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(Intuitionistic) Fuzzy Grade of a Hypergroupoid: A Survey of Some Recent Researches

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Abstract

This paper aims to present a short survey on two numerical functions determined by a hypergroupoid, called the fuzzy grade and the intuitionistic fuzzy grade of a hypergroupoid. It starts with the main construction of the sequences of join spaces and (intuitionistic) fuzzy sets associated with a hypergroupoid. After some computations of the above grades, we discuss some similarities and differences between the two grades for the complete hypergroups and for the i.p.s. hypergroups. We conclude with some open problems.

Key words: (Intuitionistic) Fuzzy Grade, Join Space, I.p.s. Hypergroup, Complete Hypergroup.

MSC2012: 20N20; 03E72.

1 Introduction

The study of hyperstructures connected with fuzzy sets represents a growing and new line of research spanning over the last twenty years. Till now one can distinguish three principal approaches of this theme: the study of new crisp hyperoperations obtained by means of fuzzy sets; the study of fuzzy subhyperstructures (i.e. fuzzy sets those level sets are crisp hyperstructures); the study of structures endowed with fuzzy hyperoperations and called fuzzy hyperstructures.

We concentrate here on the first line of reserach. Its origin is due to Corsini [12] in 1993, when he defined a join space from a nonempty set endowed with a fuzzy set. Another definition of a join space induced by a fuzzy set was given in 1997 by Ameri and Zahedi [1]. The Corsini's direction was then investigated by Corsini and Leoreanu [16]. Ten years later, the same Corsini [13] considered the inverse problem: to define a fuzzy set from a hypergroupoid. Iterating both constructions he obtained a sequence of join spaces and fuzzy sets that stops when there exist two consecutive non-isomorphic join spaces. The length of the sequence, i.e. the number of the non-isomorphic join spaces in it, was called by Corsini and Cristea [14, 15] the fuzzy grade of a hypergroupoid. At the beginning the sequence was studied for some particular hypergroups: the complete hypergroups and the i.p.s. hypergroups. Recently this grade has been investigated with a general and innovative method by Cristea [22, 23], Ştefănescu and Cristea [30], Angheluţă and Cristea [2, 3] making use of an ordered n-tuple determined by an equivalence relation. We will see the details in the next section.

One of the principal concerns of several researchers is that to obtain new hyperstructures using modern tools for investigation of uncertain problems, like intuitionistic fuzzy sets, rough sets, soft sets, interval-valued fuzzy sets. Following Corsini's idea, Cristea and Davvaz [24] associated with any hypergroupoid a sequence of join spaces and intuitionistic fuzzy sets with the length called the intuitionistic fuzzy grade. Although the fuzzy grade and the intuitionistic fuzzy grade have similar definitions, the corresponding associated sequences of join spaces have different properties. In order to highlight the differences between them, Davvaz, Hassani-Sadrabadi and Cristea [25, 26, 27] have investigated this problem for the complete hypergroups and i.p.s. hypergroups of order less than 8. In this survey our main goal is to realize a comparison between the two sequences.

The second section of the paper gives a survey of the construction of the sequences of join spaces associated with a hypergroupoid or with a non-empty set endowed with an (intuitionistic) fuzzy set. Section 3 is dedicated to some computations of the two grades: the fuzzy and the intuitionistic fuzzy grade of a hypergroupoid. In Section 4 we focus our comparison between the two grades on the cases of the complete hypergroups and of the i.p.s. hypergroups of order less than 8. In the last section we conclude with the statement of some open problems.

2 Sequences of the join spaces associated with a hypergroupoid

We briefly recall the construction of the sequences of join spaces and fuzzy sets/intuitionistic fuzzy sets associated with a hypergroupoid. For a comprehensive overview of hypergroups theory the reader is refereed to [11, 17, 31] and for fuzzy and intuitionistic fuzzy sets theory see [32, 5, 6, 7, 8]. We will use the terminology and the notations from [13, 23, 30].

Let X be a nonempty set. A *fuzzy set* A in X is characterized by a *membership function* $\mu_A : X \longrightarrow [0,1]$, where for any $x \in X$, the value $\mu_A(x)$ represents the *grade of membership* of x in A. More generally, any function $\mu : X \longrightarrow [0,1]$ is called a *fuzzy subset* of X. We denote by $FS(X)$ the set of all fuzzy subsets of X.

An *Atanassov's intuitionistic fuzzy set* A in X is a triplet of the form $A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}$, where, for any $x \in X$, the *degree of membership* of x (namely $\mu_A(x)$) and the *degree of non-membership* of x (namely $\lambda_A(x)$) verify the relation $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$; for simplicity, we denote it by $A = (\mu, \lambda)$. We denote by $IFS(X)$ the set of all intuitionistic fuzzy subsets of X.

