Teaching as a decision-making model: strategies in mathematics from a practical requirement

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Abstract

The need in the current social context to adopt teaching methods that can stimulate students and lead them towards autonomy, awareness and independence in studying could conflict with the needs of students with specific learning disorders, especially in higher education, where self-learning and self-orientation are required. In this sense, the choice of effective teaching strategies becomes a decision-making problem and must, therefore, be addressed as such. This article discusses some mathematical models for choosing effective methods in mathematics education for students with specific learning disorders. It moves from the case study of a student with specific reading and writing disorders enrolled in the mathematical analysis course 1 of the degree course in architecture and describes the personalised teaching strategy created for him.

Keywords: decision-making; inquiry model; social skills; personalised didactic strategy; Analytic Hierarchy Process.
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1 Introduction

In recent years the institutions have increased their interest in a very worrying phenomenon that concerns Italy and generally speaking the countries in the Western world: the growth of 'disaffection' towards mathematics due to a traditional didactic approach to the subject (Piochi, 2008). Young people coming out of secondary schools often have the idea that mathematics consists of mechanical processes, seeing it as an arid and pre-packaged discipline whose understanding and description seem impersonal. The mathematics one learns at school is very often a set of basic notions, axioms and definitions given by the teacher and practically impossible to discuss, causing the view of a subject that is “*already done*” and immutable (Castelnuovo, 1963). The experience of mathematicians, on the other hand, is very different: mathematics is something extremely changeable whose results are the result of hard work, debate and controversy. So, axioms and definitions first presented in textbooks come into reality only at the end, when the whole structure of the problem is understood. Then, the following question arises: what is mathematics? Definitions such as “*mathematics is the science of numbers and forms*” accepted 200 years ago is now reductive and ineffective because mathematics has developed so rapidly and intensely that no definition can take into account all the multiple aspects (Baccaglini-Frank, Di Martino, Natalini, Rosolini, 2018 (A)). The list of applications of this discipline in daily life could be endless, and so could the list of motivations that could be given to pupils to convince them to study.

About this matter it is really important the following statement: "*No doubt, mathematical knowledge is crucial to produce and maintain the most important aspects of our present life. This does not imply that the majority of people should know mathematics.*" (Vinner, 2000). Mathematics can also cause terror in students (the phenomenon of “*fear for mathematics*” (Bartilomo and Favilli, 2005)) or a state of dissatisfaction with the common conception that “*you have to be made for it*” so much that even great professionals boast that they have never understood anything about mathematics. So, is it necessary to teach mathematics to everyone? The answer is simple: apart from the fact that having a basis in mathematics is a cultural question regardless of the future job, mathematics teaches to evaluate multiple aspects of a question, and provides knowledge and skills in order to consciously face a discussion defending one's own positions with responsibility and respect for the arguments of others (National Indications, 2007).

The key role of mathematics education in the development of rational thinking and with it the responsibilities of mathematics teachers at all levels is therefore underlined. Already in 1958, the theme of the congress of the Belgian

Mathematical Society was entitled "*The human responsibility of the mathematics teacher*" (Castelnuovo, 1963).

The most effective way to bring students closer to mathematics is, therefore, the image of a "*method for dealing with problems, a language, a box of tools that allows us to strengthen our intuition*" (Baccaglini-Frank, Di Martino, Natalini, Rosolini, 2018 (A)).

2 Mathematical education: theories and models

2.1 The concept of error and the inquiry model: mathematics as a humanistic discipline

Mathematics is one of the disciplines in which many 'students' manifest difficulties that compromise the relationship with the subject. A student who comes out of secondary school has a long series of 'failures' accompanied by the conviction that she can never do mathematics because she is not good at it. The problem lies in identifying errors and difficulties in mathematical learning with the conviction that the absence of errors certifies the absence of difficulties and on the other hand the absence of difficulties guarantees the absence of errors (Zan, 2007 (A)).

