Ratio Mathematica Volume 40, 2021, pp. 7-25

A survey on fuzzy semigroups

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Abstract

Fuzzy semigroup is an algebraic extension of semigroup. It has found application in fuzzy coding theory, fuzzy finite state machines and fuzzy languages. In this paper, a comprehensive literature review on fuzzy semigroup theory is realized. We will begin with a review of fuzzy groups which heavily inspired the notion of fuzzy semigroups put forth by the algebraists. Subsequently, a brief tour of semigroup theory is considered as a precursor to the emerging subject.

Keywords: Fuzzy group; semigroup; fuzzy semigroup.

2020 AMS subject classifications: 20N25; 03E72. [1]

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[1]Received on January 12th, 2021. Accepted on May 12th, 2021. Published on June 30th, 2021. doi: 10.23755/rm.v40i1.557. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Semigroup theory is a thriving field of modern abstract algebra. As the name suggests, a semigroup is a generalization of a group; because a semigroup need not in general have an element which has an inverse. The earliest major contributions to the theory of semigroups are strongly motivated by comparisons with groups and rings. Semigroup theory can be considered as one of the most successful off-springs of ring theory in the sense that the ring theory gives a clue how to develop the ideal theory of semigroups. The algebraic structure enjoyed by a semigroup is a non-empty set together with an associative binary operation. However, the fuzzy algebraic structures and their extensions are very important. Nowadays, a lot of extensions of fuzzy algebraic structures have been introduced by many authors and have been applied to real life problems in different fields of science.

Many crisp concepts of algebraic structures have been extended to the non-classical structures. Fuzzy groups were first considered by Rosenfeld (1971). In 1971, he defined fuzzy subgroup and established some of its properties. His definition of fuzzy group is a turning point for pure mathematicians. Since then, the study of fuzzy algebraic structure has been pursued in many directions such as groups, rings, modules, vector space and so on. Aktas and Cagman (2007) gave a definition of soft groups and derived their basic properties. Rough groups were defined by Biswas and Nanda (1994), and some other authors have studied the algebraic properties of rough set as well. Demirci (2001) introduced the concept of smooth groups by using fuzzy binary operation. Multigroups were first described by Marty and several scholars put forth different definitions in an attempt to generalize group concept (see Dresher and Ore (1938), Griffiths (1938), Schein (1987) Barlotti and Strambach (1991), Nazmul et al. (2013), Tella and Daniel (2013)).

Moreover, the algebraic extensions of a semigroup have been been studied by many authors. Among others are the notion of ternary semigroups known to Banach (cf. Los (1955)) who is credited with an example of a ternary semigroup which does not reduce to a semigroup. kazim and Naseeruddin (1972) introduced left almost semigroups (LA-semigroups). The structure is also known as AG-groupoid and modular groupoid and has a variety of applications in topology, matrices, flock theory, finite mathematics and geometry. Sen (1981) introduced the concept of Γ-semigroups as a generalization of semigroups.

The purpose of this paper is to promote research and disseminate fuzzy proficiency by presenting a comprehensive and up to date literature review of the fuzzy semigroup theory.

2 A brief review of fuzzy groups

The important concept of a fuzzy set put forth by Zadeh (1965) has opened up keen insights and applications in a wide range of scientific fields. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, groupoids and topology.

The study of fuzzy algebraic structures started in the pioneering paper of Rosenfeld (1971). Rosenfeld introduced the notion of fuzzy groups and successfully extended many results from groups to fuzzy groups. Though some other definitions of fuzzy groups are available in the literature (for example, Anthony and Sherwood (1979) redefined the fuzzy groups in terms of a t-norm which replaced the minimum operation), Rosenfeld's definition seems to be the most conventional and accepted one.

Most of the recent contributions in the field are the validations of Rosenfeld's definition where a fuzzy subset A of a group X is called a fuzzy subgroup of X if and only if $\mu_A(xy) \geq min\{\mu_A(x), \mu_A(y)\}$ and $\mu_A(x^{-1}) \geq \mu_A(x)$. Das (2014) defined a level subgroup of a fuzzy subgroup A of a group X as an ordinary subgroup A_t of X, where $t \in [0, 1]$.