For any nonempty set H endowed with a fuzzy subset $\alpha \in FS(H)$, Corsini[12] defined a join space $(H, \circ_\alpha)$, where the hyperproduct is defined as it follows:

$$x \circ_\alpha y = \{z \in H \mid \alpha(x) \wedge \alpha(y) \leq \alpha(z) \leq \alpha(x) \vee \alpha(y)\} \tag{1}$$

Conversely, from any hypergroupoid $(H, \circ)$ Corsini[13] defined a fuzzy subset $\widetilde{\mu} \in FS(H)$ by the following formula:

$$\widetilde{\mu}(u) = \frac{\displaystyle\sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}}{q(u)} \tag{2}$$

where $Q(u) = \{(a,b) \in H^2 \mid u \in a \circ b\}, \quad q(u) = |Q(u)|.$

If $Q(u) = \emptyset$, we set $\widetilde{\mu}(u) = 0$.

In 2010 Cristea and Davvaz[24] defined in a similar way an intuitionistic fuzzy subset associated with a hypergroupoid $(H, \circ)$:

$$\overline{\mu}(u) = \frac{\displaystyle\sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}}{n^2}, \qquad \overline{\lambda}(u) = \frac{\displaystyle\sum_{(x,y)\in \overline{Q}(u)} \frac{1}{|x \circ y|}}{n^2} \tag{3}$$

Iterating equations (1) and (2) one obtains a sequence $({}^iH = (H, \circ_i), \widetilde{\mu}_{i-1})_{i\geq 1}$ of join spaces and fuzzy sets associated with H. The sequence stops when there exist two consecutive isomorphic join spaces. The length of this sequence, that is the number of the non-isomorphic join spaces in the sequence, is called the *fuzzy grade* of the hypergroupoid H. More exactly we have the following definition.

Definition 2.1. [14] A hypergroupoid H has the *fuzzy grade* $m \in \mathbb{N}^*$, and we write $f.g.(H) = m$ if, for any $1 \leq i < m$, the join spaces ${}^iH$ and ${}^{i+1}H$ associated with H are not isomorphic and, for any $s > m$, ${}^sH$ is isomorphic with ${}^mH$.
We say that the hypergroupoid H has the *strong fuzzy grade* m, and we write $s.f.g.(H) = m$, if $f.g.(H) = m$ and, for all $s, s > m$, ${}^sH = {}^mH$.

Similarly, iterating equations (1) and (3) one obtains two sequences $({}_iH = (H, \circ_{\overline{\mu}_i \wedge \overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i, \overline{\lambda}_i))_{i\geq 0}$ and $({}^iH = (H, \circ_{\overline{\mu}_i \vee \overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i, \overline{\lambda}_i))_{i\geq 0}$ of join spaces and intuitionistic fuzzy sets associated with H. The sequences stop when there exist two consecutive isomorphic join spaces and their lengths are called the *lower/upper intuitionistic fuzzy grade* of the hypergroupoid H. Let's see better their definitions.

Definition 2.2. [24] A set H endowed with an intuitionistic fuzzy set $A = (\mu, \lambda)$ has the *lower intuitionistic fuzzy grade* m and we write $l.i.f.g.(H) = m$ if, for any $0 \leq i < m$, the join spaces $(H, \circ_{\overline{\mu}_i \wedge \overline{\lambda}_i})$ and $(H, \circ_{\overline{\mu}_{i+1} \wedge \overline{\lambda}_{i+1}})$ associated with H are not isomorphic and, for any $s \geq m$, ${}_sH$ is isomorphic with ${}_{m-1}H$.

Definition 2.3. [24] A set H endowed with an intuitionistic fuzzy set $A = (\mu, \lambda)$ has the *upper intuitionistic fuzzy grade* m and we write $u.i.f.g.(H) = m$ if, for any $0 \leq i < m$, the join spaces $(H, \circ_{\overline{\mu}_i \vee \overline{\lambda}_i})$ and $(H, \circ_{\overline{\mu}_{i+1} \vee \overline{\lambda}_{i+1}})$ associated with H are not isomorphic and, for any $s \geq m$, ${}^sH$ is isomorphic with ${}^{m-1}H$.

The construction of the above sequences can start in two different ways:

1. from a hypergroupoid $(H, \circ)$;

2. from a nonempty set H endowed with a fuzzy set or an intuitionistic fuzzy set.

It is easy to prove that in both cases one obtains the same fuzzy grade. But what can we say about the other two sequences of join spaces and intuitionistic fuzzy sets?