This identification leads to the didactic objective of obtaining the greatest number of correct answers by nourishing the "*compromise of correct answers*" (Gardner, 2002): on the one hand the teacher chooses activities that are not "*too*" difficult and on the other hand the students elaborate the answers expected by the teacher in a reproductive way. Of course, this method does not guarantee any learning, revealing itself dangerous and counterproductive (Di Martino, 2017). Moreover, with it the fear of making mistakes arise and also the conviction that mathematics is not for everyone (for instance: you can't study mathematics if you do not have a good memory!) (Zan, Di Martino, 2004). In order to face the identification of difficulty-error, there is, therefore, a need to revolutionise the conception of error and to convey to students that *"making a mistake at school may not be perceived as something negative to avoid at all, because it could be an opportunity for new learning (and teaching) opportunities to be exploited*" (Borasi,1996).

The Inquiry model is a teaching-learning model that proposes a positive and fundamental role of errors in mathematics teaching. This model sees knowledge as a dynamic process of investigation where cognitive conflict and doubt represent the motivations to continuously search for a more and more refined understanding. Therefore, instead of eliminating ambiguities and contradictions to avoid confusion or errors, these elements must be highlighted to stimulate and give shape to ideas and discussions. Questions such as "*what would happen if this result were true?"* or "*under what circumstances could this error be corrected?*" lead to a reformulation of the problem where the error is only the starting point for a deeper understanding. Communication in the classroom plays a fundamental role, and so does the conception of mathematics as a humanistic discipline: the teacher provides the necessary support for the student's autonomous search for understanding, who in turn is an active member of a research community (Tematico, Pasucci, 2014). Learning turns out to be a process of constructing meanings, and in this way, the students also understand that what is written in textbooks is the result of debates and arguments and not simply something for its own sake ("*falling from the sky*").

2.2 Cooperative learning: development of disciplinary and social skills

Some studies highlight the need to build learning teaching models that take into account students' emotions, perceptions and culture based on the idea that human learning has a specific social character (Radford, 2006). The collaborative group and peer tutoring are two models that take on both the disciplinary dimension and the affective and social dimension and facilitate discussion in the classroom. In fact, in most cases, the teacher cannot give everyone the opportunity to express themselves, nor is he able to solicit the interventions of those who are not used to intervene.

Collaborative learning, instead, sees the involvement of all the students in two successive moments: first within the individual group and then in the final discussion in class. The necessary conditions for such learning are positive interdependence and the assignment of roles: the first is reached when the members of the group understand that there can be no individual success without collective success; the second condition allows the distribution of social and disciplinary competences among the various members of the group favouring collaboration and interdependence. The recognition of roles also helps to overcome problems such as low self-esteem or a sense of ineffectiveness, allowing social skills to grow: knowing how to make decisions, how to express one's own opinions and listen to those of others, how to mediate and share, how to encourage, help and resolve conflicts are skills that the school must teach with the same care with which disciplinary skills are taught.

Dialogue among peers guarantees greater freedom and spontaneity: the majority of students identify that among peers there is no fear of expressing doubts and perplexities, the main motivation that justifies the effectiveness of such models (Baldrighi, Pesci, Torresani, 2003; Pesci, 2011).

2.3 Recovery and enhancement interventions: breaking the educational contract

The variety of possible processes, the fact that behind correct answers there can be difficulties and that some mistakes can come out of significant thought processes, brings important elements to support the criticism of the identification between mistakes and difficulties. For example, the incorrect resolution of a problem is not necessarily due to the inability to manage the mathematical structure of the proposed situation but is probably due to a lack of understanding of the problem itself. The understanding of a text is not always immediate because it involves the student's personal knowledge of common words and scripts. Understanding is, therefore reduced to a selective reading that aims to identify the numerical data and the right operations suggested by keywords (Zan, 2012).

Recovery interventions must therefore be based on the analysis of the processes that led the student to make mistakes, shifting the attention from the observation of errors to the observation of failed behaviour with the sole objective of change. The student, in turn, must take responsibility for her own recovery and therefore there is a need for teaching that makes her feel that she is the protagonist of new situations and not simply the executor of procedures to be applied to repetitive exercises (Zan, 2007 (B)).

The teacher must propose exercises and problems that do not favour a mechanical approach but question the rules that pupils are used to use and that form part of the so-called *teaching contract* (D'Amore, 2007; D'Amore, Gagatsis,1997). The idea of a didactic contract was born to explain the causes of elective failure in mathematics, that is, the kind of failure reserved only for mathematics by students who instead do well in other subjects. The didactic contract holds the interactions between student and teacher and is made up of "*the set of teacher's behaviours expected by the student and the set of student's behaviours expected by the teacher*" (Brousseau, 1986).

