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Reliability estimation of Weibull-exponential distribution via Bayesian approach

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Abstract

Weibull-exponential distribution is considered. Bayesian method of estimation is employed in order to estimate the reliability function of Weibull-exponential distribution by using non-informative and beta priors. In this paper, the Bayes estimators of the reliability function have been obtained under squared error, precautionary and entropy loss functions.

Keywords: Weibull-exponential distribution. Reliability. Bayesian method. Non-informative and beta priors. Squared error, precautionary and entropy loss functions.

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1. Introduction

The Weibull-exponential distribution was proposed by Oguntunde et al. [1]. They obtained some of its basic Mathematical properties. This distribution is useful as a life testing model and is more flexible than the exponential distribution. The probability function $f(x;\theta)$ and distribution function $F(x;\theta)$ of Weibull-exponential distribution are respectively given by

$$f(x;\theta) = a\lambda\theta\left(1-e^{-\lambda x}\right)^{a-1} e^{a\lambda x} exp\left[-\theta\left(e^{\lambda x}-1\right)^{a}\right] \quad ; x \ge 0. \tag{1}$$

$$F(x;\theta) = 1-e^{-\theta\left(e^{-\lambda x}-1\right)^{a}} \quad ; x \ge 0,\ \theta > 0. \tag{2}$$

Let $R(t)$ denote the reliability function, that is, the probability that a system will survive a specified time t comes out to be

$$R(t) = e^{-\theta\left(e^{\lambda t}-1\right)^{a}} \quad ; t > 0,\ \theta > 0. \tag{3}$$

And the instantaneous failure rate or hazard rate, h(t) is given by

$$h(t) = a\lambda\theta e^{a\lambda t}\left(1-e^{-\lambda t}\right). \tag{4}$$

From equation (1) and (3), we get

$$f(x;R(t)) = \frac{a\lambda e^{a\lambda x}}{\left(e^{\lambda t}-1\right)^{a}}\left(1-e^{-\lambda x}\right)^{a-1}\left[-log\,R(t)\right]\left[R(t)\right]^{\left(\frac{e^{\lambda x}-1}{e^{\lambda t}-1}\right)^{a}} \quad ; \ 0<R(t)\le 1. \tag{5}$$

The joint density function or likelihood function of (5) is given by

$$f(\underline{x}\,|\,R(t)) = \frac{(a\lambda)^{n} e^{a\lambda\sum_{i=1}^{n}x_i}}{\left(e^{\lambda t}-1\right)^{na}}\left(\prod_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{a-1}\right)\left[-log\,R(t)\right]^{n}\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}} \tag{6}$$

The log likelihood function is given by

$$log\ f\left(\underline{x}\,|\,R(t)\right)=log\left(\frac{(a\lambda)^{n}}{\left(e^{\lambda t}-1\right)^{na}}\right)+a\lambda\sum_{i=1}^{n}x_i+log\left(\prod_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{a-1}\right)$$
$$+n\,log\left[-log\,R(t)\right]+\frac{1}{\left(e^{\lambda t}-1\right)^{a}}\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{a}log\left[R(t)\right] \tag{7}$$

Differentiating (7) with respect to R (t) and equating to zero, we get the maximum likelihood estimator of R (t) as

$$\hat{R}(t)=exp\left[-n\left\{\left(e^{\lambda t}-1\right)^{a}\Big/\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{a}\right\}\right]. \tag{8}$$

2. Bayesian method of estimation

The Bayesian estimation procedure have been developed generally under squared error loss function

$$L\left(\hat{R}(t),R(t)\right)=\left(\hat{R}(t)-R(t)\right)^{2}. \tag{9}$$

where $\hat{R}(t)$ is an estimate of $R(t)$. The Bayes estimator under the above loss function, say $\hat{R}(t)_S$, is the posterior mean, i.e.,

$$\hat{R}(t)_S=E\left[R(t)\right]. \tag{10}$$

The squared error loss function is often used also because it does not lead extensive numerical computation but several authors (Zellner [2], Basu & Ebrahimi [3]) have recognized the inappropriateness of using symmetric loss function. Canfield [4] points out that the use of symmetric loss function may be inappropriate in the estimation of reliability function. Norstrom [5] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss function with quadratic loss function as a special case. A very useful and simple asymmetric precautionary loss function is

$$L\left(\hat{R}(t), R(t)\right) = \frac{\left(\hat{R}(t) - R(t)\right)^2}{\hat{R}(t)} \tag{11}$$