Wu (1981) studied fuzzy normal subgroup. Also, fuzzy normal subgroups were studied by Liu (1982) and Kumar et al. (1992). In line with this, Ajmal and Jahan (2012) introduced the notion of a characteristic fuzzy subgroup of a group and related results.

Mukherjee and Bhattacharya (1984) introduced the concept of fuzzy cosets and their relation with fuzzy normal subgroups. Moreover, the authors proved fuzzy generalizations of some important theorems like Lagranges and Cayleys theorems. Also, the authors initiated the notions of a fuzzy normalizer of a fuzzy subgroup and fuzzy solvable in Mukherjee and Bhattacharya (1986) and Mukherjee and Bhattacharya (1987).

The effect of group homomorphism on fuzzy groups was studied by Rosenfeld Rosenfeld (1971) and proved that a homomorphic image of a fuzzy subgroup is a fuzzy subgroup provided the fuzzy subgroup has $\bigvee$-property, while a homomorphic pre image of a fuzzy subgroup is always a fuzzy subgroup. Anthony and Sherwood Anthony and Sherwood (1979) later proved that even without the $\bigvee$-property the homomorphic image of a fuzzy subgroup is a fuzzy subgroup. Sidky and Mishref (1990) proved that if $f : X \longrightarrow Y$ is a group homomorphism and A is a fuzzy subgroup of X "with respect to a continuous t-norm T, then $f(A)$

is a fuzzy subgroup of Y with respect to T". Since $\bigwedge$ is a continuous t-norm (Anthony and Sherwood), it follows that $f(A) \in FG(Y)$ whenever $A \in X$. It was proved by Akgul (1988) that $f^{-1}(B)$ is a fuzzy subgroup of X whenever B is a fuzzy subgroup of Y. Fang (1994) introduced the concepts of fuzzy homomorphism and fuzzy isomorphism by a natural way, and study some of their properties. Ajmal (1994) defined a notion of 'containment' of an ordinary kernel of a group homomorphism in a fuzzy subgroup and provided the long-awaited solution of the problem of showing a one-to-one correspondence between the family of fuzzy subgroups of a group, containing the kernel of a given homomorphism, and the family of fuzzy subgroups of the homomorphic image of the given group. Yong (2004) constructed a quotient group induced by a fuzzy normal subgroup and proved the corresponding isomorphism theorems.

Demirci and Racasens (2004) initiated fuzzy equivalence relation associated with a fuzzy subgroup and showed that a fuzzy subgroup is normal if only if the operation of the group is compatible with its associated fuzzy equivalence relation. kondo (2004) modified the idea of Demirci and Recasens and defined a fuzzy congruence on a group.

Ngcibi et al. (2010) obtained a formula for the group $Z_{p^m} \times Z_{p^n}$ when $n = 1, 2, 3$ and Sehgal et al. (2016) extended the concept for all values of n.

3 A tour of semigroup theory

The term semigroup was first coined in a French group theory textbook (de Seguier (1904)) with a more stringent definition than the modern one, before being introduced to the English-speaking mathematical world by Leonard Dickson the following year Dickson (1905). Three decades after, the only semigroup theory being done was that done in near-obscurity (at least from the Western perspective) by a Russian mathematician, Anton Kazimirovich Suschkewitsch. Suschkewitsch (1928) was essentially doing semigroup theory before the rest of the world knew that there was such a thing, thus many of his results were rediscovered by later researchers who were unaware of his achievements.

The study of semigroups exploded after the publication of a series of highly influential papers in the early 1940s. Ree (1940) obtained the structure of finite simple semigroups and proved that the minimal ideal (Green's relation) of a finite semigroup is simple. Clifford (1941) introduced semigroups admitting relative inverses. Dubreil (1941) studied semigroup theory from the concept of lattice of equivalence relations on sets. Preston (1954) defined and developed the concept of inverse semigroups. Furthermore, Preston described congruences on completely 0-simple semigroup and free inverse semigroups were also studied by the author (see Preston (1961) and Preston (1973)). Munn (1955), having carried out research in a different direction, introduced the notion of semigroup algebras.

kimura (1957) studied semigroups very widely and vividly and carried studies on idempotent semigroups. He further researched idempotent semigroups which satisfies some identities. Moreover, idempotent semigroups was earlier studied by McLean (1954).