This explains the students' belief that a problem or exercise always has a solution because it is the teacher's job to make sure that there is only one answer to the proposed question and that all the data is necessary (Baruk, 1985).

In Bagni (1997) the following goniometry test is proposed to fourth-year students in three classes of scientific high school (students aged 17 to 18).
Determine the values of x belonging to ℝ for which it results:

a) $\sin x = 1/2$	b) $\cos x = 1/2$
c) $\sin x = 1/3$	d) $\mathrm{tg}x = 2$
e) $\sin x = \pi/3$	f) $\cos x = \pi/2$
g) $\sin x = \sqrt{3}$	h) $\cos x = \sqrt{3}/3$

Table 2.1 Experiment in Bagni, 1997.

Remember that the goniometric functions are often introduced by making initial reference to the values they assume in correspondence to relatively common angles of use, so we have the well-known table shown in the next page (Table 2.2.)

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π	…
sinx	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	…
cosx	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1	…
tgx	0	$\sqrt{3}/3$	1	$\sqrt{3}$	*n.d.*	$-\sqrt{3}$	-1	$-\sqrt{3}/3$	0	…
…	…	…	…	…	…	…	…	…	…	…

Table 2.2 Values assumed by common angles where "*n.d.*" is "*not define*".

Let us now examine the test: agreed time 30 minutes and pupils were not allowed to use protractor tables nor scientific calculator. It has been conceived with:

- two "*traditional*" questions (a), (b);
- two possible questions, but with the results not included between the values of x "*of common use*" (c), (d);
- two impossible questions (e), (f) but with values (of *sin*x and *cos*x) that recall the measurements in radians of "common use" angles (π/3, π /2);
- two questions (g), (h) where the first impossible and the second possible. They propose instead values (of sin*x*, cos*x*) that are included in the table referred to the angles "*of common use*" but in relation to other goniometric functions (tg*x*, cotg*x*).

Well, as far as the answers to the questions (e), (f) are concerned, the didactic contract has led some pupils to look for solutions anyway; and the "*solutions*" that most spontaneously presented themselves to their mind are the ones that they see associated, in the case of the sinus function, the two values π/3 and $\sqrt{3}/2$ and, in the case of the cosine function, the two values π /2 and 0. So we have, for instance, the following errors:

$$\sin x = \pi/3 \quad \text{so} \quad x = \sqrt{3}/2$$
$$\cos x = \pi/2 \quad \text{so} \quad x = 0$$

As far as the answers to questions (g), (h) are concerned, the reference to the tangent function was clearly expressed in the answers of some students: also in this case, some students, not finding the proposed values among those corresponding to the most frequently used x values (for the sine and cosine functions, in the table above), were induced to look for another correspondence in which the proposed values are involved. We then find errors such as:

$$\text{if } \sin x = \sqrt{3}\text{, then } x = \pi/3$$
$$\text{if } \sin x = \sqrt{3}\text{, then } x = \pi/3 + k\pi$$

What has now been pointed out obliges us to conclude that the need that leads the student to always and in any case look for a result for each proposed exercise is unstoppable: *breaking the teaching contract can be used as a* *teaching strategy* to overcome the mechanical approach used by the students and enhance knowledge (Bagni, 1997).

3 Concept image ad concept definition

These notions were developed to analyse the learning processes of mathematical definitions (Tall and Vinner, 1981). Concept image is the whole cognitive structure related to the concept and includes all mental images, the properties and processes of recall and manipulation associated with a concept, bringing into play its meaning and use. It is built through years of experience of all kinds, changing with the encounter of new stimuli and the growth of the individual. The concept definition is the set of words used to specify a concept and turns out to be personal and can often differ from the formal definition because it represents the reconstruction made by the student and the form of the words he uses to explain his concept image. It can change from time to time and for each individual the concept definition can generate its own concept image which can be called in this case concept definition image. The acquisition of a concept occurs when a good relationship is developed between the concept name, the concept image and the concept definition. Students tend to learn definitions in a mechanical way and this can lead to conflict factors when concept image or concept definition are invoked at the same time which conflict with another part of the concept image or concept definition acquired on the same concept. To explore this topic a questionnaire was administered to 41 students with an A or B grade in mathematics. They were asked: "*Which of the following functions are continuous? If possible, give your reason for your answer*."

$$f_1(x) = x^2$$

$$f_2(x) = 1/x \quad (x \neq 0)$$

$$f_3(x) = \begin{cases} 0 & (x \leq 0) \\ x & (x \geq 0) \end{cases}$$

$$f_4(x) = \begin{cases} 0 & (x \leq 0) \\ 1 & (x \geq 0) \end{cases}$$

$$f_5(x) = \begin{cases} 0 & (x \text{ rational}) \\ 1 & (x \text{ irrational}) \end{cases}$$

(No picture was drawn in the last case).