If we start the construction from a hypergroupoid $(H, \circ)$, we obtain only one sequence of join spaces, because the first join spaces ${}_0H = (H, \circ_{\overline{\mu}\wedge\overline{\lambda}})$ and ${}^0H = (H, \circ_{\overline{\mu}\vee\overline{\lambda}})$ are always isomorphic. Indeed, since, for any $x \in H$, the sum $\overline{\mu}(x) + \overline{\lambda}(x)$ is always constant, it follows that $\overline{\mu}(x) = \overline{\mu}(y)$ if and only if $\overline{\lambda}(x) = \overline{\lambda}(y)$.
Therefore in this case $l.i.f.g.(H) = u.i.f.g.(H)$ and it is simply called the *intuitionistic fuzzy grade* of H (shortly $i.f.g.(H)$).

On the other side, if we start the construction from a nonempty set H endowed with an intuitionistic fuzzy set, one may obtain two different sequences with different lengths. We only illustrate this case by the following example.

Example 2.1. [24] On $H = \{a, b, c, d\}$ we define the Atanassov's intuitionistic fuzzy set:

$$\begin{array}{llll} \overline{\mu}(a) = 0.25 & \overline{\mu}(b) = 0.25 & \overline{\mu}(c) = 0.30 & \overline{\mu}(d) = 0.10 \\ \overline{\lambda}(a) = 0.40 & \overline{\lambda}(b) = 0.40 & \overline{\lambda}(c) = 0.50 & \overline{\lambda}(d) = 0.90. \end{array}$$

To start with, we construct the first sequence of join spaces and we determine its $l.i.f.g.(H)$.

For the associated join space

$\circ_{\overline{\mu}\wedge\overline{\lambda}}$	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,d\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,d\}$
c	$\{a,b,c\}$	$\{a,b,c\}$	c	H
d	$\{a,b,d\}$	$\{a,b,d\}$	H	d

we calculate the Atanassov's intuitionistic fuzzy set associated with H as in (3) and we obtain the following values:

$$\begin{array}{llll} \overline{\mu}_1(a) = 31/96 & \overline{\mu}_1(b) = 31/96 & \overline{\mu}_1(c) = 17/96 & \overline{\mu}_1(d) = 17/96 \\ \overline{\lambda}_1(a) = 12/96 & \overline{\lambda}_1(b) = 12/96 & \overline{\lambda}_1(c) = 26/96 & \overline{\lambda}_1(d) = 26/96, \end{array}$$

therefore $\overline{\mu}_1 \wedge \overline{\lambda}_1(a) = \overline{\mu}_1 \wedge \overline{\lambda}_1(b) < \overline{\mu}_1 \wedge \overline{\lambda}_1(c) = \overline{\mu}_1 \wedge \overline{\lambda}_1(d)$ and thereby it results the following join space

$\circ_{\overline{\mu}_1\wedge\overline{\lambda}_1}$	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	H	H
b	$\{a,b\}$	$\{a,b\}$	H	H
c	H	H	$\{c,d\}$	$\{c,d\}$
d	H	H	$\{c,d\}$	$\{c,d\}$

For any $x \in H$, we compute $\overline{\mu}_2(x) = 4/16$ and $\overline{\lambda}_2(x) = 2/16$; thus, for any $x, y \in H$, $x \circ_{\overline{\mu}_2\wedge\overline{\lambda}_2} y = H$. So the first sequence of join spaces associated with H has 3 elements, that is $l.i.f.g.(H) = 3$.

Now, in order to determine the $u.i.f.g.(H)$, we start with the join space

$\circ_{\overline{\mu}\vee\overline{\lambda}}$	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	H
b	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	H
c	$\{a,b,c\}$	$\{a,b,c\}$	c	$\{c,d\}$
d	H	H	$\{c,d\}$	d

Note that $\langle {}_0H, \circ_{\overline{\mu}\wedge\overline{\lambda}}\rangle$ and $\langle {}^0H, \circ_{\overline{\mu}\vee\overline{\lambda}}\rangle$ are not isomorphic. Since

$$\begin{array}{llll} \overline{\mu}_1(a) = 13/48 & \overline{\mu}_1(b) = 13/48 & \overline{\mu}_1(c) = 13/48 & \overline{\mu}_1(d) = 9/48 \\ \overline{\lambda}_1(a) = 9/48 & \overline{\lambda}_1(b) = 9/48 & \overline{\lambda}_1(c) = 9/48 & \overline{\lambda}(d) = 13/48 \end{array}$$

it follows that $\overline{\mu}_1 \vee \overline{\lambda}_1(x) = 13/48$, whenever $x \in H$, which means that, for any $x, y \in H$, $x \circ_{\overline{\mu}_1\vee\overline{\lambda}_1} y = H$. Therefore the second sequence of join spaces associated with H has 2 elements, that is $u.i.f.g.(H) = 2$.

