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Fuzzy homotopy analysis method for solving fuzzy autonomous differential equation

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Abstract

In this paper, we have presented the theory and the applications of the fuzzy homotopy analysis method to find the fuzzy semi-analytical solutions of the second order fuzzy autonomous ordinary differential equation. This method allows for the solution of the fuzzy initial value problems to be calculated in the form of an infinite fuzzy series in which the fuzzy components can be easily calculated. Some numerical results have been given to illustrate the used method. The obtained numerical results have been compared with the fuzzy exact-analytical solutions.

Keywords: fuzzy homotopy analysis method; fuzzy autonomous differential equation; fuzzy series solution.
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1. Introduction

The topic of the fuzzy semi-analytical methods(fuzzy series method) for solving the fuzzy initial value problems(FIVPs) has been rapidly growing in recent years, whereas the fuzzy series solutions of FIVP have been studied by several authors during the past few years. Several fuzzy semi-analytical methods have been proposed to obtain the fuzzy series solution of the linear and non-linear FIVB which are mostly first order problems. Some of these methods have been proposed to obtain the fuzzy series solutions of the high order FIVB.

Fuzzy homotopy analysis method was used for the first time to solve the fuzzy differential equations in 2012. Researchers and scientists are continuing to develop this method for solving various types of the fuzzy initial value problems because it represents an efficient and effective technique.

In the following we will review some of the findings of the researchers regarding this method. In 2012, Hashemi, Malekinagad and Marasi[4] suggested and applied the Fuzzy homotopy analysis method for solving a system of fuzzy differential equations with fuzzy initial conditions. In 2013, Abu-Arqub, El-Ajou1 and Momani[6] studied and developed the Fuzzy homotopy analysis method to obtain the analytical solutions of the fuzzy initial value problems. In 2014, Jameel, Ghoreishi and Ismail[8] introduced and applied the Fuzzy homotopy analysis method to obtain the approximate-analytical solutions of the high order fuzzy initial value problems. In 2015, Al-Jassar[9] introduced and presented fuzzy semi-analytical methods (including the fuzzy homotopy analysis method) to obtain the numerical and approximate-analytical solutions of the linear and non-linear fuzzy initial value problems. In 2016, Otadi and Mosleh[12] studied and developed the fuzzy homotopy analysis method to obtain numerical and approximate-analytical solutions of the hybrid fuzzy ordinary differential equations with the fuzzy initial conditions. As well, In 2016, Lee, Kumaresan and Ratnavelu[11] suggested a solution of the fuzzy fractional differential equations with fuzzy initial conditions by using the fuzzy homotopy analysis method. In 2017, Padma and Kaliyappan[14] introduced and presented fuzzy semi-analytical methods(including the fuzzy homotopy analysis transform method) to obtain the numerical and approximate-analytical solutions of the fuzzy fractional initial value problems. In 2018, Sevindir, Cetinkaya and Tabak[16] introduced and presented fuzzy semi-analytical methods(including the fuzzy homotopy analysis method) to obtain the numerical and approximate-analytical solutions of the first order fuzzy initial value problems. Also, In 2018, Jameel, Saaban and Altaie[15] suggested and applied a new concepts for solving the first order non-linear fuzzy initial value problems by using the fuzzy optimal homotopy asymptotic method. In 2020, Nematallah and Najafi[18] introduced and applied the fuzzy homotopy analysis method to obtain the fuzzy semi-analytical solution of the fuzzy fractional initial value problems based on the concepts of generalized Hukuhara differentiability. As well, In 2020, Ali and Ibraheem[17] studied and developed some fuzzy analytical and numerical solutions of the linear first order fuzzy initial value problems by using fuzzy homotopy analysis method based on the Padè Approximate method.

In this work, we have studied and applied the fuzzy homotopy analysis method to find the fuzzy series solution(fuzzy approximate-analytical solution) of the second order fuzzy autonomous ordinary differential equation with real variable coefficients(real-valued function coefficients). The fuzzy semi-analytical solutions that we have obtained during this work are accurate solutions and very close to the fuzzy exact-analytical solutions, based on the comparison that we have introduced between the results that we have obtained and the fuzzy exact-analytical solutions.

2. Basic definitions in fuzzy set theory

In this section, we will present some of the fundamental definitions and the primitive concepts related to the fuzzy set theory, which are very necessary for understanding this subject.

Definition (1), [1] (Fuzzy Set)

The fuzzy set $\widetilde{A}$ can be defined as:

$$\widetilde{A} = \{ (x , \mu_{\widetilde{A}}(x)) \ : \ x \in X ; \ 0 \leq \mu_{\widetilde{A}}(x) \leq 1 \} \tag{1}$$

where X is the universal set and $\mu_{\widetilde{A}}(x)$ is the grade of membership of x in $\widetilde{A}$.

Definition (2), [7] (α – Level Set)

The α - level (or α - cut) set of a fuzzy set $\widetilde{A}$ can be defined as:

$$A_\alpha = \{ x \in X : \ \mu_{\widetilde{A}}(x) \geq \alpha \ ; \ \alpha \in [0,1] \} . \tag{2}$$

Definition (3), [9] (Fuzzy Number)

A fuzzy number $\tilde{u}$ is an ordered pair of functions $(\underline{u}(\alpha) , \overline{u}(\alpha))$, $0 \leq \alpha \leq 1$, with the following conditions:

1) $\underline{u}(\alpha)$ is a bounded left continuous and non-decreasing function on $[0,1]$.

2) $\overline{u}(\alpha)$ is a bounded left continuous and non-increasing function on $[0,1]$.

3) $\underline{u}(\alpha) \le \overline{u}(\alpha)$, $0 \le \alpha \le 1$. (3)

Remark (1), [9] :

1) The crisp number u is simply represented by :

$\underline{u}(\alpha) = \overline{u}(\alpha) = u$, $0 \le \alpha \le 1$. (4)

2) The set of all the fuzzy numbers is denoted by E^1.

Remark (2), [13]:

The distance between two arbitrary fuzzy numbers $\tilde{u} = (\underline{u}, \overline{u})$ and $\tilde{v} = (\underline{v}, \overline{v})$ can be defined as:

$$D(\tilde{u}, \tilde{v}) = \left[\int_0^1 (\underline{u}(\alpha) - \underline{v}(\alpha))^2 d\alpha + \int_0^1 (\overline{u}(\alpha) - \overline{v}(\alpha))^2 d\alpha \right]^{\frac{1}{2}} \quad (5)$$

Remark (3), [13]:

(E^1, D) is a complete metric space.

Definition (4), [9] (Fuzzy Function)

A mapping $F : T \to E^1$ for some interval $T \subseteq E^1$ is called a fuzzy function or fuzzy process with non-fuzzy variable (crisp variable), and we denote α -level sets by:

$$[F(t)]_\alpha = [\underline{F}(t;\alpha) , \overline{F}(t;\alpha)] \quad (6)$$

Where $t \in T$, $\alpha \in [0,1]$. we refer to $\underline{F}$ and $\overline{F}$ as the lower and upper branches on F.

Definition (5), [9] (H-Difference)

Let $u , v \in E^1$. If there exist $w \in E^1$ such that $u = v + w$ then w is called the H-difference (Hukuhara-difference) of u and v and it is denoted by $w= u \ominus v$, where $u \ominus v \ne u + (-1) v$.

Definition (6), [13] (Fuzzy Derivative)

Let $F : T \to E^1$ for some interval $T \subseteq E^1$ and $t_0 \in T$. We say that F is H-differential(Hukuhara-differential) at t_0 , if there exists an element $F'(t_0) \in E^1$ such that for all $h > 0$ (sufficiently small) , $\exists$ $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and the limits(in the metric D)

$$\lim_{h\to 0}\frac{F(t_0+h)\ominus F(t_0)}{h}=\lim_{h\to 0}\frac{F(t_0)\ominus F(t_0-h)}{h}=F'(t_0) \qquad (7)$$

Then $F'(t_0)$ is called the fuzzy derivative(H-derivative) of F at t_0 . where D is the distance between two fuzzy numbers.