Figure 3.1 Images from Tall and Vinner, 1981.

We see that the concept image of this topic comes from a variety of resources such as the colloquial use of the term "*continuous*" in phrases such as "*It rained all day long*". So, often the use of the term "*continuous function*" implies the idea that the graph of the function can be drawn continuously. The answers are summarised in the table shown in Figure 3.2.

	1	2	3	4	5
continuous	41	6	27	1	8
discontinuous	0	35	12	28	26
no answer	0	0	2	2	7

Figure 3.2 Tables from Tall and Vinner, 1981. It Summarises the results of the experiment.

The reasons given to justify the discontinuity of $f_2(x)$ and $f_4(x)$ are of the type: "*The graph is not in a single piece*", "*There is no single formu*la". In these answers, we see that many students invoked a concept image including a graph without any interruption or a function defined by a "*single formula*". Instead, there are many continuous functions that conflict with the concept images just mentioned as the following:

$$f(x) = \begin{cases} 0\ (x < 0\ or\ x^2 < 2) \\ 1(x > 0\ or\ x^2 > 2) \end{cases}$$

whose graph is:

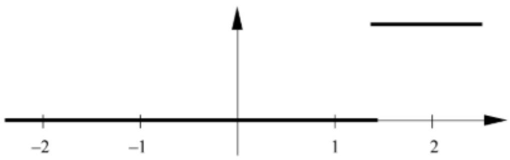

Figure 3.3 Image from Tall and Vinner, 1981. It represents the function defined above.

The idea that emerges from similar issues is that *mathematical concepts should be learned in the everyday, not technical, way of thinking*, starting with many examples and non-examples through which the concept image is formed and then arriving at a formal definition. Students should use the formal definition, but in order to internalise the concept it is necessary to aim at cognitive conflicts between concept image and concept definition. To do this it is necessary to give tasks that do not refer only to the concept image for a correct resolution, inducing the students to use the definition (Baccaglini-Frank, Di Martino, Natalini and Rosolini , 2018 (B)).

4 Teaching as a decision problem

Today more than ever, the world of education has to work on the construction of personalities that can favour to all the students with freedom of choice and reactivity. The social context in which we live is complex because it comprehends factors of unpredictability and uncertainty: the educational systems have the job to provide a path that aims to thought and action autonomy.

The school, therefore, has an orientation character, where the term "*orientation*" indicates a continuous and personal process that involves awareness, learning and education in choice (Biagioli, 2003). In particular, placing orientation as the main purpose of teaching "*means developing strategies, methodologies and contents aimed from the acquisition of awareness to understanding the complex society and the mechanisms that govern the world of studies and work*" (Guerrini, 2017). To give the proper and necessary instruments to the student, in order to activate the auto-orientation processes, the teacher has to chose what the best didactic strategy is. Therefore, on an operational point of view, the teaching is a decisional problem and has to be faced as it is. Indeed, we can speak of decision when in a situation there are: *alternatives* (being able to act in several different ways), *probability* (the possibility that the results relating to each alternative will be achieved) and the *consequences* associated with the results. Such factors are characteristic of the school world. So, to realise the best didactic strategy it is necessary to start with a representation of the problem: only through the calculation of the expectations and the evaluation of the results, it is possible to choose the right option.

Decisions can be studied in terms of absolute rationality or limited rationality. The first model ideally combines rationality and information by preferring the best alternative; the second recognises the objective narrowness of the human mind by proposing the selection of the most satisfactory alternative (Lanciano, 2019-20).

It is important to emphasise that the consequences of a decision are determined also by the context in which the decision-making process is developed. On the basis of the decision maker's knowledge of the state of nature. we distinguish various types of decisions:

- decisions in a situation of certainty: when the decision-maker knows the state of nature;
- decisions in risk situations: when the decision-maker does not directly know each state of nature, but has a probability measure for them;
- decisions in situations of uncertainty: when the decision maker has neither information on the state of nature nor the probability associated with it.

