrac{(n+3)}{2}$. □

Corolary 5.1. *The class equation of D_n(n odd) is $1 + 2 + 2 + \ldots + 2 + n = 2n$, the summation runs over $(n-1)/2$ times.*

Theorem 5.2. *If n is even, the number of conjugacy classes in D_n is $\dfrac{n+6}{2}$.*

Proof. Let $r_i (0 \leq i \leq n-1)$ be a rotation in D_n. Then

$$\begin{aligned} cl(r_i) &= \{r_j r_i r_j^{-1}, s_j r_i s_j^{-1} \ : \ 0 \leq j \leq n-1\} \ = \{r_j r_i r_{-j}, s_j r_i s_j \ : \ 0 \leq j \leq n-1\} \\ &= \{r_i, s_j r_i s_j \ : \ 0 \leq j \leq n-1\} \\ &= \{r_i, s_{j-i} s_j \ : \ 0 \leq j \leq n-1\} \\ &= \{r_i, r_{-i}\} \end{aligned}$$

Since n is even $r_i = r_{-i}$ if and only if $i = 0$ or $\dfrac{n}{2}$. Therefore

$cl(r_0) = \{r_0\}, cl(r_{n/2}) = \{r_{n/2}\}$ and $cl(r_i) = \{r_i, r_{-i}\},$ a two element set, for all $1 \leq i \leq n-1$ and $i \neq \dfrac{n}{2}$.

Also,

$$\begin{aligned} cl(s_0) &= \{r_j s_0 r_j^{-1}, s_j s_0 s_j^{-1} \ : \ 0 \leq j \leq n-1\} \\ &= \{r_j s_0 r_j^{-1}, s_j s_0 s_j \ : \ 0 \leq j \leq n-1\} \\ &= \{r_j s_0 r_{-j}, s_j s_0 s_j \ : \ 0 \leq j \leq n-1\} \\ &= \{s_{2j} \ : \ 0 \leq j \leq n/2-1\} \end{aligned}$$

Again,

$$\begin{aligned} cl(s_1) &= \{r_j s_1 r_j^{-1}, s_j s_1 s_j^{-1} \ : \ 0 \le j \le n-1\} \\ &= \{r_j s_1 r_j^{-1}, s_j s_1 s_j \ : \ 0 \le j \le n-1\} \\ &= \{r_j s_1 r_{-j}, s_j s_1 s_j \ : \ 0 \le j \le n-1\} \\ &= \{s_{2j+1} \ : \ 0 \le j \le n-1\} \\ &= \{s_{2j+1} \ : \ 0 \le j \le n/2-1\} \end{aligned}$$

Hence, if n is even,

$$\Big\{ \{s_{2j} \ : \ 0 \le j < n/2\}, \{s_{2j+1} \ : \ 0 \le j < n/2\}, \{r_0\}, \{r_{n/2}\}, \{r_i, r_{-i}\} : \ 1 \le i \le (n-2)/2 \Big\}$$

are the conjugacy classes of D_n. Thus if n is even, the number of conjugacy class of D_n is $\dfrac{(n-2)}{2} + 4 = \dfrac{(n+6)}{2}$. □

Corolary 5.2. *The class equation of $D_n(n \text{ even })$ is $1 + 1 + 2 + 2 + \ldots + 2 + n/2 + n/2 = 2n$, the summation runs over $(n-2)/2$ times.*

Corolary 5.3. *Each conjugacy class of D_n contains either rotations alone or reflections alone.*

Corolary 5.4. *The number of conjugacy classes of D_n which contain rotations alone is $\dfrac{(n+1)}{2}$ if n is odd and $\dfrac{(n+2)}{2}$ if n is even.*

Corolary 5.5. *The number of conjugacy classes of D_n which contain reflections alone is 1, namely D_n, if n is odd and is 2, namely $\Big\{ \{s_{2j} \ : \ 0 \le j < n/2\}, \{s_{2j+1} \ : \ 0 \le j < n/2\} \Big\}$, if n is even.*

6 Normal subgroups of D_n

In this section we will describe all mixed and non-mixed normal subgroups of D_n.

Theorem 6.1. *The number of non-mixed normal subgroups of D_n is $d(n)$.*

Proof. Since R_n is a cyclic normal subgroup of D_n, by theorem 2.11, the non-mixed subgroups and non-mixed normal subgroup of D_n are same. Hence the number of non-mixed normal subgroups of D_n is $d(n)$. □

Theorem 6.2. *The number of mixed normal subgroups of D_n is 1 if n odd and 3 if n even.*

Proof. Since normal subgroups are union of conjugacy classes, a mixed normal subgroup contain at least one conjugacy class having reflection. If n is odd, there is only one conjugacy class having reflection, namely $\{s_j : 0 \leq j \leq n-1\}$. Therefore D_n is the only mixed normal subgroup of D_n if n is odd. If n even, $\{s_{2j} : 0 \leq j < n/2\}$ and $\{s_{2j+1} : 0 \leq j < n/2\}$ are the only conjugacy classes having reflection. Therefore $\{s_{2j}, r_{2j} : 0 \leq j < n/2\}, \{s_{2j+1}, r_{2j} : 0 \leq j < n/2\}$ and D_n are the only mixed normal subgroups of D_n if n is even. Therefore the number of mixed normal subgroups of D_n is 3 if n is even.

□

Corolary 6.1. *The number of normal subgroups of D_n is $d(n) + 1$ if n odd and $d(n) + 3$ if n even.*

7 Conclusion

In this paper, it is proved that the number of mixed basis and non-mixed basis for $D_n(n \geq 3)$ are $n\varphi(n)$ and $\frac{n\varphi(n)}{2}$ respectively, where $\varphi(n)$is the number of non- negative integers less than n and relatively prime to n. Also it is shown that the number of different bases for $D_n(n \geq 3)$ is $\frac{3n}{2}\varphi(n)$. If n is odd, the number of conjugacy classes in D_n is $\frac{n+3}{2}$ and if n is even, the number of conjugacy classes in D_n is $\frac{n+6}{2}$. Finally we have shown that the number of non-mixed normal subgroups of D_n is $d(n)$ and the number of mixed normal subgroups of D_n is 1 if n odd and 3 if n even.

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