Definition (7), [9] (Nth Order Fuzzy Derivative)

Let $F' : T \to E^1$ for some interval $T \subseteq E^1$ and $t_0 \in T$. We say that F' is H-differential(Hukuhara-differential) at t_0, if there exists an element $F^{(n)}(t_0) \in E^1$ such that for all $h > 0$ (sufficientlysmall), $\exists\, F^{(n-1)}(t_0+h) \ominus F^{(n-1)}(t_0)$, $F^{(n-1)}(t_0)\ominus F^{(n-1)}(t_0 - h)$ and the limits(in the metric D)

$$\lim_{h\to 0}\frac{F^{(n-1)}(t_0+h)\ominus F^{(n-1)}(t_0)}{h}=$$
$$\lim_{h\to 0}\frac{F^{(n-1)}(t_0)\ominus F^{(n-1)}(t_0-h)}{h}=F^{(n)}(t_0) \qquad (8)$$

Then $F^{(n)}(t_0)$ is called the nth order fuzzy derivative (H-derivative of order n) of F at t_0.

Theorem(1), [9]:

Let $F : T \to E^1$ for some interval $T \subseteq E^1$ be an nth order Hukuhara differentiable functions at $t \in T$ and denote

$$[\,F(t)\,]_\alpha = \left[\underline{F}(t\,;\alpha),\ \overline{F}(t\,;\alpha)\right], \forall\,\alpha \in [0,1].$$

Then the boundary functions $\underline{F}(t\,;\alpha)$, $\overline{F}(t\,;\alpha)$ are both nth order Hukuhara differentiable functions and

$$[\,F^{(n)}(t)\,]_\alpha = \left[\underline{F}^{(n)}(t\,;\alpha),\ \overline{F}^{(n)}(t\,;\alpha)\right], \forall\,\alpha \in [0,1]. \qquad (9)$$

3. Fuzzy autonomous differential equation

A fuzzy ordinary differential equation is said to be autonomous if it is independent of it's independent crisp variable t. This is to say an explicit nth order fuzzy autonomous differential equation is of the following form[13] :

$$x^{(n)}(t) = f\left(x(t), x'(t), x''(t), \ldots, x^{(n-1)}(t)\right),\ t \in [t_0, h] \qquad (10)$$

with the fuzzy initial conditions :

$$x(t_0) = x_0\ ,\ x'(t_0) = x'_0\ ,\ x''(t_0) = x''_0\ , \ldots ,\ x^{(n-1)}(t_0) = x_0^{(n-1)}$$

where :

x is a fuzzy function of the crisp variable t ,

$f\left(x(t), x'(t), x''(t), \dots, x^{(n-1)}(t)\right)$ is a fuzzy function of the crisp variable t and the fuzzy variable x ,

$x^{(n)}(t)$ is the fuzzy derivative of the $x(t)$, $x'(t)$, $x''(t), \dots,$ $x^{(n-1)}(t)$,

and $x(t_0)$, $x'(t_0)$, $x''(t_0), \dots,$ $x^{(n-1)}(t_0)$ are fuzzy numbers.

The fuzzy differential equations that are dependent on t are called non-autonomous, and a system of fuzzy autonomous differential equations is called a fuzzy autonomous system.

The main idea in solving the fuzzy autonomous differential equation is to convert it into a system of non-fuzzy(crisp) differential equations, and then solve this system by the known and commonly used methods of solving the non-fuzzy differential equations.

Now it is possible to replace (10) by the following equivalent system of the nth order crisp ordinary differential equations:

$$\underline{x}^{(n)}(t) = \underline{f}\left(x, x', x'', \dots, x^{(n-1)}\right)$$

$$= F(\underline{x}, \underline{x}', \underline{x}'', \dots, \underline{x}^{(n-1)}, \overline{x}, \overline{x}', \overline{x}'', \dots, \overline{x}^{(n-1)});$$

$$\underline{x}(t_0) = \underline{x}_0, \ \underline{x}'(t_0) = \underline{x}'_0, \ \underline{x}''(t_0) = \underline{x}''_0, \dots, \ \underline{x}^{(n-1)}(t_0) = \underline{x}_0^{(n-1)}, \quad (11)$$

$$\overline{x}^{(n)}(t) = \overline{f}\left(x, x', x'', \dots, x^{(n-1)}\right)$$

$$= G(\underline{x}, \underline{x}', \underline{x}'', \dots, \underline{x}^{(n-1)}, \overline{x}, \overline{x}', \overline{x}'', \dots, \overline{x}^{(n-1)});$$

$$\overline{x}(t_0) = \overline{x}_0, \ \overline{x}'(t_0) = \overline{x}'_0, \ \overline{x}''(t_0) = \overline{x}''_0, \dots, \ \overline{x}^{(n-1)}(t_0) = \overline{x}_0^{(n-1)} \quad (12)$$

Where

$$F(\underline{x}, \underline{x}', \underline{x}'', \dots, \underline{x}^{(n-1)}, \overline{x}, \overline{x}', \overline{x}'', \dots, \overline{x}^{(n-1)}) =$$

$$\operatorname{Min}\left\{ f(t, u) : u \in \left[\underline{x}, \underline{x}', \underline{x}'', \dots, \underline{x}^{(n-1)}, \overline{x}, \overline{x}', \overline{x}'', \dots, \overline{x}^{(n-1)}\right]\right\}, \quad (13)$$

$$G(\underline{x}, \underline{x}', \underline{x}'', \dots, \underline{x}^{(n-1)}, \overline{x}, \overline{x}', \overline{x}'', \dots, \overline{x}^{(n-1)}) =$$

$$\operatorname{Max}\left\{ f(t, u) : u \in \left[\underline{x}, \underline{x}', \underline{x}'', \dots, \underline{x}^{(n-1)}, \overline{x}, \overline{x}', \overline{x}'', \dots, \overline{x}^{(n-1)}\right]\right\}. \quad (14)$$

The parametric form of system (13-14) is given by:

$\underline{x}^{(n)}(t,\alpha) = F(\underline{x}(t,\alpha), \underline{x}'(t,\alpha), \underline{x}''(t,\alpha), \dots, \underline{x}^{(n-1)}(t,\alpha), \overline{x}(t,\alpha), \overline{x}'(t,\alpha), \overline{x}''(t,\alpha), \dots, \overline{x}^{(n-1)}(t,\alpha))$

$\underline{x}(t_0,\alpha) = \underline{x}_0(\alpha)$, $\underline{x}'(t_0,\alpha) = \underline{x}'_0(\alpha)$, $\underline{x}''(t_0,\alpha) = \underline{x}''_0(\alpha)$, ..., $\underline{x}^{(n-1)}(t_0,\alpha) = \underline{x}_0^{(n-1)}(\alpha)$ (15)

$\overline{x}^{(n)}(t,\alpha) = G(\underline{x}(t,\alpha), \underline{x}'(t,\alpha), \underline{x}''(t,\alpha), \dots, \underline{x}^{(n-1)}(t,\alpha), \overline{x}(t,\alpha), \overline{x}'(t,\alpha), \overline{x}''(t,\alpha), \dots, \overline{x}^{(n-1)}(t,\alpha))$

$\overline{x}(t_0,\alpha) = \overline{x}_0(\alpha)$, $\overline{x}'(t_0,\alpha) = \overline{x}'_0(\alpha)$, $\overline{x}''(t_0,\alpha) = \overline{x}''_0(\alpha)$, ..., $\overline{x}^{(n-1)}(t_0,\alpha) = \overline{x}_0^{(n-1)}(\alpha)$ (16)

Where $t \in [t_0, h]$ and $\alpha \in [0,1]$.

The following theorem ensures the existence and uniqueness of the fuzzy solution of the nth order fuzzy autonomous differential equation.

Theorem(2), [13] :

If we return to problem (10),

$x^{(n)}(t) = f\left(x(t), x'(t), x''(t), \dots, x^{(n-1)}(t)\right)$, $t \in [t_0, h]$

Let $f_i : T \to E^1$, $1 \le i \le n$ be a continuous fuzzy functions, T= $[t_0, h]$ and assume that there exist a real numbers $k_i > 0$ such that

$$D\left(f_i(t, z_i), f_i(t, w_i)\right) \le k_i D(z_i, w_i) \quad (17)$$

For all $t \in T$ and all $z_i, w_i \in E^1$.