The decision maker can adopt two kinds of approaches:

- Normative approach. which bases the choice with reference to rational decision-making ideals;
- Descriptive approach which analyses how to make a decision based on the context.

So, the teacher has to consider on the basis of the objectives and the context the various alternatives, and for each one of them, the possible consequences. For each pair (alternative, circumstance) the teacher obtains a result according to a utility function.

However, the decision is subjective: it is based on the criterion of obtaining a maximum value for the utility function. Moreover, even if the choice is rational, it is made in terms of limited rationality because, in general, there are few alternatives, but it increases as the teacher expands his/her culture and experience (Delli Rocili, Maturo, 2013; Maturo, Zappacosta, 2017).

4.1 A model for evaluating educational alternatives

Multi-criteria decision analysis (MCDA) provides support to the decision maker, or a group of decision makers, when many conflicting assessments have to be considered, especially in data synthesis phase while working with complex and heterogeneous pieces of information.

Let $A = \{A_1, A_2, ..., A_m\}$ be the set of the alternatives, i. e. the possible educational strategies. Let $O = \{O_1, O_2, ..., O_n\}$ be the set of the objectives that we want to achieve. Let $D = \{D_1, D_2, ..., D_k\}$ be the set of the decision making processes. The first phase consists of the establishment of a procedure that is able to assign to each couple (alternative A_i, objective O_j) a p_{ij} score. In this way, the responsible for the decision measure the grade in which the alternative A_i satisfies the objective O_j. Assume that p_{ij} is in [0, 1], where:

- $p_{ij} = 0$ if the objective O_j is not at all satisfied by A_i;
- $p_{ij} = 1$ if the objective O_j is completely satisfied by A_i.

At the end of the procedure we obtain a matrix $P = [p_{ij}]$ of the scores which is the starting point of the elaborations that lead to the choice of the alternative, or at least to their ordering, possibly even partial (Maturo, Ventre, 2009a, 2009b). There may be constraints: it could be necessary to establish for each objective O_j a threshold $j > 0$, with the constraint $p_{ij} \geq j$, for each i.
Furthermore, through a convex linear combinations of alternatives A_i it is possible to take into consideration mixed strategies that will have the following form:

$$A(h_1, h_2, ..., h_m) = h_1 A_1 + h_2 A_2 + ... + h_m A_m$$

with:

- $h_1, h_2, ..., h_m$ non-negative real numbers;
- the h_i's are such that $h_1 + h_2 + ... + h_m = 1$;

The number h_i can represent the fraction of time in which the teaching strategy A_i is adopted. If we consider also the mixed strategies, then the single alternatives A_i are called pure strategies. The mixed strategies are particularly considered in presence of "*at risk*" alternatives: these situations have high scores for certain objectives and low for others (possibly below the threshold).

It is appropriate to construct a ranking of the alternative educational plans, i. e., a linear ordering of the alternatives that takes into account the objectives which contribute to the most suitable formation of the student. Such a ranking can be usefully obtained by means of the application of the Analytic hierarchy process, a procedure due to T. L. Saaty (1980, 2008).

4.2 The Analytic Hierarchy Process: attributions of weights and scores

The Analytic Hierarchy Process (AHP) is both a method and a technique that allows to compare alternatives of different qualitative and quantitative nature, not easily comparable in a direct way, through the assignment of numerical values that specify their priority. The first thing to do is represent the elements of the decision problem through the construction a hierarchical structure. Indeed, the Analytic Hierarchy Process is based on the representation of the problem in terms of a directed graph G = (V, A). Let us recall that (Knuth, 1973):

- a *directed graph*, or *digraph*, is a pair G = (V, A), where V is a non-empty set whose elements are called *vertices* and A is a set of ordered pairs of vertices, called *arcs*;
- a vertex is indicated with a Latin letter; for every arc (u, v) u is called the *initial vertex* and v the *final vertex* or *end vertex*;
- an ordered n-tuple of vertices (v_1, v_2, ..., v_n), n > 1, is called a *path* with length n -1, if, and only if, every pair (v_i, v_{i+1}), i = 1, 2, …, n-1, is an arc of G.