Yamada (1997) analyzed idempotent semigroups. Green (1951) authored a classical paper on the structure of semigroups and with Rees to study those semigroups in which $x^r = x$. (Green and Rees, 1952)

For over three decades, Howie (1976)-Howie (1995) worked on embedding theorems for semigroups in his book Howie (1976). He collaborated with Munn and Weirert and edited a proceeding of the conference on semigroups and their applications. Howie et al. (1992). His contributions to semigroup theory is very significant. In this period of three decades, semigroup theorists like Petrich (1973)-Petrich (1984), McAlister and McFadden (1974)-McAlister (1974), Alan (1998), Lawson (1998) and Lajos (1971) have done lots of research on special class of semigroups in the vein of inverse semigroups, free semigroups, etc. and their properties. Okninski (1998) published a book on semigroups of matrices.

Several researchers have worked on the types of semigroup mentioned earlier and developed more properties with applications across a broad spectrum of areas (see Eilenberg (1974), Eilenberg (1976), Hopcropt and Ullman (1979), Howie (1991), Lallement (1979), Straubing (1994)).

In a conference, Meakin (2005) delivered a lecture on groups and semigroups exploring their connections and contrasts. He clearly acknowledged that in the past several decades, group theory and semigroup theory have developed in different directions. Cayley's theorem enables one to view groups as groups of permutations of some set while semigroups are represented as semigroups of functions from a set to itself. However, significant research has been carried out both in group theory and semigroup theory beyond the early viewpoints. In reality, several concepts in modern semigroup theory are closely related to group theory. For instance, automata theory and formal language theory turn out to be related (see Hopcropt and Ullman (1979), Howie (1991)).

Very recently, Gould and Yang (2014) presented a piece of research work titled ”Every group is a maximal subgroup of a naturally occurring free idempotent generated semigroup”. The structures of generalized inverse semigroups by kudryavtseva and Lausa (2014) is also a recent work on inverse semigroups. Haggins (1992) carried out a research on permutations of a semigroup that maps to inverses. The variety of unary semigroups with associate inverse subsemigroup by Billhardt et al. (2014) is however an additional view on inverse subsemigroups. Thus, semigroup theory has developed rapidly to become the extremely prolific area of research for scholars.

4 Development of fuzzy semigroup theory

In this section, we systematically provides research work done on fuzzy semigroup analogue of some basic notions from semigroup theory as well as record some elementary properties and applications of fuzzy semigroups.

4.1 Fuzzy Semigroup

The study of fuzzy algebraic structures started with the introduction of the concepts of fuzzy subgroup (subgroupoid) and fuzzy (left, right) ideal in the pioneering paper by Rosenfeld Rosenfeld (1971). In 1979, fuzzy semigroups were introduced by ((Kuroki (1981), Kuroki (1982)), which is a generalization of classical semigroups. He had published a series of papers Kuroki (1981)-Kuroki (1997), in which he laid the foundation of an algebraic theory of semigroup in the fuzzy framework. In literature, many related works vis-*á*-vis fuzzy ideals of semigroups can be found in (Lajos (1979), Mclean and Kummer (1992), Xue-Ping et al. (1992), Ahsan et al. (1995), Zhi-Wen and Xue-Ping (1995), Dib and Galhum (1997), Xiang-Yun (1999b), Das (1999), Xiang-Jun (2001b)-Xiang-Jun (2002), Ahsan et al. (2001), Lee and Shun (2001), Ahsan et al. (2002), Jun and Seok-Zun (2016a)-Jun and Seok-Zun (2016b), kazanci and Yamak (2008), Zhan and Jun (2010), khan et al.).