Then the above nth order FIVB has a unique fuzzy solution on T in each case.

4. Fuzzy homotopy analysis method

A fuzzy homotopy analysis method is one of the fuzzy semi- analytical methods used to obtain the fuzzy series solution(fuzzy approximate-analytical solution) of the FIVBs. This technique utilizes homotopy in order to generate a convergent fuzzy series of fuzzy linear equations from fuzzy non-linear ones. This means that this technique is based on generating a convergent fuzzy series of fuzzy solutions to approximate the fuzzy analytical solution of the FIVB.

The basic mathematical concepts of the fuzzy homotopy analysis method are the same as the basic mathematical concepts of the homotopy analysis method, but with the use of the concepts of the fuzzy set theory. This means that solving any FIVB by using fuzzy homotopy analysis method is based on converting the FIVB into a system of non-fuzzy(crisp) initial value problems by using the steps that we explained in section(3), and then using the homotopy analyss method to solve this system.

The fuzzy homotopy analysis method provides us with both the freedom to choose proper base fuzzy functions for approximating a non-linear fuzzy problem and a simple way to ensure the convergence of the fuzzy series solution.

5. Description of the method

To describe the basic mathematical ideas of the fuzzy homotopy analysis method, we consider the following nth order FIVB :

$$[\mathrm{N}(\mathrm{x}(\mathrm{t})]_{\alpha} = 0 \quad , \tag{18}$$

Where N is the fuzzy non-linear operator , t denotes the independent crisp variable , $\mathrm{x}(\mathrm{t})$ is an unknown fuzzy function .

By the concepts of section(3), We can conclude that:

$$[\mathrm{N}(\mathrm{x}(\mathrm{t})]_{\alpha} = [\ [\mathrm{N}(\mathrm{x}(\mathrm{t})]_{\alpha}^{\mathrm{L}} , [\mathrm{N}(\mathrm{x}(\mathrm{t})]_{\alpha}^{\mathrm{U}}\ ] \tag{19}$$

Since $0 = [\ 0 , 0]$, we can get:

$$[\mathrm{N}(\mathrm{x}(\mathrm{t})]_{\alpha}^{\mathrm{L}} = 0 \tag{20i}$$

$$[\mathrm{N}(\mathrm{x}(\mathrm{t})]_{\alpha}^{\mathrm{U}} = 0 \tag{20ii}$$

Now, we construct the zero-order fuzzy deformation equation:

$$[(1-\mathrm{w})\mathrm{L}\ (\theta(\mathrm{t}\ ;\ \mathrm{w}) - \mathrm{x}_0(\mathrm{t}))]_{\alpha} = [\mathrm{w}\ \mathrm{h}\ \mathrm{N}(\theta(\mathrm{t}\ ;\ \mathrm{w}))]_{\alpha} , \tag{21}$$

Where $\mathrm{w} \in [0 , 1]$ is the homotopy embedding parameter, $\mathrm{h} \in [-1 , 0)$ is the convergence control parameter , L is the fuzzy linear operator , $\mathrm{x}_0(\mathrm{t})$ is the fuzzy initial guess of $\mathrm{x}(\mathrm{t})$ and $\theta(\mathrm{t};\mathrm{w})$ is a fuzzy function.

By the concepts of section(3), We can get:

$$(1-\mathrm{w})\mathrm{L}(\ [\theta(\mathrm{t}\ ;\ \mathrm{w})]_{\alpha}^{\mathrm{L}} - [\mathrm{x}_0(\mathrm{t})]_{\alpha}^{\mathrm{L}}\ ) = \mathrm{w}\ \mathrm{h}(\ [\mathrm{N}(\theta(\mathrm{t}\ ;\ \mathrm{w}))]_{\alpha}^{\mathrm{L}}\ ) \tag{22i}$$

$$(1-\mathrm{w})\mathrm{L}(\ [\theta(\mathrm{t}\ ;\ \mathrm{w})]_{\alpha}^{\mathrm{U}} - [\mathrm{x}_0(\mathrm{t})]_{\alpha}^{\mathrm{U}}\ ) = \mathrm{w}\ \mathrm{h}(\ [\mathrm{N}(\theta(\mathrm{t}\ ;\ \mathrm{w}))]_{\alpha}^{\mathrm{U}}\ ) \tag{22ii}$$

Obviously , when $w = 0$ and $w = 1$, both

$$[\theta(t\,;0)]_\alpha = [x_0(t)]_\alpha\,, \tag{23}$$

$$[\theta(t\,;1)]_\alpha = [x(t)]_\alpha \tag{24}$$

Hold, therefore when w is increasing from 0 to 1 , the fuzzy solutions $[\theta(t\,;\,w)]_\alpha^L$ and $[\theta(t\,;\,w)]_\alpha^U$ varies from the fuzzy initial guess $[x_0(t)]_\alpha$ to the fuzzy solution$[x(t)]_\alpha$.

Thus, we have:

$$[\theta(t\,;0)]_\alpha^L = [x_0(t)]_\alpha^L \tag{25i}$$

$$[\theta(t\,;0)]_\alpha^U = [x_0(t)]_\alpha^U \tag{25ii}$$

$$[\theta(t\,;1)]_\alpha^L = [x(t)]_\alpha^L \tag{26i}$$

$$[\theta(t\,;1)]_\alpha^U = [x(t)]_\alpha^U \tag{26ii}$$

By expanding $[\theta(t\,;w)\,]_\alpha$ in Taylor series with respect to w , one has:

$$[\theta(t\,;w)]_\alpha = [x_0(t)]_\alpha + \sum_{m=1}^{\infty}[x_m(t)w^m]_\alpha \tag{27}$$

Where

$$[x_m(t)]_\alpha = \frac{1}{m!}\,\frac{\partial^m\,[\theta(t\,;w)]_\alpha}{\partial w^m}\bigg|_{w=0} \tag{28}$$

By the concepts of parametric form in section(3), We can conclude that:

$$[\theta(t\,;w)]_\alpha^L = [x_0(t)]_\alpha^L + \sum_{m=1}^{\infty}[x_m(t)]_\alpha^L\, w^m \tag{29i}$$

$$[\theta(t\,;w)]_\alpha^U = [x_0(t)]_\alpha^U + \sum_{m=1}^{\infty}[x_m(t)]_\alpha^U\, w^m \tag{29ii}$$

Where

$$[x_0(t)]_\alpha^L = \frac{1}{m!}\,\frac{\partial^m\,[\theta(t\,;\,w)]_\alpha^L}{\partial w^m}\bigg|_{w=0} \tag{30i}$$

$$[x_0(t)]_\alpha^U = \frac{1}{m!}\,\frac{\partial^m\,[\theta(t\,;\,w)]_\alpha^U}{\partial w^m}\bigg|_{w=0} \tag{30ii}$$

If the fuzzy linear operator , the fuzzy initial guess , the auxiliary parameter h , and the auxiliary fuzzy function are so properly chosen , then the fuzzy series (27) converges at $w = 1$, and one has:

$$[\theta(t\,;1)]_\alpha = [x(t)]_\alpha = [x_0(t)]_\alpha + \sum_{m=1}^{\infty}[x_m(t)]_\alpha \tag{31}$$

Where

$$[\theta(t\,;1)]_\alpha^L = [x(t)]_\alpha^L \; = [x_0(t)]_\alpha^L + \sum_{m=1}^{\infty}[x_m(t)]_\alpha^L \qquad (32i)$$

$$[\theta(t\,;1)]_\alpha^U = [x(t)]_\alpha^U \; = [x_0(t)]_\alpha^U + \sum_{m=1}^{\infty}[x_m(t)]_\alpha^U \qquad (32ii)$$

Which must be one of the fuzzy solutions of the problem(18).