Sometimes the computations of the values of the membership functions $\widetilde{\mu}_i, \overline{\mu}_i, \overline{\lambda}_i$, $i \geq 0$ could take much time, thus it is useful to determine the (intuitionistic) fuzzy grade of a hypergroupoid without calculating all these values. In order to realize this we introduce some notations.

With any join space ${}^iH$ in the sequence of join spaces and fuzzy sets corresponding to H, one may associate an *ordered chain* $({}^iC_1, {}^iC_2, ..., {}^iC_r)$ and an *ordered r-tuple* $({}^ik_1, {}^ik_2, ..., {}^ik_r)$, where

1. $\forall j \geq 1$: $x, y \in {}^iC_j \Longleftrightarrow \widetilde{\mu}_{i-1}(x) = \widetilde{\mu}_{i-1}(y)$,
2. for $x \in {}^iC_j$ and $z \in {}^iC_k$, if $j < k$ then $\widetilde{\mu}_{i-1}(x) < \widetilde{\mu}_{i-1}(z)$
3. ${}^ik_j = |{}^iC_j|$, for all j.

We have similar notations for the sequence $({}_iH = (H, \circ_{\overline{\mu}_i\wedge\overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i, \overline{\lambda}_i))_{i\geq 0}$.

With any join space ${}_iH$ one may associate an *ordered chain* $({}^iC_1, {}^iC_2, ..., {}^iC_r)$ and an *ordered r-tuple* $({}^ik_1, {}^ik_2, ..., {}^ik_r)$, where

1. $\forall j \geq 1$: $x, y \in {}^iC_j \Longleftrightarrow \overline{\mu}_i \wedge \overline{\lambda}_i(x) = \overline{\mu}_i \wedge \overline{\lambda}_i(y)$,

2. for $x \in {}^iC_j$ and $z \in {}^iC_k$, if $j < k$ then $\overline{\mu}_i \wedge \overline{\lambda}_i(x) < \overline{\mu}_i \wedge \overline{\lambda}_i(z)$

3. ${}^ik_j = |{}^iC_j|$, for all j.

One of the crucial questions we have to ask is: **When two consecutive join spaces in these sequences are isomorphic?**

A first answer was given by Corsini and Leoreanu [16] in 1995.

Theorem 2.1. [16] *Let ${}^iH$ and ${}^{i+1}H$ be two consecutive join spaces associated with H determined by the membership functions $\widetilde{\mu}_{i-1}$ and $\widetilde{\mu}_i$, where ${}^iH = \bigcup_{l=1}^{r_1} C_l$, ${}^{i+1}H = \bigcup_{l=1}^{r_2} C'_l$ and $(k_1, k_2, ..., k_{r_1})$ is the r_1-tuple associated with ${}^iH$, $(k'_1, k'_2, ..., k'_{r_2})$ is the r_2-tuple associated with ${}^{i+1}H$.*

The join spaces ${}^iH$ and ${}^{i+1}H$ are isomorphic if and only if $r_1 = r_2$ and $(k_1, \ldots, k_{r_1}) = (k'_1, \ldots, k'_{r_1})$ or $(k_1, \ldots, k_{r_1}) = (k'_{r_1}, \ldots, k'_1)$.

A similar theorem can be formulated also for the join spaces in the sequence $({}_iH = (H, \circ_{\overline{\mu}_i \wedge \overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i, \overline{\lambda}_i))_{i \geq 0}$.

In order to use this result we need to determine both r-tuples associated with ${}^iH$ and ${}^{i+1}H$. But Ştefănescu and Cristea [30] have given a sufficient condition such that two consecutive join spaces are not isomorphic, that is the sequence doesn't stop, using only one r-tuple.

Theorem 2.2. [30] *Let $(k_1, k_2, ..., k_r)$ be the r-tuple associated with the join space ${}^iH$. If $(k_1, \ldots, k_r) = (k_r, k_{r-1}, \ldots, k_1)$, then the join spaces ${}^iH$ and ${}^{i+1}H$ are not isomorphic.*

A second crucial problem arises: **does there exist a hypergroupoid H such that $s.f.g.(H)/i.f.g.(H) = n$, whenever n is a natural number?** The anser is yes, and an example is given in the following fundamental result.

Theorem 2.3. [30, 24] *Let $H = \{x_1, x_2, \ldots, x_s\}$, where $s = 2^n$, $n \in \mathbb{N}^* \setminus \{1, 2\}$, be the commutative hypergroupoid defined by the hyperproduct*

$$\begin{aligned} &x_i \circ x_i = x_i, 1 \leq i \leq n, \\ &x_i \circ x_j = \{x_i, x_{i+1}, \ldots, x_j\}, 1 \leq i < j \leq n. \end{aligned}$$

Then $s.f.g.(H) = i.f.g.(H) = n$. Moreover, H is a join space.