Shen (1990) initiated the concepts of fuzzy regular subsemigroups, fuzzy weakly regular subsemigroups fuzzy completely regular subsemigroups, fuzzy weakly completely regular subsemigroup and investigated some of their algebraic properties. Based on the definition of fuzzy regular subsemigroup given by Shen (1990), Xue-Ping and Wang-Jin (1993) defined fuzzy (left, intra-) regular subsemigroup in semigroups and studied some related properties. Furthermore, point-wise depiction of fuzzy regularity of semigroups was introduced by Zhi-

Wen and Xue-Ping (1993). They also proposed the concept of a fuzzy weakly left (right, intra-) regular subsemigroup and exhibited some algebraic properties. Shabir et al. (2010) in the twentieth century characterized regular semigroups by (α, β) - fuzzy ideals.

Samhan (1993) discussed the fuzzy congruence relation generated by a given fuzzy relation on a semigroup. He also studied the lattice of fuzzy congruence relation on a semigroup and gave some lattice theoretic properties. Kuroki (1997) introduced the notion of a quotient semigroup induced by a fuzzy congruence relation on a semigroup and obtained homomorphism theorems with respect to the fuzzy congruence. Earlier before his published paper in Kuroki (1997), Kuroki (1995) studied fuzzy congruences on T^{*}- pure semigroups. Moreover, he had earlier proposed the concept of an idempotent-separating fuzzy congruence on inverse semigroups before Das (1997) developed fuzzy congruences in an inverse semigroup and established some important results. The notions of fuzzy kernel and fuzzy trace of a fuzzy congruence on an inverse semigroup were introduced by Al-Thurkair (1993). He established a one-to-one correspondence between fuzzy congruence pair and fuzzy congruences on an inverse semigroup. Xiang-Yun (1999a) introduced fuzzy Rees congruences on semigroups and obtained that a homomorphic image of a fuzzy Rees congruences semigroup is a fuzzy Rees congruences semigroup. Tan (2001) studied fuzzy congruences on a regular semigroup. Zhang (2000) introduced the concept of fuzzy group congruences on a semigroup and investigated some of its properties. Two years after, he examined fuzzy congruences on completely 0-simple semigroups. Recently, Ma and Tian (2011) introduced the notion of fuzzy congruence triple on a completely simple semigroup and used it to characterize fuzzy congruence on a completely simple semigroup.

The concept of fuzzy semiprimality in a semigroup as an extension of semiprimality in a semigroup was introduced by Kuroki (1982). He described a semigroup that is a semilattice of simple semigroups in terms of fuzzy semiprimality. Kuroki (1993) characterized a completely regular semigroup and a semigroup that is semilattice of groups in terms of fuzzy semiprime quasi-ideals. Xiang-Jun (2000) defined and studied prime fuzzy ideals of a semigroup. Subsequently, Xiang-Jun (2001a) introduced and studied the quasi-prime and weakly quasi-prime fuzzy left ideals of a semigroup. kehayopulu et al. (2001) worked on characterization of prime and semiprime ideals of semigroups in terms of fuzzy subsets. Shabir (2015) characterized semigroups in which each fuzzy ideal is prime. kazanci and Yamak (2009) defined φ-semiprime fuzzy ideals of a fuzzy semigroup and described all of φ-semigroups in which every φ-fuzzy ideal is φ-semiprime. Manikantan and Peter (2015) proposed some new kind of fuzzy subsets of a semigroup by using fuzzy magnified translation, fuzzy translation, fuzzy multiplication and extension of a fuzzy subset and obtained some results on fuzzy semiprime ideals of semigroups.

Among other authors who reported work done on fuzzy semigroup are Lizasoain and Gomez (2017) who showed that the direct of two fuzzy transformation is again a fuzzy transformation semigroup if and only if the lattice is distributive. Budimirovic et al. (2014) introduced fuzzy semigroups with respect to a fuzzy equality. Sen and Choudhury (2006) studied the intersection graphs of fuzzy semigroups and showed related results.