The Bayes estimator of $R(t)$ under precautionary loss function is denoted by $\hat{R}(t)_P$, and is obtained by solving the following equation

$$\hat{R}(t)_P = \left[E\left(R(t)\right)^2\right]^{\frac{1}{2}}. \tag{12}$$

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\frac{\hat{R}(t)}{R(t)}$. In this case, Calabria and Pulcini [6] points out that a useful asymmetric loss function is the entropy loss

$$L(\delta) \propto \left[\delta^p - p\ log_e(\delta) - 1\right],$$

where $\delta = \frac{\hat{R}(t)}{R(t)}$, and whose minimum occurs at $\hat{R}(t) = R(t)$ when $p > 0$, a positive error $\left(\hat{R}(t) > R(t)\right)$ causes more serious consequences than negative error, and vice-versa. For small $|p|$ value, the function is almost symmetric when both $\hat{R}(t)\ and\ R(t)$ are measured in a logarithmic scale, and approximately

$$L(\delta) \propto \frac{p^2}{2}\left[log_e \hat{R}(t) - log_e R(t)\right]^2.$$

Also, the loss function $L(\delta)$ has been used in Dey et al. [7] and Dey and Liu [8], in the original form having $p = 1$. Thus $L(\delta)$ can be written as

$$L(\delta) = b\left[\delta - log_e(\delta) - 1\right]; \quad b > 0. \tag{13}$$

The Bayes estimator of $R(t)$ under entropy loss function is denoted by $\hat{\theta}_E$ and is obtained as

$$\hat{R}(t)_E = \left[E\left(\frac{1}{R(t)} \right) \right]^{-1}. \tag{14}$$

For the situation where we have no prior information about $R(t)$, we may use non-informative prior distribution

$$h_1(R(t)) = \frac{1}{R(t)\ logR(t)};\quad 0 < R(t) \le 1. \tag{15}$$

The most widely used prior distribution for $R(t)$ is a beta distribution with parameters $\alpha, \beta > 0$, given by

$$h_2(R(t)) = \frac{1}{B(\alpha,\beta)} \left[R(t) \right]^{\alpha-1} \left[1 - R(t) \right]^{\beta-1};\quad 0 < R(t) \le 1. \tag{16}$$

3. Bayes estimators of $R(t)$ under $h_1(R(t))$

Under $h_1(R(t))$, the posterior distribution is defined by

$$f(R(t) \mid \underline{x}) = \frac{f(\underline{x} \mid R(t)) h_1(R(t))}{\int_0^1 f(\underline{x} \mid R(t)) h_1(R(t)) dR(t)} \tag{17}$$

Substituting the values of $h_1(R(t))$ and $f(\underline{x} \mid R(t))$ from equations (15) and (6) in (17), we get

$$f\left(R(t)\,|\,\underline{x}\right)=\frac{\left[\frac{(a\lambda)^n}{\left(e^{\lambda t}-1\right)^{na}}e^{a\lambda\sum_{i=1}^{n}x_i}\left(\prod_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{a-1}\right)\left[-log\,R(t)\right]^n \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a}\frac{1}{R(t)\,\,log\,R(t)}\right]}{\int_0^1\left[\frac{(a\lambda)^n}{\left(e^{\lambda t}-1\right)^{na}}e^{a\lambda\sum_{i=1}^{n}x_i}\left(\prod_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{a-1}\right)\left[-log\,R(t)\right]^n \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a}\frac{1}{R(t)\,\,log\,R(t)}\right]dR(t)}$$

$$=\frac{\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a-1}\left[\text{-}log\,R(t)\right]^{n-1}}{\int_0^1\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a-1}\left[\text{-}log\,R(t)\right]^{n-1}dR(t)}$$

or, $$f\left(R(t)\,|\,\underline{x}\right)=\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)}\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a-1}\left[\text{-}log\,R(t)\right]^{n-1} \tag{18}$$

Theorem 1. Assuming the squared error loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_S=\left(1+\frac{\left(e^{\lambda t}-1\right)^a}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^a}\right)^{-n} \tag{19}$$

Proof. From equation (10), on using (18),

$$\hat{R}(t)_S=E\left[R(t)\right]$$

$$=\int_0^1 R(t)\, f\left(R(t)\,|\,\underline{x}\right) dR(t)$$

$$=\int_0^1 R(t)\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)}\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a-1}\left[\text{-}log\, R(t)\right]^{n-1} dR(t)$$

$$=\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)}\int_0^1\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a}\left[\text{-}log\, R(t)\right]^{n-1} dR(t)$$

$$=\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)}\frac{\Gamma(n)}{\left(\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+1\right)^n}$$

or, $$\hat{R}(t)_S=\left(1+\frac{\left(e^{\lambda t}-1\right)^a}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^a}\right)^{-n}.$$

Theorem 2. Assuming the precautionary loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_P=\left[1+\frac{2\left(e^{\lambda t}-1\right)^a}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^a}\right]^{-\frac{n}{2}}. \tag{20}$$