3 Computation of the fuzzy grade/intuitionistic fuzzy grade of a hypergroupoid

Let $(H, \circ)$ be a hypergroupoid. We define the following equivalence on H

$$xR_{\widetilde{\mu}}y \Longleftrightarrow \widetilde{\mu}(x) = \widetilde{\mu}(y).$$

We determine $f.g.(H)$ when $|H/R_{\widetilde{\mu}}| \in \{2, 3\}$.

Theorem 3.1. [2] *Let $(H, \circ)$ be a finite hypergroupoid.*

1. *If $|H/R_{\widetilde{\mu}}| = 2$, that is (k_1, k_2) is the pair associated with H, then $s.f.g.(H) = 2$ whenever $k_1 = k_2$, and $f.g.(H) = 1$ otherwise.*
2. *If $|H/R_{\widetilde{\mu}}| = 3$, that is (k_1, k_2, k_3) is the triple associated with H, and*
   (a) *if $k_1 = k_2 = k_3$, then $f.g.(H) = 2$*
   (b) *if $k_1 = k_3 \neq k_2$, then $f.g.(H) = 2$ whenever $k_2 \neq 2k_3$ and $s.f.g.(H) = 3$, otherwise*
   (c) *if $k_1 < k_2 = k_3$, then $f.g.(H) = 1$*
   (d) *if $k_1 = k_2 < k_3$, then $f.g.(H) = 1$ whenever $P = 2k_3^3 - 8k_1^3 - k_1^2k_3 + 5k_1k_3^2 > 0$ and $f.g.(H) = 3$, otherwise*
   (e) *if $k_1 \neq k_2 \neq k_3$, there is no precise order between $\widetilde{\mu}_1(x), \widetilde{\mu}_1(y), \widetilde{\mu}_1(z)$.*

A similar result we obtain for the quotient $H/R_{\overline{\mu}\wedge\overline{\lambda}}$.

Theorem 3.2. [3] *Let $(H, \circ)$ be a finite hypergroupoid.*

1. *If $|H/R_{\overline{\mu}\wedge\overline{\lambda}}| = 2$, that is (k_1, k_2) is the pair associated with H, then $i.f.g.(H) = 2$ whenever $k_1 = k_2$, and $i.f.g.(H) = 1$, otherwise.*
2. *If $|H/R_{\overline{\mu}\wedge\overline{\lambda}}| = 3$, that is (k_1, k_2, k_3) is the triple associated with H, and*
   (a) *if $k_1 = k_2 = k_3$, then $i.f.g.(H) = 2$*
   (b) *if $k_1 = k_3 \neq k_2$, then $i.f.g.(H) = 2$ whenever $k_2 \neq 2k_3$ and $i.f.g.(H) = 3$, otherwise*
   (c) *if $k_1 < k_2 = k_3$, then $i.f.g.(H) = 1$ whenever $2k_2^2 > 3k_1k_2 + 3k_1^2$ and $i.f.g.(H) = 3$, otherwise*
   (d) *if $k_1 = k_2 < k_3$, then $i.f.g.(H) = 1$ whenever $k_3 \neq 2k_1$ and $i.f.g.(H) = 2$, otherwise*
   (e) *if $k_1 \neq k_2 \neq k_3$, then there is no precise order between $\overline{\mu}_1(x) \wedge \overline{\lambda}_1(x), \overline{\mu}_1(y) \wedge \overline{\lambda}_1(y), \overline{\mu}_1(z) \wedge \overline{\lambda}_1(z)$.*

Angheluţă and Cristea [3] have studied the intuitionistic fuzzy grade of a hypergroupoid in the case of some particular ternaries. We integrate here this theorem with similar results for the fuzzy grade. In the following the ternary (k_1, k_2, k_3) associated with H is intended respected with the equivalence $R_{\widetilde{\mu}}$ (when we talk about the fuzzy grade) and $R_{\overline{\mu}\wedge\overline{\lambda}}$, respectively, (when we talk about the intuitionistic fuzzy grade).

Proposition 3.1. *Let H be a hypergroupoid of cardinality $2k$, $k \geq 3$, with the ternary associated with H of the type $(k, k-1, 1)$. Then $s.f.g.(H) = i.f.g.(H) = 1$.*

Generalizing, we obtain the same result for the ternary $(pk, p(k-1), p)$, with $p, k \in \mathbb{N} \setminus \{0, 1\}$.

Proof. We prove only that $s.f.g.(H) = 1$. For the proof of the fact that $i.f.g.(H) = 1$, see [3].