If $h = -1$, (21) becomes

$$[(1-w)L\,(\theta(t\,;\,w) - x_0(t))]_\alpha + [w\,N(\theta(t\,;\,w))]_\alpha = 0\,, \qquad (33)$$

Where

$$(1-w)L(\,[\theta(t\,;\,w)]_\alpha^L - [x_0(t)]_\alpha^L) + w\,(\,[N(\theta(t\,;\,w))]_\alpha^L\,) = 0 \qquad (34i)$$

$$(1-w)L(\,[\theta(t\,;\,w)]_\alpha^U - [x_0(t)]_\alpha^U) + w\,(\,[N(\theta(t\,;\,w))]_\alpha^U\,) = 0 \qquad (34ii)$$

Which is used mostly in the fuzzy homotopy analysis method.

We define the fuzzy vectors

$$[\,\vec{x}_i\,]_\alpha = \{\,[\,x_0(t)]_\alpha\,,\ [x_1(t)]_\alpha\,, [\,x_2(t)]_\alpha\,, \dots, [\,x_i(t)]_\alpha\,\} \qquad (35)$$

Where

$$[\,\vec{x}_i\,]_\alpha^L = \{\,[\,x_0(t)]_\alpha^L\,, [\,x_1(t)]_\alpha^L\,,\ [\,x_2(t)]_\alpha^L\,, \dots, [\,x_i(t)]_\alpha^L\,\} \qquad (36i)$$

$$[\,\vec{x}_i\,]_\alpha^U = \{\,[\,x_0(t)]_\alpha^U\,, [\,x_1(t)]_\alpha^U\,,\ [\,x_2(t)]_\alpha^U\,, \dots, [\,x_i(t)]_\alpha^U\,\} \qquad (36ii)$$

Now, by differentiating (21) m- times with respect to the parameter w and then setting $w = 0$ and finally dividing them by $m!$, we have the mth-order fuzzy deformation equation:

$$L(\,[x_m(t)]_\alpha - \chi_m[x_{m-1}(t)]_\alpha\,) = h(\,[R_m(\vec{x}_{m-1})\,]_\alpha\,) \qquad (37)$$

Where $[\,R_m(\vec{x}_{m-1})\,]_\alpha =$

$$\frac{1}{(m-1)!}\,\frac{\partial^{m-1}[N(\theta(t\,;w))]_\alpha}{\partial w^{m-1}}\bigg|_{w=0} \qquad (38)$$

By the concepts of parametric form in section(3), We get:

$$L(\,[x_m(t)]_\alpha^L - \chi_m[x_{m-1}(t)]_\alpha^L\,) = h\,(\,[R_m(\vec{x}_{m-1})]_\alpha^L\,) \qquad (39i)$$

$$L(\,[x_m(t)]_\alpha^U - \chi_m[x_{m-1}(t)]_\alpha^U\,) = h\,(\,[R_m(\vec{x}_{m-1})]_\alpha^U\,) \qquad (39ii)$$

Where

$$[R_m(\vec{x}_{m-1})]_\alpha^L = \frac{1}{(m-1)!} \frac{\partial^{m-1}([N(\theta(t;w))]_\alpha^L)}{\partial w^{m-1}}\bigg|_{w=0} \quad (40i)$$

$$[R_m(\vec{x}_{m-1})]_\alpha^U = \frac{1}{(m-1)!} \frac{\partial^{m-1}([N(\theta(t;w))]_\alpha^U)}{\partial w^{m-1}}\bigg|_{w=0} \quad (40ii)$$

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \quad (41)$$

6. Applied example

In this section, one fuzzy problem has been solved in order to clarify the efficiency and the accuracy of the method. According to Liao's book [3], the optimal value of h was found to be approximately $-1 \leq h < 0$. In addition, the practical examples in[3,5,10] showed that the optimal value of h can be determined while solving the problem by experimenting with a number of different values of h. The optimal value of h depends greatly on the nature of the problem, but still h = -1 is an optimal value and achieves a rapid convergence.

Example 1: Consider the second order fuzzy autonomous differential equation

$$x''(t) + x(t) = 0 \,, \quad (42)$$

Subject to the fuzzy initial conditions :

$$[x(0)]_\alpha = [0,0] \,, \; [x'(0)]_\alpha = [0.01\alpha + 0.02 \,, -0.01\alpha + 0.04] \,, \; \alpha \in [0,1] \,.$$

Solution:

The fuzzy linear operator is :

$$[L(\theta(t;w))]_\alpha = [\, [L(\theta(t;w))]_\alpha^L \,, [L(\theta(t;w))]_\alpha^U \,] \quad (43)$$

Where

$$[L(\theta(t;w))]_\alpha^L = \left[\frac{\partial^2\theta(t;w)}{\partial t^2}\right]_\alpha^L \quad (44i)$$

$$[L(\theta(t;w))]_\alpha^U = \left[\frac{\partial^2\theta(t;w)}{\partial t^2}\right]_\alpha^U \quad (44ii)$$

We define the fuzzy non-linear operator as :

$$[N(\theta(x;w))]_\alpha = [\, [N(\theta(x;w))]_\alpha^L \,, [N(\theta(x;w))]_\alpha^U \,] \quad (45)$$

Where

$$[\mathrm{N}(\theta(\mathrm{x};\mathrm{w}))]^{\mathrm{L}}_{\alpha} = \frac{\partial^2[\theta(\mathrm{t};\mathrm{w})]^{\mathrm{L}}_{\alpha}}{\partial \mathrm{t}^2} + [\theta(\mathrm{t};\mathrm{w})]^{\mathrm{L}}_{\alpha} \tag{46i}$$

$$[\mathrm{N}(\theta(\mathrm{x};\mathrm{w}))]^{\mathrm{U}}_{\alpha} = \frac{\partial^2[\theta(\mathrm{t};\mathrm{w})]^{\mathrm{U}}_{\alpha}}{\partial \mathrm{t}^2} + [\theta(\mathrm{t};\mathrm{w})]^{\mathrm{U}}_{\alpha} \tag{46ii}$$

The fuzzy series solution is :

$$[\mathrm{x}(\mathrm{t})]_{\alpha} = [\,[\mathrm{x}(\mathrm{t})]^{\mathrm{L}}_{\alpha}, [\mathrm{x}(\mathrm{t})]^{\mathrm{U}}_{\alpha}\,] \tag{47}$$

Where

$$[\mathrm{x}(\mathrm{t})]^{\mathrm{L}}_{\alpha} = [\mathrm{x}_0(\mathrm{t})]^{\mathrm{L}}_{\alpha} + [\mathrm{x}_1(\mathrm{t})]^{\mathrm{L}}_{\alpha} + [\mathrm{x}_2(\mathrm{t})]^{\mathrm{L}}_{\alpha} + [\mathrm{x}_3(\mathrm{t})]^{\mathrm{L}}_{\alpha} + \cdots \tag{48i}$$

$$[\mathrm{x}(\mathrm{t})]^{\mathrm{U}}_{\alpha} = [\mathrm{x}_0(\mathrm{t})]^{\mathrm{U}}_{\alpha} + [\mathrm{x}_1(\mathrm{t})]^{\mathrm{U}}_{\alpha} + [\mathrm{x}_2(\mathrm{t})]^{\mathrm{U}}_{\alpha} + [\mathrm{x}_3(\mathrm{t})]^{\mathrm{U}}_{\alpha} + \cdots \tag{48ii}$$

By Taylor series expansion , the fuzzy initial approximation is:

$$[\mathrm{x}_0(\mathrm{t})]_{\alpha} = [\,[\mathrm{x}_0(\mathrm{t})]^{\mathrm{L}}_{\alpha}, [\mathrm{x}_0(\mathrm{t})]^{\mathrm{U}}_{\alpha}\,] \tag{49}$$

Where

$$[\mathrm{x}_0(\mathrm{t})]^{\mathrm{L}}_{\alpha} = 0.01\,\alpha\mathrm{t} + 0.02\,\mathrm{t} \tag{50i}$$

$$[\mathrm{x}_0(\mathrm{t})]^{\mathrm{U}}_{\alpha} = -0.01\,\alpha\mathrm{t} + 0.04\,\mathrm{t} \tag{50ii}$$