Furthermore, in our context, we assume the following conditions be satisfied from a directed graph:

- the vertices are distributed in a fixed integer number $n \geq 2$ of levels; each level is indexed from 1 to n;
- there is only one vertex of level 1, called the *root* of the directed graph;
- for every vertex v different from the root there is at least one path having the root as the initial vertex and v as the final vertex;
- every vertex u of level i < n is the initial vertex of at least one arc and there are no arcs with the initial vertex of level n;
- if an arc has the initial vertex of level i<n, then it has the end vertex in the level i+1.

Let us describe, considering for example n=3, functional aspects of each level:

- the level 1 vertex is called the *general objective* and denoted GO. It indicates the objective of the entire decision making process;
- the vertices of level 2 are called *criteria*. With this level we indicate the parameters used to evaluate the alternatives;
- vertices of level 3 are called *alternatives* that represent the various ways of reaching the GO.

So we have the structure shown in the next page (Figure 4.1).

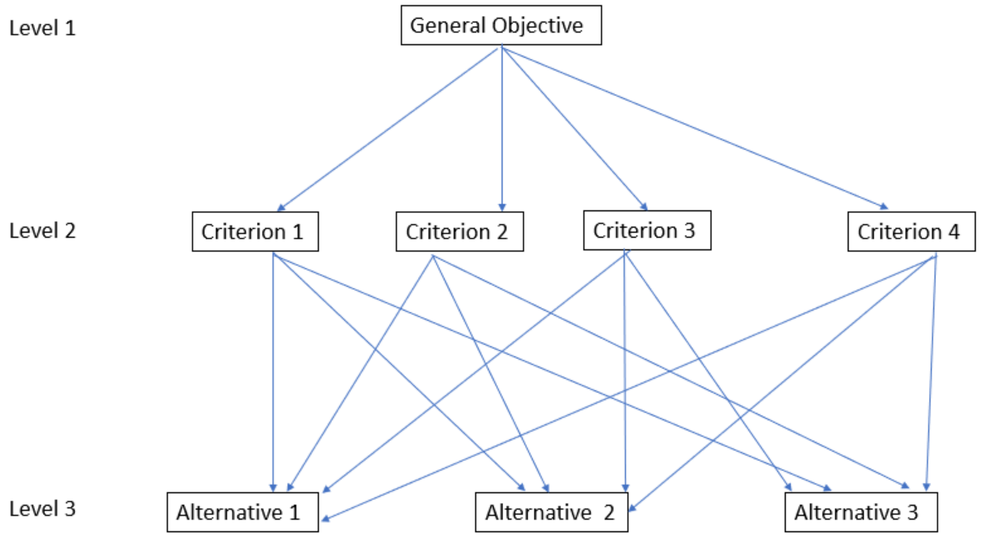

Figure 4.1 The Digraph of the Functional Aspects.

For instance, to build up a decisional model for the mathematical didactic the elements of the hierarchy might be the following:

- General objective: to guarantee a route that, through education, ensures to all students the formation of knowledge and the development of action and communication skills;
- Criterion 1: acquisition of the objectives set out in the educational plan;
- Criterion 2: ability to measure oneself against peers in a safe and rational way while respecting the ideas of others;
- Criterion 3: internalisation of the objectives set by the didactic plan. With the term "*internalisation*" we indicate the ability to re-elaborate knowledge from a critical and personal point of view;
- Criterion 4: ability to cope with trials, planned or not, without negative moods, anxiety and terror of judgement;
- Alternative 1: new didactics, inspired by the inquiry model that puts the pupil and her emotions at the center of the context with cooperative learning experiences;
- Alternative 2: traditional didactic, that is a model of theaching-learning that prefers frontal lessons. In terms of learning this alternative hypothesises that the acquisition and internalisation take place at the same time and that the error is the manifestation of the failure to complete one of the two processes.
- Alternative 3: distance learning, inspired by the inquiry model mediated through the use of technological tools characterised by a total absence of sharing the same physical space between student and teacher.