Let H be a hypergroupoid such that $|H| = 2k$, $k \geq 3$, and $(k, k-1, 1)$ be the ternary associated with H determined by the equivalence $R_{\widetilde{\mu}}$. Thus H can be written as the union $H = C_1 \cup C_2 \cup C_3$, with $|C_1| = k, |C_2| = k-1, |C_3| = 1$. Using equation (2), for any $x \in C_1, y \in C_2, z \in C_3$, one obtains

$$\widetilde{\mu}_1(x) = \frac{4k^2 - k - 1}{3k^2(2k-1)}$$

$$\widetilde{\mu}_1(y) = \frac{8k^3 + 2k^2 - 12k + 4}{2k(2k-1)(3k^2-1)}$$

$$\widetilde{\mu}_1(z) = \frac{4k-2}{k(4k-1)}$$

After some computations that we omit here for lack of space, it results the following relation

$$\widetilde{\mu}_1(x) \leq \widetilde{\mu}_1(y) \leq \widetilde{\mu}_1(z),$$

which means that $(k, k-1, 1)$ is the ternary associated with the join space ${}^1H$ and thereby $s.f.g.(H) = 1$. □

Proposition 3.2. *Let $(2k, k, 1)$, with $k \geq 2$, be the ternary associated with a hypergroupoid H. Then*

1. *$s.f.g.(H) = 1$.*

2. *$i.f.g.(H) = 3$ and the sequence of join spaces is cyclic, that is ${}_{3l}H \simeq {}_3H$, ${}_{3l+1}H \simeq {}_1H$, ${}_{3l+2}H \simeq {}_2H$, for any $l \in \mathbb{N}^*$.*

Proof. Considering $(2k, k, 1)$, with $k \geq 2$, be the ternary associated with a hypergroupoid H respected with the equivalence $R_{\widetilde{\mu}}$, we decompose H as the union $H = C_1 \cup C_2 \cup C_3$, with $|C_1| = 2k, |C_2| = k, |C_3| = 1$. Using again equation (2), for any $x \in C_1, y \in C_2, z \in C_3$, one obtains

$$\widetilde{\mu}_1(x) = \frac{15k + 11}{6(2k+1)(3k+1)}$$

$$\widetilde{\mu}_1(y) = \frac{21k^2 + 58k + 25}{3(k+1)(3k+1)(5k+6)}$$

$$\widetilde{\mu}_1(z) = \frac{13k^2 + 10k + 1}{(k+1)(3k+1)(6k+1)}$$

For any $k \geq 2$, it follows that

$$\widetilde{\mu}_1(x) \leq \widetilde{\mu}_1(y) \leq \widetilde{\mu}_1(z),$$

that is $s.f.g.(H) = 1$.

For the second part of the theorem see [3]. □

Proposition 3.3. *Let $(h, k, 1)$, with $2 \leq h \leq k$, be the ternary associated with a hypergroupoid H.*

1. *Then $s.f.g.(H) = 1$.*
2. *If $h = 2$ and*
   (a) *$k = 2$, then $i.f.g.(H) = 3$;*
   (b) *$k = 3$, then $i.f.g.(H) = 2$;*
   (c) *$k \geq 4$, then $i.f.g.(H) = 3$ and the sequence of join spaces associated with H is cyclic.*
3. *If $h > 2$, then $i.f.g.(H) = 1$, whenever $k \geq h$.*

Proof. Since $(h, k, 1)$, with $2 \leq h \leq k$, is the ternary associated with a hypergroupoid H respected with the equivalence $R_{\widetilde{\mu}}$, we write H as the union $H = C_1 \cup C_2 \cup C_3$, with $|C_1| = h, |C_2| = k, |C_3| = 1$. Calculating the values of the membership function $\widetilde{\mu}_1$ by using the formula (2), we obtain, for any $x \in C_1, y \in C_2, z \in C_3$, that

$$\widetilde{\mu}_1(x) = \frac{h^2+3k^2+4hk+3h+5k}{(h+k)(h+k+1)(h+2k+2)}$$

$$\widetilde{\mu}_1(y) = \frac{3h^2k^2+7h^2k+4hk^3+9hk^2+7hk+k^4+4k^3+3k^2+2h^2}{(h+k)(h+k+1)(k+1)(2hk+2h+k^2+2k)}$$

$$\widetilde{\mu}_1(z) = \frac{5hk+3h+3k^2+4k+1}{(h+k+1)(k+1)(2h+2k+1)}$$

Similarly as in the previous two lemmas, we prove that

$$\widetilde{\mu}_1(x) \leq \widetilde{\mu}_1(y) \leq \widetilde{\mu}_1(z)$$

and thus $s.f.g.(H) = 1$.