To find $[\mathrm{x}_1(\mathrm{t})]_{\alpha} = [\,[\mathrm{x}_1(\mathrm{t})]^{\mathrm{L}}_{\alpha}, [\mathrm{x}_1(\mathrm{t})]^{\mathrm{U}}_{\alpha}\,]$

From(29), we can find

$$[\theta(\mathrm{t};\mathrm{w})]^{\mathrm{L}}_{\alpha} = [\mathrm{x}_0(\mathrm{t})]^{\mathrm{L}}_{\alpha} + \mathrm{w}[\mathrm{x}_1(\mathrm{t})]^{\mathrm{L}}_{\alpha} \tag{51i}$$

$$[\theta(\mathrm{t};\mathrm{w})]^{\mathrm{U}}_{\alpha} = [\mathrm{x}_0(\mathrm{t})]^{\mathrm{U}}_{\alpha} + \mathrm{w}[\mathrm{x}_1(\mathrm{t})]^{\mathrm{U}}_{\alpha} \tag{51ii}$$

From(37), we can find

$$\mathrm{L}([\mathrm{x}_1(\mathrm{t})]^{\mathrm{L}}_{\alpha} - 0) = \mathrm{h}[\mathrm{R}_1]^{\mathrm{L}}_{\alpha} \tag{52i}$$

$$\mathrm{L}([\mathrm{x}_1(\mathrm{t})]^{\mathrm{U}}_{\alpha} - 0) = \mathrm{h}[\mathrm{R}_1]^{\mathrm{U}}_{\alpha} \tag{52ii}$$

Then from(38), we can get :

$$[\mathrm{R}_1]^{\mathrm{L}}_{\alpha} = [\mathrm{N}(\theta(\mathrm{t};\mathrm{w}))]^{\mathrm{L}}_{\alpha}|_{\mathrm{w}=0} \tag{53i}$$

$$[\mathrm{R}_1]^{\mathrm{U}}_{\alpha} = [\mathrm{N}(\theta(\mathrm{t};\mathrm{w}))]^{\mathrm{U}}_{\alpha}|_{\mathrm{w}=0} \tag{53ii}$$

Then , we apply the following steps :

$$\frac{\partial[\theta(t;w)]_{\alpha}^{L}}{\partial t} = \frac{\partial[x_0(t)]_{\alpha}^{L}}{\partial t} + w\frac{\partial[x_1(t)]_{\alpha}^{L}}{\partial t} \tag{54i}$$

$$\frac{\partial[\theta(t;w)]_{\alpha}^{U}}{\partial t} = \frac{\partial[x_0(t)]_{\alpha}^{U}}{\partial t} + w\frac{\partial[x_1(t)]_{\alpha}^{U}}{\partial t} \tag{54ii}$$

$$\frac{\partial^2[\theta(t;w)]_{\alpha}^{L}}{\partial t^2} = \frac{\partial^2[x_0(t)]_{\alpha}^{L}}{\partial t^2} + w\frac{\partial^2[x_1(t)]_{\alpha}^{L}}{\partial t^2} \tag{55i}$$

$$\frac{\partial^2[\theta(t;w)]_{\alpha}^{U}}{\partial t^2} = \frac{\partial^2[x_0(t)]_{\alpha}^{U}}{\partial t^2} + w\frac{\partial^2[x_1(t)]_{\alpha}^{U}}{\partial t^2} \tag{55ii}$$

$$[N(\theta(x;w))]_{\alpha}^{L} = \frac{\partial^2[x_0(t)]_{\alpha}^{L}}{\partial t^2} + w\frac{\partial^2[x_1(t)]_{\alpha}^{L}}{\partial t^2} + [x_0(t)]_{\alpha}^{L} + w[x_1(t)]_{\alpha}^{L} \tag{56i}$$

$$[N(\theta(x;w))]_{\alpha}^{U} = \frac{\partial^2[x_0(t)]_{\alpha}^{U}}{\partial t^2} + w\frac{\partial^2[x_1(t)]_{\alpha}^{U}}{\partial t^2} + [x_0(t)]_{\alpha}^{U} + w[x_1(t)]_{\alpha}^{U} \tag{56ii}$$

$$[R_1]_{\alpha}^{L} = \frac{\partial^2[x_0(t)]_{\alpha}^{L}}{\partial t^2} + [x_0(t)]_{\alpha}^{L} \tag{57i}$$

$$[R_1]_{\alpha}^{U} = \frac{\partial^2[x_0(t)]_{\alpha}^{U}}{\partial t^2} + [x_0(t)]_{\alpha}^{U} \tag{57ii}$$

$$[R_1]_{\alpha}^{L} = 0.01\ \alpha t + 0.02\ t \tag{58i}$$

$$[R_1]_{\alpha}^{U} = -0.01\ \alpha t + 0.04\ t \tag{58ii}$$

$$L([x_1(t)]_{\alpha}^{L}) = 0.01\ \alpha ht + 0.02\ ht \tag{59i}$$

$$L([x_1(t)]_{\alpha}^{U}) = -0.01\ \alpha ht + 0.04\ ht \tag{59ii}$$

$$[x_1(t)]_{\alpha}^{L} = \iint (0.01\ \alpha ht + 0.02\ ht)\ dt\,dt \tag{60i}$$

$$[x_1(t)]_{\alpha}^{U} = \iint (-0.01\ \alpha ht + 0.04\ ht)\ dt\,dt \tag{60ii}$$

$$[x_1(t)]_{\alpha}^{L} = 0.001667\ \alpha ht^3 + 0.003333\ ht^3 \tag{61i}$$

$$[x_1(t)]_{\alpha}^{U} = -0.001667\ \alpha ht^3 + 0.006667\ ht^3 \tag{61ii}$$

Now, to find $[x_2(t)]_{\alpha} = [\,[x_2(t)]_{\alpha}^{L}, [x_2(t)]_{\alpha}^{U}\,]$

From(29), we can find

$$[\theta(t;w)]_{\alpha}^{L} = [x_0(t)]_{\alpha}^{L} + w[x_1(t)]_{\alpha}^{L} + w^2[x_2(t)]_{\alpha}^{L} \tag{62i}$$

$$[\theta(t;w)]_{\alpha}^{U} = [x_0(t)]_{\alpha}^{U} + w[x_1(t)]_{\alpha}^{U} + w^2[x_2(t)]_{\alpha}^{U} \tag{62ii}$$

From(37), we can find

$$L([x_2(t)]^L_\alpha - [x_1(t)]^L_\alpha) = h[R_2]^L_\alpha \tag{63i}$$

$$L([x_2(t)]^U_\alpha - [x_1(t)]^U_\alpha) = h[R_2]^U_\alpha \tag{63ii}$$

Then from (38) , we can get :

$$[R_2]^L_\alpha = \frac{\partial[N(\theta(t;w))]^L_\alpha}{\partial w}\Big|_{w=0} \tag{64i}$$

$$[R_2]^U_\alpha = \frac{\partial[N(\theta(t;w))]^U_\alpha}{\partial w}\Big|_{w=0} \tag{64ii}$$