4.2 Elemantary Properties of Fuzzy Semigroup

Here, we refer readers to Mordeson et al. (2003) for more details.

4.2.1 Fuzzy set

Let S be a non empty set. A fuzzy set in S is a function $f : S \longrightarrow [0, 1]$.

4.2.2 Semigroup

A semigroup is an algebraic structure $(S, .)$ consisting of a non empty set together with an associative binary operation $" \cdot "$.

4.2.3 Fuzzy ideals in semigroups

Let S be a semigroup and f, g be two fuzzy subsets of S. The product of $f \circ g$ is defined by

$$f \circ g\,(x) = \begin{cases} \bigvee_{x=yz}\{f(y) \wedge g(z)\}, & if\ \exists\, y, z \in S\ such\ that\ x = yz, \\ 0, & otherwise. \end{cases}$$

for all $x \in S$.

A fuzzy subset f of S is called a fuzzy subsemigroup of S if $f(ab) \geq f(a) \bigwedge f(b)$ for all $a, b \in S$, and is called a fuzzy left (right) ideal of S if $f(ab) \geq f(b) \quad (f(ab) \geq f(a))$ for all $a, b \in S$. A fuzzy subset f of S is called a fuzzy two-sided ideal (or a fuzzy ideal) of S if it is both a fuzzy left and a fuzzy right of S.

Lemma 4.1. *Let f be a fuzzy subset of a semigroup S. Then the following properties hold.*

(i) f is a fuzzy subsemigroup of S if and only if $f \circ f \subseteq f$.

(ii) f is a left ideal of S if and only if $S \circ f \subseteq f$.

(iii) f is a right ideal of S if and only if $f \circ S \subseteq f$.

(iv) f is a two-sided ideal of S if and only if $S \circ f \subseteq f$ and $f \circ S \subseteq f$.

Proof. See Mordeson et al. (2003) □

Lemma 4.2. *Let S be a semigroup. Then the following properties hold.*

(i) Let f and g be two fuzzy subsemigroups of S. Then $f \cap g$ is a fuzzy subsemigroup of S.

(ii) Let f and g be (left, two-sided) ideal of S. Then $f \cap g$ is also a fuzzy left (right two-sided) ideal of S.

Proof. See Mordeson et al. (2003) □

Lemma 4.3. *If f is a fuzzy left (right) ideal of S. Then $f \cup (S \circ f)$ $(f \cup (f \circ S))$ is a fuzzy two-sided ideal of S.*

Proof. See Mordeson et al. (2003) □

4.2.4 Fuzzy regular subsemigroup and homomorphism

If f is a fuzzy subsemigroup of S and $\forall\, x \in S$, there exists $x^{'} \in R_x$ such that $f(x^{'}) \geq f(x)$ provided $f(x) \neq 0$, then f is called a fuzzy regular subsemigroup of S.

Proposition 4.1. *f is a fuzzy regular subsemigroup of S if and only if $\forall\, t \in (0, 1]$, f_t is a regular subsemigroup of S provided $f_t \neq \emptyset$.*

Proof. See Mordeson et al. (2003) □

Proposition 4.2. *If f is a fuzzy regular subsemigroup of S, then $f \circ f = f$.*

Proof. See Mordeson et al. (2003) □

Proposition 4.3. *Let α be a semigroup surjection homomorphism from S onto T.*

(i) If f is a fuzzy regular subsemigroup of S, then $\alpha(f)$ is a fuzzy regular subsemigroup of T.

(ii) If g is a fuzzy regular subsemigroup of T, then $\alpha^{-1}(g)$ is a fuzzy regular subsemigroup of S.

Proof. See Mordeson et al. (2003) □

4.2.5 Fuzzy congruences on semigroups and fuzzy factor semigroups

A fuzzy equivalence relation on a semigroup S which is compatible is called a fuzzy congruence relation on S.

Theorem 4.1. *Let μ and ν be fuzzy congruences on semigroup S. Then the following conditions are equivalent.*

(i) $\mu \circ \nu$ is a fuzzy congruence.

(ii) $\mu \circ \nu$ is a fuzzy equivalence.