For the second part of the theorem see [3].

$\square$

4 (Intuitionistic) fuzzy grade of special types of hypergroups

In this section we deal with two classes of hypergroups: the complete hypergroups and the hypergroups with partial scalar identities, shortly called i.p.s. hypergroups. We determine the fuzzy and intuitionistic fuzzy grades of all these hypergroups of order less than 8. This part of the paper is based on the articles of Corsini and Cristea [14, 15], Cristea [21], Cristea and Davvaz [24], Angheluţă and Cristea [2, 3], Davvaz, Hassani-Sadrabadi and Cristea [25, 26, 27].

4.1 The complete hypergroups

We start with the characterization of the complete hypergroups.

A hypergroup $(H, \circ)$ is a *complete hypergroup* if it can be written as the union of its subsets $H = \cup_{g \in G} A_g$, where

1. $(G, \cdot)$ is a group;

2. for any $(g_1, g_2) \in G^2$, $g_1 \neq g_2$, we have $A_{g_1} \cap A_{g_2} = \emptyset$;

3. if $(a, b) \in A_{g_1} \times A_{g_2}$, then $a \circ b = A_{g_1 g_2}$.

Let $G = \{g_1, \ldots, g_m\}$ be a finite group. Then with $H = \bigcup_{i=1}^{m} A_{g_i}$ we may associate an m-tuple $[k_1, \ldots, k_m]$, where $k_i = |A_{g_i}|$.
Using these notations and based on the fact that, for any $u \in H$, $\exists! g_u \in G : u \in A_{g_u}$, Corsini [13] has determined the formula for the membership function $\widetilde{\mu}$ associated with a complete hypergroup as the following one:

$$\widetilde{\mu}(u) = \frac{1}{|A_{g_u}|}.$$

Similarly, the formula of the membership functions $\overline{\mu}, \overline{\lambda}$ for a complete hypergroup is simpler that those in the general case, as Cristea and Davvaz proved [24].
Defining on H the following equivalence

$$u \sim v \Longleftrightarrow \exists g \in G : u, v \in A_g,$$

one obtains that:

$$\bar{\mu}(u) = \frac{|Q(u)|}{|A_{g_u}|} \cdot \frac{1}{n^2}, \quad \bar{\lambda}(u) = \left(\sum_{v \notin \hat{u}} \frac{|Q(v)|}{|A_{g_v}|} \right) \cdot \frac{1}{n^2}.$$

We notice that, if G_1 and G_2 are non isomorphic groups of the same cardinality and H_1, H_2 are the complete hypergroups obtained by them, then $f.g.(H_1) = f.g.(H_2)$, but their $i.f.g.$ may be different.

Cristea [21] determined the fuzzy grade of all complete hypergroups of order less than 7. Later on, Davvaz together with Hassani-Sadrabadi and Cristea [27] determined their intuitionistic fuzzy grade. We recall here these results.

Theorem 4.1. *Let H be a complete hypergroup of order $n \leq 6$.*

1. *There are two non isomorphic complete hypergroups of order three that have $s.f.g.(H) = i.f.g.(H) = 1$.*

2. *Among the five non isomorphic complete hypergroups of order four, three of them have $s.f.g.(H) = i.f.g.(H) = 1$ and two of them have $s.f.g.(H) = i.f.g.(H) = 2$.*

3. *There are 12 non isomorphic complete hypergroups of order 5. All of them have $s.f.g.(H) = 1$. Nine of them have $i.f.g.(H) = 1$ and the other three have $i.f.g.(H) = 3$.*

4. *There are* 21 *non isomorphic complete hypergroups of order* 6*:*
   - 17 *with* $s.f.g.(H) = 1$ *and* 4 *with* $s.f.g.(H) = 2$.
   - 16 *with* $i.f.g.(H) = 1$, 3 *with* $i.f.g.(H) = 2$ *and* 2 *with* $i.f.g.(H) = 3$.

Angheluţă and Cristea [2] have studied the fuzzy grade of some particular complete hypergroups. It remains an open problem, and not a very easy one, to determine their intuitionistic fuzzy grade.

Theorem 4.2. *Let* H *be a complete hypergroup of type*

1. $[\underbrace{p, p, ..., p}_{k \quad times}, kp]$, *where* $n = |H| = 2kp$. *Then* $s.f.g.(H) = 2$.

2. $[\underbrace{p, p, \ldots, p}_{s \quad times}, \underbrace{k, k, \ldots, k}_{t \quad times}, ps]$, $2 \leq p < k < ps$, $n = |H| = 2ps + kt$.

   - *for* $n = 4ps$, $s.f.g.(H) = 3$,
   - *for* $kt \neq 2ps$, $f.g.(H) = 2$.