Proof. From equation (12), on using (18),

$$\hat{R}(t)_P=\left[E\left(R(t)\right)^2\right]^{\frac{1}{2}}$$

$$=\left[\int_0^1 \left(R(t)\right)^2 \ f\left(R(t\,|\,\underline{x})\right) dR(t)\right]^{\frac{1}{2}}$$

$$=\left[\int_0^1 \left(R(t)\right)^2 \frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)}\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a-1}\left[\text{-}log\,R(t)\right]^{n-1} dR(t)\right]^{\frac{1}{2}}$$

$$=\left[\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)}\int_0^1 \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a+1}\left[\text{-}log\,R(t)\right]^{n-1} dR(t)\right]^{\frac{1}{2}}$$

$$=\left[\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\Gamma(n)}\ \frac{\Gamma(n)}{\left(\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+2\right)^n}\right]^{\frac{1}{2}}$$

$$=\left[\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)^n}{\left(\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+2\right)^n}\right]^{\frac{1}{2}}$$

or, $$\hat{R}(t)_P=\left[1+\frac{2\left(e^{\lambda t}-1\right)^a}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^a}\right]^{-\frac{n}{2}}.$$

Theorem 3. Assuming the entropy loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_E = \left[1 - \frac{\left(e^{\lambda t} - 1\right)^a}{\sum_{i=1}^{n}\left(e^{\lambda x_i} - 1\right)^a}\right]^n \tag{21}$$

Proof. From equation (14), on using (18),

$$\hat{R}(t)_E = \left[E\left(\frac{1}{R(t)}\right)\right]^{-1}$$

$$= \left[\int_0^1 \frac{1}{R(t)} f\left(R(t \mid \underline{x})\right) dR(t)\right]^{-1}$$

$$= \left[\int_0^1 \frac{1}{R(t)} \frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1}\right)^a\right)^n}{\Gamma(n)} \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1}\right)^a - 1} \left[\text{-}log\, R(t)\right]^{n-1} dR(t)\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1}\right)^a\right)^n}{\Gamma(n)} \int_0^1 \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1}\right)^a - 2} \left[\text{-}log\, R(t)\right]^{n-1} dR(t)\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1}\right)^a\right)^n}{\Gamma(n)} \frac{\Gamma(n)}{\left(\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1}\right)^a\right) - 1\right)^n}\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n} \left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1} \right)^a \right)^n}{\left(\left(\sum_{i=1}^{n} \left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1} \right)^a \right) - 1 \right)^n} \right]^{-1}$$

or, $$\hat{R}(t)_E = \left[1 - \frac{\left(e^{\lambda t} - 1\right)^a}{\sum_{i=1}^{n} \left(e^{\lambda x_i} - 1\right)^a} \right]^n .$$

4. Bayes estimators of $R(t)$ under $h_2(R(t))$

Under $h_2(R(t))$, the posterior distribution is defined by

$$f(R(t) | \underline{x}) = \frac{f(\underline{x} | R(t)) h_2(R(t))}{\int_0^1 f(\underline{x} | R(t)) h_2(R(t)) dR(t)} \tag{22}$$

Substituting the values of $h_2(R(t))$ and $f(\underline{x} | R(t))$ from equations (16) and (6) in (22), we get

$$f(R(t) | \underline{x}) = \frac{\left[\begin{array}{l} \frac{(a\lambda)^n}{\left(e^{\lambda t} - 1\right)^{na}} e^{a\lambda \sum_{i=1}^{n} x_i} \left(\prod_{i=1}^{n} \left(1 - e^{-\lambda x_i}\right)^{a-1} \right) \left[-\log R(t) \right]^n \\ \left[R(t) \right]^{\sum_{i=1}^{n} \left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1} \right)^a} \frac{1}{B(\alpha, \beta)} \left[R(t) \right]^{\alpha - 1} \left[1 - R(t) \right]^{\beta - 1} \end{array} \right]}{\int_0^1 \left[\begin{array}{l} \frac{(a\lambda)^n}{\left(e^{\lambda t} - 1\right)^{na}} e^{a\lambda \sum_{i=1}^{n} x_i} \left(\prod_{i=1}^{n} \left(1 - e^{-\lambda x_i}\right)^{a-1} \right) \left[-\log R(t) \right]^n \\ \left[R(t) \right]^{\sum_{i=1}^{n} \left(\frac{e^{\lambda x_i} - 1}{e^{\lambda t} - 1} \right)^a} \frac{1}{B(\alpha, \beta)} \left[R(t) \right]^{\alpha - 1} \left[1 - R(t) \right]^{\beta - 1} \end{array} \right] dR(t)}$$

$$= \frac{\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}+\alpha-1}\left[-log\,R(t)\right]^{n}\left[1-R(t)\right]^{\beta-1}}{\int_0^1 \left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}+\alpha-1}\left[-log\,R(t)\right]^{n}\left[1-R(t)\right]^{\beta-1} dR(t)}$$

or, $$f\left(R(t)\,|\,\underline{x}\right) = \frac{\left[R(t)\right]^{\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}+\alpha-1}\left[-log\,R(t)\right]^{n}\left[1-R(t)\right]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}\right)+\alpha+k\right)^{n+1}\right]} \quad (23)$$