A decision-maker assigns a score to each arc following the AHP procedure (Saaty, 1980, 2008; see also: Maturo, Ventre, 2009a, 2009b). So, the second step consists in determining the ratios of preference of the elements of a level over any of the higher, i.e. previous, level: we, therefore, compare alternatives A1, A2 and A3 with respect to each criterion and the individual criteria with respect to the general objective. In order to determine the value of each comparison, the following scale of evaluations is used:

Value a_{ij}	**Interpretation**
1	*i* and *j* are equally important
3	*i* is a little more important than *j*
5	*i* is quite more important than *j*
7	*i* is definitely more important than *j*
9	*i* is absolutely more important than *j*
1/3	*i* is a little less important than *j*
1/5	*i* is quite less important than *j*
1/7	*i* is definitely less important than *j*
1/9	*i* is absolutely less important than *j*

Table 4.1 Scale of evaluations.

Matrices are called *pairwise comparison matrices* and they represent quantitative preferences between criteria or between alternatives and satisfy the followings:

- if alternative *i* assumes the value x in comparison with alternative *j* with respect to a criterion, then the comparison of alternative *j* with alternative *i* with respect to the same criterion assumes the value 1/x. Analogous is the procedure to assign values when comparing couples of criteria;
- since equally important alternatives correspond to value 1, the diagonal of the matrices are composed entirely of unit values.

In our case, we obtain the matrices shown in the next page (from Table 4.2 to Table 4.6) whose values have been assigned due to the following considerations:

- acquisition and internalisation are two different processes and only through a mutual combination of them the full formation of the student can be guaranteed;
- traditional didactic is far from the social character of human learning;
- the relationship with others is a fundamental space for personal and social development, necessary for the student to learn to respect rules and roles;
- distance learning offers insufficient physical interaction between student-teacher and student-student: expressions and gestures make the difference in the learning process;
- exercising young people to face tests in a lucid way is a fundamental aspect for the construction of a personality that faces in a competitive way the working challenges of a competitive society.

O	**C1**	**C2**	**C3**	**C4**
C1	*1*	*5*	*1*	*1*
C2	*1/5*	*1*	*1/5*	*1/7*
C3	*1*	*5*	*1*	*1*
C4	*1*	*7*	*1*	*1*

Table 4.2 Matrix M1: comparison of criteria by the ratio of preference with respect to the general objective

C1	**A1**	**A2**	**A3**
A1	*1*	*5*	*1*
A2	*1/5*	*1*	*1/3*
A3	*1*	*3*	*1*

Table 4.3 Matrix M2: comparison of alternatives by the ratio of preference with respect to the criterion 1

C2	**A1**	**A2**	**A3**
A1	*1*	*5*	*7*
A2	*1/5*	*1*	*3*
A3	*1/7*	*1/3*	*1*

Table 4.4 Matrix M3: comparison of alternatives by the ratio of preference with respect to the criterion 2

C3	**A1**	**A2**	**A3**
A1	*1*	*7*	*5*
A2	*1/5*	*1*	*3*
A3	*1/7*	*1/3*	*1*

Table 4.5 Matrix M4: comparison of alternatives by the ratio of preference with respect to the criterion 3

C4	**A1**	**A2**	**A3**
A1	*1*	*3*	*7*
A2	*1/3*	*1*	*3*
A3	*1/7*	*1/3*	*1*

Table 4.6 Matrix M5: comparison of alternatives by the ratio of preference with respect to the criterion 4

After constructing the pairwise comparison matrices, we have to fix the weights of the elements in each level. This step is fundamental to determine whether the matrices are relevant through a scale of values ranging from 0 to 1. These weights must meet the normality condition:

$$w_1+w_2+\ldots+w_n=1$$

This procedure supposes that, if the decision maker knew all the actual weights of the elements of the pairwise comparison matrix, then it would be:

$$A=\left({}^{w_i}/_{w_j}\right)=\begin{pmatrix} {}^{w_1}/_{w_1} & \cdots & {}^{w_1}/_{w_n} \\ \vdots & \ddots & \vdots \\ {}^{w_n}/_{w_1} & \cdots & {}^{w_n}/_{w_n} \end{pmatrix}$$