Then , we apply the following steps :

$$\frac{\partial[\theta(t;w)]^L_\alpha}{\partial t} = \frac{\partial[x_0(t)]^L_\alpha}{\partial t} + w\frac{\partial[x_1(t)]^L_\alpha}{\partial t} + w^2\frac{\partial[x_2(t)]^L_\alpha}{\partial t} \tag{65i}$$

$$\frac{\partial[\theta(t;w)]^U_\alpha}{\partial t} = \frac{\partial[x_0(t)]^U_\alpha}{\partial t} + w\frac{\partial[x_1(t)]^U_\alpha}{\partial t} + w^2\frac{\partial[x_2(t)]^U_\alpha}{\partial t} \tag{65ii}$$

$$\frac{\partial^2[\theta(t;w)]^L_\alpha}{\partial t^2} = \frac{\partial^2[x_0(t)]^L_\alpha}{\partial t^2} + w\frac{\partial^2[x_1(t)]^L_\alpha}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]^L_\alpha}{\partial t^2} \tag{66i}$$

$$\frac{\partial^2[\theta(t;w)]^U_\alpha}{\partial t^2} = \frac{\partial^2[x_0(t)]^U_\alpha}{\partial t^2} + w\frac{\partial^2[x_1(t)]^U_\alpha}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]^U_\alpha}{\partial t^2} \tag{66ii}$$

$$[N(\theta(x;w))]^L_\alpha = \frac{\partial^2[x_0(t)]^L_\alpha}{\partial t^2} + w\frac{\partial^2[x_1(t)]^L_\alpha}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]^L_\alpha}{\partial t^2} + [x_0(t)]^L_\alpha + w[x_1(t)]^L_\alpha + w^2[x_2(t)]^L_\alpha \tag{67i}$$

$$[N(\theta(x;w))]^U_\alpha = \frac{\partial^2[x_0(t)]^U_\alpha}{\partial t^2} + w\frac{\partial^2[x_1(t)]^U_\alpha}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]^U_\alpha}{\partial t^2} + [x_0(t)]^U_\alpha + w[x_1(t)]^U_\alpha + w^2[x_2(t)]^U_\alpha \tag{67ii}$$

$$\frac{\partial[N(\theta(t;w))]^L_\alpha}{\partial w} = \frac{\partial^2[x_1(t)]^L_\alpha}{\partial t^2} + 2w\frac{\partial^2[x_2(t)]^L_\alpha}{\partial t^2} + [x_1(t)]^L_\alpha + 2w[x_2(t)]^L_\alpha \tag{68i}$$

$$\frac{\partial[N(\theta(t;w))]^U_\alpha}{\partial w} = \frac{\partial^2[x_1(t)]^U_\alpha}{\partial t^2} + 2w\frac{\partial^2[x_2(t)]^U_\alpha}{\partial t^2} + [x_1(t)]^U_\alpha + 2w[x_2(t)]^U_\alpha \tag{68ii}$$

$$[R_2]^L_\alpha = \frac{\partial^2[x_1(t)]^L_\alpha}{\partial t^2} + [x_1(t)]^L_\alpha \tag{69i}$$

$$[R_2]^U_\alpha = \frac{\partial^2[x_1(t)]^U_\alpha}{\partial t^2} + [x_1(t)]^U_\alpha \tag{69ii}$$

$$[R_2]^L_\alpha = 0.010002\ \alpha ht + 0.019998\ ht + 0.001667\ \alpha ht^3 + 0.003333\ ht^3\ , \tag{70i}$$

$$[R_2]^U_\alpha = -0.010002\ \alpha ht + 0.040002\ ht - 0.001667\ \alpha ht^3 + 0.006667\ ht^3\ . \tag{70ii}$$

$L([x_2(t)]^L_\alpha - [x_1(t)]^L_\alpha) = 0.010002\ \alpha h^2 t + 0.019998\ h^2 t + 0.001667\ \alpha h^2 t^3 + 0.003333\ h^2 t^3$ (71i)

$L([x_2(t)]^U_\alpha - [x_1(t)]^U_\alpha) = -0.010002\ \alpha h^2 t + 0.040002\ h^2 t - 0.001667\ \alpha h^2 t^3 + 0.006667\ h^2 t^3$ (71ii)

$[x_2(t)]^L_\alpha - [x_1(t)]^L_\alpha = \iint (0.010002\ \alpha h^2 t + 0.019998\ h^2 t + 0.001667\ \alpha h^2 t^3 + 0.003333\ h^2 t^3)\ dtdt$,
(72i)

$[x_2(t)]^U_\alpha - [x_1(t)]^U_\alpha = \iint (-0.010002\alpha h^2 t + 0.040002 h^2 t - 0.001667\alpha h^2 t^3 + 0.006667 h^2 t^3)\ dtdt$.
(72ii)

$[x_2(t)]^L_\alpha - [x_1(t)]^L_\alpha =$

$0.001667\ \alpha h^2 t^3 + 0.003333\ h^2 t^3 + 0.000083\ \alpha h^2 t^5 + 0.000167\ h^2 t^5$, (73i)

$[x_2(t)]^U_\alpha - [x_1(t)]^U_\alpha = -0.001667\ \alpha h^2 t^3 + 0.006667\ h^2 t^3 - 0.000083\ \alpha h^2 t^5 + 0.000333\ h^2 t^5$.
(73ii)

$[x_2(t)]^L_\alpha = 0.001667\ \alpha h^2 t^3 + 0.003333\ h^2 t^3 + 0.000083\ \alpha h^2 t^5 + 0.000167\ h^2 t^5 + 0.001667\ \alpha h t^3 + 0.003333\ h t^3$ (74i)

$[x_2(t)]^U_\alpha = -0.001667\ \alpha h^2 t^3 + 0.006667\ h^2 t^3 - 0.000083\ \alpha h^2 t^5 + 0.000333\ h^2 t^5 - 0.001667\ \alpha h t^3 + 0.006667\ h t^3$ (74ii)

Now, to find $[x_3(t)]_\alpha = [\ [x_3(t)]^L_\alpha , [x_3(t)]^U_\alpha]$:

From(29), we can find

$[\theta(t;w)]^L_\alpha = [x_0(t)]^L_\alpha + w[x_1(t)]^L_\alpha + w^2[x_2(t)]^L_\alpha + w^3[x_3(t)]^L_\alpha$ (75i)

$[\theta(t;w)]^U_\alpha = [x_0(t)]^U_\alpha + w[x_1(t)]^U_\alpha + w^2[x_2(t)]^U_\alpha + w^3[x_3(t)]^U_\alpha$ (75ii)

From(37), we can find

$L([x_3(t)]^L_\alpha - [x_2(t)]^L_\alpha) = h[R_3]^L_\alpha$ (76i)

$L([x_3(t)]^U_\alpha - [x_2(t)]^U_\alpha) = h[R_3]^U_\alpha$ (76ii)

Then from(38), we can get :

$$[R_3]_\alpha^L = \frac{1}{2}\frac{\partial^2[N(\theta(t;w))]_\alpha^L}{\partial w^2}\Big|_{w=0} \tag{77i}$$

$$[R_3]_\alpha^U = \frac{1}{2}\frac{\partial^2[N(\theta(t;w))]_\alpha^U}{\partial w^2}\Big|_{w=0} \tag{77ii}$$

Then , we apply the following steps :

$$\frac{\partial[\theta(t;w)]_\alpha^L}{\partial t} = \frac{\partial[x_0(t)]_\alpha^L}{\partial t} + w\frac{\partial[x_1(t)]_\alpha^L}{\partial t} + w^2\frac{\partial[x_2(t)]_\alpha^L}{\partial t} + w^3\frac{\partial[x_3(t)]_\alpha^L}{\partial t} \tag{78i}$$

$$\frac{\partial[\theta(t;w)]_\alpha^U}{\partial t} = \frac{\partial[x_0(t)]_\alpha^U}{\partial t} + w\frac{\partial[x_1(t)]_\alpha^U}{\partial t} + w^2\frac{\partial[x_2(t)]_\alpha^U}{\partial t} + w^3\frac{\partial[x_3(t)]_\alpha^U}{\partial t} \tag{78ii}$$

$$\frac{\partial^2[\theta(t;w)]_\alpha^L}{\partial t^2} = \frac{\partial^2[x_0(t)]_\alpha^L}{\partial t^2} + w\frac{\partial^2[x_1(t)]_\alpha^L}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]_\alpha^L}{\partial t^2} + w^3\frac{\partial^2[x_3(t)]_\alpha^L}{\partial t^2} \tag{79i}$$

$$\frac{\partial^2[\theta(t;w)]_\alpha^U}{\partial t^2} = \frac{\partial^2[x_0(t)]_\alpha^U}{\partial t^2} + w\frac{\partial^2[x_1(t)]_\alpha^U}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]_\alpha^U}{\partial t^2} + w^3\frac{\partial^2[x_3(t)]_\alpha^U}{\partial t^2} \tag{79ii}$$