Theorem 4. Assuming the squared error loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_S = \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}\right)+\alpha+1+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}\right)+\alpha+k\right)^{n+1}}\right] \quad (24)$$

Proof. From equation (10), on using (23),

$$\hat{R}(t)_S = E\left[R(t)\right]$$

$$= \int_0^1 R(t)\, f\left(R(t)\,|\,\underline{x}\right) dR(t)$$

$$= \int_0^1 R(t) \frac{\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}\right)+\alpha-1}\left[-log\,R(t)\right]^{n}\left[1-R(t)\right]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^{a}\right)+\alpha+k\right)^{n+1}\right]} dR(t)$$

$$= \frac{\int_0^1 \left[R(t)\right]^{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha} \left[\text{-log } R(t)\right]^n \left[1-R(t)\right]^{\beta-1} dR(t)}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k}\left(1 \Big/ \left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}\right]}$$

or, $$\hat{R}(t)_S = \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k}\left(1 \Big/ \left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+1+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k}\left(1 \Big/ \left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}}\right].$$

Theorem 5. Assuming the precautionary loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_P = \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k}\left(1 \Big/ \left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+2+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k}\left(1 \Big/ \left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}}\right]^{\frac{1}{2}} \tag{25}$$

Proof. From equation (12), on using (23),

$$\hat{R}(t)_P = \left[E\left(R(t)\right)^2\right]^{\frac{1}{2}}$$

$$= \left[\int_0^1 \left(R(t)\right)^2 f\left(R(t \mid \underline{x})\right) dR(t)\right]^{\frac{1}{2}}$$

$$=\left[\int_0^1 \left(R(t)\right)^2 \frac{\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha-1}\left[-\log R(t)\right]^n\left[1-R(t)\right]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}\right]}dR(t)\right]^{\frac{1}{2}}$$

$$=\left[\frac{\int_0^1\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+1}\left[-\log R(t)\right]^n\left[1-R(t)\right]^{\beta-1}dR(t)}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}\right]}\right]^{\frac{1}{2}}$$

or, $$\hat{R}(t)_P=\left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+2+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}}\right]^{\frac{1}{2}}.$$

Theorem 6. Assuming the entropy loss function, the Bayes estimate of $R(t)$, is of the form

$$\hat{R}(t)_E=\left[\frac{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k\binom{\beta-1}{k}\left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha-1+k\right)^{n+1}}\right] \qquad (26)$$

Proof. From equation (14), on using (23),

$$\hat{R}(t)_E = \left[E\left(\frac{1}{R(t)} \right) \right]^{-1}$$

$$= \left[\int_0^1 \frac{1}{R(t)} f\left(R(t \mid \underline{x}) \right) dR(t) \right]^{-1}$$

$$= \left[\int_0^1 \frac{1}{R(t)} \frac{\left[R(t)\right]^{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha-1} \left[\text{-}log\, R(t)\right]^n \left[1-R(t)\right]^{\beta-1}}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k} \left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}\right]} dR(t) \right]^{-1}$$

$$= \left[\frac{\int_0^1 \left[R(t)\right]^{\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha-2} \left[\text{-}log\, R(t)\right]^n \left[1-R(t)\right]^{\beta-1} dR(t)}{\Gamma(n+1)\left[\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k} \left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}\right]} \right]^{-1}$$

$$= \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k} \left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha-1+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k} \left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}} \right]^{-1}$$

or, $$\hat{R}(t)_E = \left[\frac{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k} \left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha+k\right)^{n+1}}{\sum_{k=0}^{\beta-1}(-1)^k \binom{\beta-1}{k} \left(1\Big/\left(\sum_{i=1}^{n}\left(\frac{e^{\lambda x_i}-1}{e^{\lambda t}-1}\right)^a\right)+\alpha-1+k\right)^{n+1}} \right] .$$

5. Conclusion

We have obtained a number of Bayes estimators of reliability function $R(t)$ of Weibull-exponential distribution. In equations (19), (20), and (21), we have obtained the Bayes estimators by using non-informative prior and in equations (24), (25), and (26), under beta prior. From the above said equation, it is clear that the Bayes estimators of $R(t)$ depend upon the parameters of the prior distribution. In this case the risk function and corresponding Bayes risks do not exist.

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