(iii) $\mu \circ \nu$ is fuzzy symmetric.

(iv) $\mu \circ \nu = \nu \circ \mu$.

Proof. See Mordeson et al. (2003) □

Let μ be a fuzzy congruence on S. Then $S/\mu = \{\mu_a \mid a \in S\}$, where $\mu_a = \mu(a, x)$ for all $x \in S$.

Theorem 4.2. *The binary relation $*$ on S/μ is well-defined.*

Proof. See Mordeson et al. (2003) □

Theorem 4.3. *Let μ be a fuzzy congruence on a semigroup S. Then $\mu^{-1}(1) = \{(a, b) \in S \times S \mid \mu(a, b) = 1\}$ is a congruence on S.*

Proof. See Mordeson et al. (2003) □

4.3 Applications of Fuzzy Semigroups

There are some important areas in which the fuzzy semigroup-theoretic approach is quite substantial and more completely utilized. The most significant such areas are the theories of fuzzy codes, fuzzy finite state machines and fuzzy languages. For greater details on the subject, the readers are also directed to the monograph by Mordeson et al. (2003).

4.3.1 Fuzzy codes

Let X be an alphabet with $1 \leq\mid X \mid< \infty$ and $X^*(X^+)$ is the free monoid (semigroup) generated by X with operation of concatenation. If A is a fuzzy submonoid of X^* and $B \in \mathcal{FP}(X^*)$ such that $B \subseteq A$, then B is its fuzzy base with $B(e) = 0$ and

(B1) $\forall\, x \in Supp(A)\backslash e,\ B^*(x) \geq A(x)$;

(B2) $\forall\, x \in Supp(A)\backslash e,\ x_i y_j \in X^*,\ i = 1, ..., n;\ j = 1, ..., m$ and $x = x_i...x_n = y_1...y_m,\ \bigwedge\{B(x_1), ..., B(x_n), B(y_1), ..., B(y_n)\} \propto \bigwedge\{[m = n], [x_1 = y_1], ..., [x_n = y_n\} \geq A(x)$, where e and $\mathcal{FP}(X^*)$ denote the empty string and the class of all fuzzy subsets of X^*. This explains the origin of the concept. A fuzzy code A over X^+ is such that $A \neq \emptyset$ and A is a fuzzy base of A^*.

4.3.2 Fuzzy finite state machine

A fuzzy finite state machine is an ordered triple $M = (Q, X, \mu)$, where Q and X are non-empty finite sets and $\mu : Q \times X \times X \longrightarrow [0, 1]$. The elements of Q are called states and those of X are called inputs. However, a fuzzy finite state machine can be regarded as a finite state machine when $M \subseteq \{0, 1\}$.

Fuzzy finite state machines can be divided into four categories:

(i) M is called a deterministic fuzzy finite state if μ is a partial fuzzy function.

(ii) M is called a non-deterministic fuzzy finite state if μ is a fuzzy relation.

(iii) M is called a complete deterministic fuzzy finite state if μ is a complete partial fuzzy function.

(iv) M is called a complete non-deterministic fuzzy finite state if μ is a complete fuzzy relation.

4.3.3 Fuzzy languages

A fuzzy formal language or a fuzzy language $\mu : T^* \longrightarrow [0,1]$ can serve to indicate the degree of meaningfulness of each string in T^*, namely, for $x \in T^*$, $\mu(x)$ near 1 implies that x is meaningful and $\mu(x)$ near 0 implies that x is not meaningful. A language L is defined to be a sequential fuzzy language if there is a finite fuzzy automata A_f and a cut-point t such that L is the set of coded words that yield at least one path from the initial state to a final state of A_f whose fuzzy measure is greater than t.

5 Conclusion

We have presented a comprehensive literature survey on the concept of fuzzy semigroups with some basic properties outlined and significant notable applications highlighted. For future research, we can hybridize non-classical structures to study their algebraic structures in semigroups.

6 Acknowledgements

The authors are very thankful to the learned referee for his invaluable suggestions for the improvement of this paper.

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