$$[N(\theta(x;w))]_\alpha^L = \frac{\partial^2[x_0(t)]_\alpha^L}{\partial t^2} + w\frac{\partial^2[x_1(t)]_\alpha^L}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]_\alpha^L}{\partial t^2} + w^3\frac{\partial^3[x_3(t)]_\alpha^L}{\partial t^2} + [x_0(t)]_\alpha^L + w[x_1(t)]_\alpha^L + w^2[x_2(t)]_\alpha^L + w^3[x_3(t)]_\alpha^L \tag{80i}$$

$$[N(\theta(x;w))]_\alpha^U = \frac{\partial^2[x_0(t)]_\alpha^U}{\partial t^2} + w\frac{\partial^2[x_1(t)]_\alpha^U}{\partial t^2} + w^2\frac{\partial^2[x_2(t)]_\alpha^U}{\partial t^2} + w^3\frac{\partial^3[x_3(t)]_\alpha^U}{\partial t^2} + [x_0(t)]_\alpha^U + w[x_1(t)]_\alpha^U + w^2[x_2(t)]_\alpha^U + w^3[x_3(t)]_\alpha^U \tag{80ii}$$

$$\frac{\partial[N(\theta(t;w))]_\alpha^L}{\partial w} = \frac{\partial^2[x_1(t)]_\alpha^L}{\partial t^2} + 2w\frac{\partial^2[x_2(t)]_\alpha^L}{\partial t^2} + 3w^2\frac{\partial^3[x_3(t)]_\alpha^L}{\partial t^2} + [x_1(t)]_\alpha^L + 2w[x_2(t)]_\alpha^L + 3w^2[x_3(t)]_\alpha^L \tag{81i}$$

$$\frac{\partial[N(\theta(t;w))]_\alpha^U}{\partial w} = \frac{\partial^2[x_1(t)]_\alpha^U}{\partial t^2} + 2w\frac{\partial^2[x_2(t)]_\alpha^U}{\partial t^2} + 3w^2\frac{\partial^3[x_3(t)]_\alpha^U}{\partial t^2} + [x_1(t)]_\alpha^U + 2w[x_2(t)]_\alpha^U + 3w^2[x_3(t)]_\alpha^U \tag{81ii}$$

$$\frac{\partial^2[N(\theta(t;w))]_\alpha^L}{\partial w^2} = 2\frac{\partial^2[x_2(t)]_\alpha^L}{\partial t^2} + 6w\frac{\partial^3[x_3(t)]_\alpha^L}{\partial t^2} + 2[x_2(t)]_\alpha^L + 6w[x_3(t)]_\alpha^L \tag{82i}$$

$$\frac{\partial^2[N(\theta(t;w))]_\alpha^U}{\partial w^2} = 2\frac{\partial^2[x_2(t)]_\alpha^U}{\partial t^2} + 6w\frac{\partial^3[x_3(t)]_\alpha^U}{\partial t^2} + 2[x_2(t)]_\alpha^U + 6w[x_3(t)]_\alpha^U \tag{82ii}$$

$$[R_3]_\alpha^L = \frac{\partial^2[x_2(t)]_\alpha^L}{\partial t^2} + [x_2(t)]_\alpha^L \tag{83i}$$

$$[R_3]_\alpha^U = \frac{\partial^2[x_2(t)]_\alpha^U}{\partial t^2} + [x_2(t)]_\alpha^U \tag{83ii}$$

$$[R_3]_\alpha^L = 0.010002\ \alpha h^2 t + 0.019998\ h^2 t + 0.003327\ \alpha h^2 t^3 + 0.006673\ h^2 t^3 + 0.010002\ \alpha h t + 0.019998\ h t + 0.000083\ \alpha h^2 t^5 +$$

$0.000167\ h^2t^5 + 0.001667\ \alpha h t^3 + 0.003333\ h t^3$
(84i)

$[R_3]^U_\alpha = -0.010002\ \alpha h^2 t + 0.040002\ h^2 t - 0.003327\ \alpha h^2 t^3 + 0.013327\ h^2 t^3 - 0.010002\ \alpha h t + 0.040002\ h t - 0.000083\ \alpha h^2 t^5 + 0.000333\ h^2 t^5 - 0.001667\ \alpha h t^3 + 0.006667\ h t^3$ (84ii)

$L([x_3(t)]^L_\alpha - [x_2(t)]^L_\alpha) = 0.010002\ \alpha h^3 t + 0.019998\ h^3 t + 0.003327\ \alpha h^3 t^3 + 0.006673\ h^3 t^3 + 0.010002\ \alpha h^2 t + 0.019998\ h^2 t + 0.000083\ \alpha h^3 t^5 + 0.000167\ h^3 t^5 + 0.001667\ \alpha h^2 t^3 + 0.003333\ h^2 t^3$
(85i)

$L([x_3(t)]^U_\alpha - [x_2(t)]^U_\alpha) = -0.010002\ \alpha h^3 t + 0.040002\ h^3 t - 0.003327\ \alpha h^3 t^3 + 0.013327\ h^3 t^3 - 0.010002\ \alpha h^2 t + 040002\ h^2 t - 0.000083\ \alpha h^3 t^5 + 0.000333\ h^3 t^5 - 0.001667\ \alpha h^2 t^3 + 0.006667\ h^2 t^3$
(85ii)

$[x_3(t)]^L_\alpha - [x_2(t)]^L_\alpha = \iint (0.010002\ \alpha h^3 t + 0.019998\ h^3 t + 0.003327\ \alpha h^3 t^3 + 0.00667\ h^3 t^3 + 0.010002\ \alpha h^2 t + 0.019998\ h^2 t + 0.000083\ \alpha h^3 t^5 + 0.000167\ h^3 t^5 + 0.001667\ \alpha h^2 t^3 + 0.003333\ h^2 t^3)\ dt\, dt$ (86i)

$[x_3(t)]^U_\alpha - [x_2(t)]^U_\alpha = \iint (-0.010002\ \alpha h^3 t + 0.040002\ h^3 t - 0.003327\ \alpha h^3 t^3 + 0.013327 h^3 t^3 - 0.010002\ \alpha h^2 t + 0.040002\ h^2 t - 0.000083\ \alpha h^3 t^5 + 0.000333\ h^3 t^5 - 0.001667\ \alpha h^2 t^3 + 0.006667\ h^2 t^3)\ dt\, dt$ (86ii)

$[x_3(t)]^L_\alpha - [x_2(t)]^L_\alpha = 0.001667\ \alpha h^3 t^3 + 0.000333\ h^3 t^3 + 0.000166\ \alpha h^3 t^5 + 0.000334\ h^3 t^5 + 0.001667\ \alpha h^2 t^3 + 0.000333\ h^2 t^3 + 0.000002\ \alpha h^3 t^7 + 0.000004\ h^3 t^7 + 0.000083\ \alpha h^2 t^5 + 0.000167\ h^2 t^5$
(87i)

$[x_3(t)]^U_\alpha - [x_2(t)]^U_\alpha = -0.001667\ \alpha h^3 t^3 + 0.006667\ h^3 t^3 - 0.000166\ \alpha h^3 t^5 + 0.000666\ h^3 t^5 - 0.001667\ \alpha h^2 t^3 + 0.006667\ h^2 t^3 - 0.000002\ \alpha h^3 t^7 + 0.000008\ h^3 t^7 - 0.000083\ \alpha h^2 t^5 + 0.000333\ h^2 t^5$
(87ii)

$[x_3(t)]^L_\alpha = 0.001667\ \alpha h^3 t^3 + 0.000333\ h^3 t^3 + 0.000166\ \alpha h^3 t^5 + 0.000334\ h^3 t^5 + 0.003334\ \alpha h^2 t^3 + 0.003666\ h^2 t^3 + 0.000002\ \alpha h^3 t^7 + 0.000004\ h^3 t^7 + 0.000334\ h^2 t^5 + 0.000166\ \alpha h^2 t^5 + 0.001667\ \alpha h t^3 + 0.003333\ h t^3$ (88i)

$[x_3(t)]^U_\alpha = -0.001667\ \alpha h^3 t^3 + 0.006667\ h^3 t^3 - 0.000166\ \alpha h^3 t^5 + 0.000666\ h^3 t^5 - 0.003334\ \alpha h^2 t^3 + 0.013334\ h^2 t^3 - 0.000002\ \alpha h^3 t^7 +$