3. $[\underbrace{k, k, \ldots, k}_{l \quad times}, \underbrace{p, p, \ldots, p}_{s \quad times}, \underbrace{s, s, \ldots, s}_{p \quad times}, \underbrace{l, l, \ldots, l}_{k \quad times}]$, $2 \leq k < p < s < l$, $n = |H| = 2(ps + kl)$.

   - *if* $kl = ps$, *then* $s.f.g.(H) = 3$,
   - *otherwise* $f.g.(H) = 2$.

4. $[\underbrace{1, \ldots, 1}_{k \quad times}, \underbrace{2, \ldots, 2}_{\frac{k}{2} \quad times}, \underbrace{3, \ldots, 3}_{\frac{k}{3} \quad times}, \ldots, \underbrace{m-1, \ldots, m-1}_{\frac{k}{m-1} \quad times}, k]$,
   $k = m.c.m.(1, 2, \ldots, m-1)$, $n = |H| = mk = 2^{s-1}k$.
   Then $s.f.g.(H) = s$.

4.2 The i.p.s. hypergroups

An *i.p.s. hypergroup* (i.e. a hypergroup with partial scalar identities) is a particular type of canonical hypergroup. We recall that a *canonical hypergroup* is a commutative hypergroup $(H, \circ)$ with a scalar identity, where every element $a \in H$ has a unique inverse a^{-1} such that it satisfies the following properties

1. if $y \in a \circ x$, then $x \in a^{-1} \circ y$;

2. $x \in x \circ a \Longrightarrow x = x \circ a$

The canonical hypergroups were introduced for the first time by Krasner [28] as the additive structures of the hyperfields and then Mittas [29] studied them independently from the others operations. Later on Corsini determined all i.p.s. hypergroups of order less than 9, proving that they are strongly canonical.

Corsini and Cristea [14, 15] calculated the fuzzy grade for all i.p.s. hypergroups of order less than 8 and their intuitionistic fuzzy grade have been determined by Davvaz, Hassan-sadrabadi and Cristea [25, 26]. Here we summarize these results.

Theorem 4.3. *1. There exists one i.p.s. hypergroup H of order 3 and $f.g.(H) = i.f.g.(H) = 1$.*

2. There exist 3 i.p.s. hypergroups of order 4 with $f.g.(H_1) = i.f.g.(H_1) = 1$, $f.g.(H_2) = 1, i.f.g.(H_2) = 2$, $f.g.(H_3) = i.f.g.(H_3) = 2$.

3. There exist 8 i.p.s. hypergroups of order 5: one has $f.g.(H) = 2$, all the others have $f.g.(H) = 1$; four of them have $i.f.g.(H) = 1$, three have $i.f.g.(H) = 2$ and one is with $i.f.g.(H) = 3$.

*4. There exist 19 i.p.s. hypergroups of order 6: 14 of them have $f.g.(H) = 1$, 4 have $f.g.(H) = 2$ and one has $f.g.(H) = 3$;
ten of them have $i.f.g.(H) = 1$, eight of them have $i.f.g.(H) = 2$ and only for one of them we find $i.f.g.(H) = 3$.*

*5. There exist 36 i.p.s. hypergroups of order 6: 27 of them with $f.g.(H) = 1$, 8 with $f.g.(H) = 2$, 1 with $f.g.(H) = 3$;
10 of them with $i.f.g.(H) = 1$, 10 with $i.f.g.(H) = 2$, 9 with $i.f.g.(H) = 3$, 5 with $i.f.g.(H) = 4$, 2 with $i.f.g.(H) = 5$.
Moreover, for 18 of them we find that the associated sequences of join spaces and intuitionistic fuzzy sets are cyclic.*

5 Future work

Based on the definition of the intuitionistic fuzzy set associated with a hypergroupoid [24], Ashgari-Larimi and Cristea [4] have defined the Atanassov's intuitionistic fuzzy index of a hypergroupoid. It would be interesting to obtain some relations between this index and the intuitionistic fuzzy grade.

We think that, finding necessary and sufficient conditions in order that the sequences of join spaces $({}_iH = (H, \circ_{\overline{\mu}_i \wedge \overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i, \overline{\lambda}_i))_{i\geq 0}$ and $({}^iH = (H, \circ_{\overline{\mu}_i \vee \overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i, \overline{\lambda}_i))_{i\geq 0}$ associated with a nonempty set H endowed with an intuitionistic fuzzy set (like in Section 2) coincide, could be another line of research on this argument.

We also intend to find in our future work some possible connection between hypergroups and automata. For example, what can we say about its (intuitionistic) fuzzy grade? Does there exist an automaton with (intuitionistic) fuzzy grade of its state hypergroup equal to a given natural number?

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