$0.000008\ h^3t^7 - 0.000166\ \alpha h^2t^5 + 0.000666\ h^2t^5 - 0.001667\ \alpha ht^3 + 0.006667\ ht^3$ (88ii)

Then, the fuzzy series solution is:

$$[x(t)]_\alpha = [\ [x(t)]_\alpha^L , [x(t)]_\alpha^U] \quad (89)$$

Where

$$[x(t)]_\alpha^L = 0.01\ \alpha t + 0.02\ t + 0.005001\ \alpha ht^3 + 0.009999\ ht^3 + 0.005001\ \alpha h^2t^3 + 0.006999\ h^2t^3 + 0.000249\ \alpha h^2t^5 + 0.000501\ h^2t^5 + 0.001667\ \alpha h^3t^3 + 0.000333\ h^3t^3 + 0.000166\ \alpha h^3t^5 + 0.000334\ h^3t^5 + 0.000002\ \alpha h^3t^7 + 0.000004\ h^3t^7 + \cdots \quad (90i)$$

$$[x(t)]_\alpha^U = -0.01\ \alpha t + 0.04\ t - 0.005001\ \alpha ht^3 + 0.020001\ ht^3 - 0.005001\ \alpha h^2t^3 + 0.020001\ h^2t^3 - 0.000249\ \alpha h^2t^5 + 0.000999\ h^2t^5 - 0.001667\ \alpha h^3t^3 + 0.006667\ h^3t^3 - 0.000166\ \alpha h^3t^5 + 0.000666\ h^3t^5 - 0.000002\ \alpha h^3t^7 + 0.000008\ h^3t^7 + \cdots \quad (90ii)$$

The fuzzy series solution at $h = -1$, will be

$$[x(t)]_\alpha = [\ [x(t)]_\alpha^L , [x(t)]_\alpha^U] \quad (91)$$

Where

$[x(t)]_\alpha^L =$

$$0.01\ \alpha t + 0.02\ t - 0.001667\ \alpha t^3 - 0.003333\ t^3 + 0.000083\ \alpha t^5 + 0.000167\ t^5 - 0.000002\ \alpha t^7 - 0.000004\ t^7 + \cdots \quad (92i)$$

$$[x(t)]_\alpha^U = -0.01\ \alpha t + 0.04\ t + 0.001667\ \alpha t^3 - 0.006667\ t^3 - 0.000083\ \alpha t^5 + 0.000333\ t^5 + 0.000002\ \alpha t^7 - 0.000008\ t^7 + \cdots$$
(92ii)

7. Discussion

When solving a fuzzy autonomous differential equation by using the fuzzy homotopy analysis method , the accuracy of the results depends greatly on the value of the parameter h, other factors also affect, including : the number of terms of the solution series, the value of the constant α and the period to which the variable t belongs. The fuzzy semi-analytical solutions that we obtained during this work are accurate solutions and very close to the fuzzy exact-analytical solutions, based on the comparison that we will make between the results that we obtained and the fuzzy exact-analytical solutions to the chosen problem.

If we go back to example(1) :

$x''(t) + x(t) = 0\ ,\ \ t \in [0\ ,\ 0.5]$ (93)

The fuzzy exact-analytical solution for this problem is :

$[x(t)]_\alpha = [\ [x(t)]_\alpha^L\ , [x(t)]_\alpha^U\ ]$ (94)

Where

$[x(t)]_\alpha^L = (\ 0.02 + 0.01\alpha\ )\text{sint}$ (95i)

$[x(t)]_\alpha^U = (\ 0.04 - 0.01\alpha\ )\text{sint}$ (95ii)

While the fuzzy semi-analytical solution that we got(at $h = -1$, $\alpha = 0.3$) is :

$[x(t)]_\alpha = [\ [x(t)]_\alpha^L\ , [x(t)]_\alpha^U\ ]$ (96)

Where

$[x(t)]_\alpha^L = 0.023\ t - \ 0.003833\ t^3\ +\ 0.000122\ t^5\ -\ 0.000005\ t^7\ \ + \cdots$, (97i)

$[x(t)]_\alpha^U =\ 0.037\ t - \ 0.006167\ t^3\ +\ 0.000308\ t^5\ -\ 0.000007\ t^7\ \ + \cdots$. (97ii)

Also, the fuzzy semi-analytical solution that we got(at $h = -1$, $\alpha = 0.4$) is :

$[x(t)]_\alpha = [\ [x(t)]_\alpha^L\ , [x(t)]_\alpha^U\ ]$ (98)

Where

$[x(t)]_\alpha^L = 0.024\ t - \ 0.004\ t^3\ +\ 0.0002\ t^5\ -\ 0.000005\ t^7\ \ + \cdots$ (99i)

$[x(t)]_\alpha^U =\ \ 0.036\ t - \ 0.006\ t^3\ +\ 0.0003\ t^5\ -\ 0.000007\ t^7\ \ + \cdots$ (99ii)

We test the accuracy of the obtained solutions by computing the absolute errors

$[\text{error}]_\alpha^L\ = |\ [x_{\text{exact}}(t)]_\alpha^L\ -\ [x_{\text{series}}(t)]_\alpha^L\ |$ (100i)

$[\text{error}]_\alpha^U\ = |\ [x_{\text{exact}}(t)]_\alpha^U\ -\ [x_{\text{series}}(t)]_\alpha^U\ |$ (100ii)

The following tables provides a comparison between the fuzzy exact-analytical solution and the fuzzy semi- analytical solution for this problem.

t	$[x_{\text{series}}(t)]_\alpha^L$	$[\text{error}]_r^L$	$[x_{\text{series}}(t)]_\alpha^U$	$[\text{error}]_r^U$

Table 1. Comparison of the results of example(1), $\alpha = 0.3$.

0	0	0	0	0
0.05	0.001149520	1.99 e-11	0.001849229	4.18 e-11
0.10	0.002296168	3.63 e-10	0.003693836	3.37 e-10
0.15	0.003437072	4.00 e-9	0.005529209	1.00 e-9
0.20	0.004569374	1.90 e-8	0.007350762	2.00 e-9
0.25	0.005690228	6.20 e-8	0.009153940	5.00 e-9
0.30	0.006796804	1.60 e-7	0.010934237	9.00 e-9
0.35	0.007886297	3.51 e-7	0.012687203	1.50 e-8
0.40	0.008955929	6.92 e-7	0.014408454	2.40 e-8
0.45	0.010002950	1.26 e-6	0.016093689	3.50 e-8
0.50	0.011024648	2.14 e-6	0.017738695	4.90 e-8

t	$[x_{series}(t)]^L_\alpha$	$[error]^L_r$	$[x_{series}(t)]^U_\alpha$	$[error]^U_r$
0	0	0	0	0
0.05	0.001199500	0	0.001799250	0
0.10	0.002396002	0	0.003594002	0
0.15	0.003586515	4.20 e-13	0.005379772	2.30 e-13
0.20	0.004768063	3.08 e-12	0.007152095	1.78 e-12
0.25	0.005937695	1.48 e-11	0.008906542	8.32 e-12
0.30	0.007092484	5.34 e-11	0.010638727	2.93 e-11
0.35	0.008229547	1.58 e-10	0.012344321	8.41 e-11
0.40	0.009346039	4.07 e-10	0.014019060	2.08 e-10
0.45	0.010439171	9.40 e-10	0.015658759	4.59 e-10
0.50	0.011506210	1.00 e-9	0.017259320	9.23 e-10

Table 2. Comparison of the results of example(1), $\alpha = 0.4$.

8. Conclusion

In this work, we have studied the fuzzy approximate-analytical solutions of the second order fuzzy autonomous differential equation. Obviously the accuracy of the results that can be obtained when solving using the fuzzy homotopy analysis method, these results may improve further when increasing the number of terms of the solution series or using another value for the parameter h. The value of the variable t greatly affects the accuracy of the results, if the value of the variable t is close to the initial value, the results will be more accurate. Also, the value of the constant α greatly affects the accuracy of the results. Certainly, the best value of the constant α cannot be determined, as it changes from one problem to another.

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