In this case the weights would be obtained from any of the rows which are all multiple of the same row $\left({}^{1}/_{w_1}, {}^{1}/_{w_2}, \ldots, {}^{1}/_{w_n}\right)$. It follows that the matrix A has rank 1. Being $w=(w_1, w_2, \ldots, w_n)^T$ we get:

$$Aw = nw.$$

Thus, from the equation above, n is an eigenvalue of A and w is one of the eigenvectors associated with n. Since the elements on the diagonal are all 1, denoted with $\lambda_1, \lambda_2, \ldots, \lambda_n = n$ the eigenvalues of A, the value of the trace of A is:

$$tr(A)=\lambda_1+\lambda_2+\ldots+\lambda_n=n$$

As *n* is an eigenvalue of A the other *n-1* eigenvalues of A must be zero. The matrix A satisfies the condition:

$$\mathbf{a}_{ij}\mathbf{a}_{jk}=\mathbf{a}_{ik}$$

for every i, j, k, called *consistency condition*, that implies transitivity of the preferences, and A is said to be *consistent* is said to be A.

In practice the decision maker does not know the vector *w*: the a_{ij} values that he assigns according to his judgement may deviate from the unknown w_i/w_j. So the decision maker may produce inconsistent pairwise comparison matrices. However the closer the a_{ij} values are to w_i/w_j, the closer the maximum eigenvalue is to n and the closer the other eigenvalues are to zero.

Therefore the vector of the weights $w'^T=(w'_1, w'_2, \ldots, w'_n)$ associated to the maximum eigenvalue (among the infinite w'^T we choose the one for which $w'_1+w'_2+\ldots+w'_n=1$) will be an estimate of the vector $w^T=(w_1, w_2, \ldots, w_n)$ the more precise the more the maximum eigenvalue λ_{max} of A is close to n, what is due to the continuity of the involved operations (Ventre, 2019). Where:

$$CI = \frac{\lambda_{max} - n}{n - 1}$$

is defined as the *consistency index* of A, and n is the order of the matrix itself. The consistency index reveals how far from consistency the matrix is. In our case with the help of MATLAB we obtain:

	M1	M2	M3	M4	M5
λ_{max}	4,0142	3,0291	3,0649	3,0649	3,0070
CI	0,0047	0,0145	0,0324	0,0342	0,0035

Table 4.7 CI Values for each matrix.

We can therefore write the local priorities obtained by proceeding with the last step of the AHP method which, through the aggregation of the relative weights of each level, provides a weighted ranking of the alternatives.

The third step implements the estimation of local assessments, i.e. the weightings that express the relative importance of the elements of a hierarchical level over any element of the next higher level.

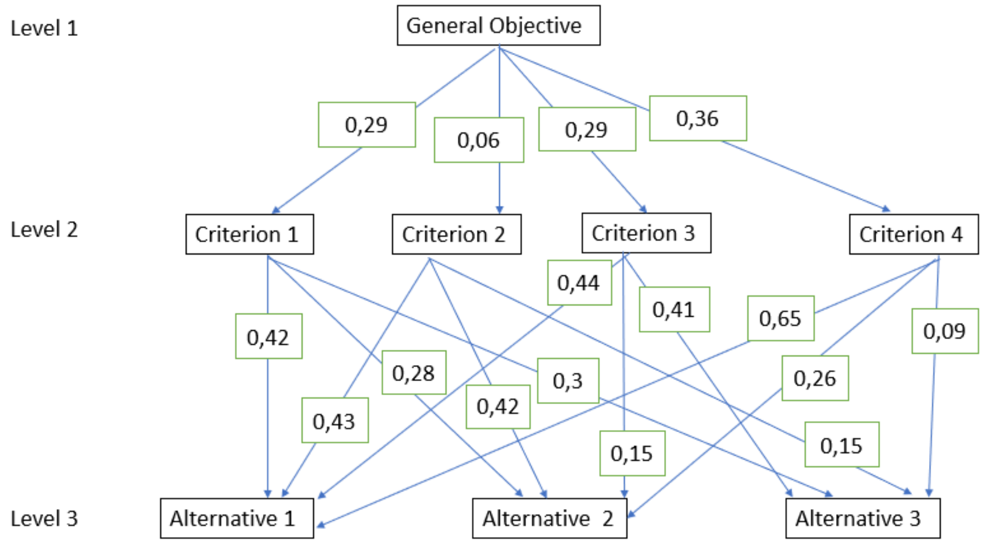

Figure 4.2 The Digraph of the Functional Aspects, with weights.

We observe that scores are nonnegative real numbers and such that the sum of the scores of the arcs coming out of the same vertex u is equal to 1. The score assigned to an arc (u, v) indicates the extent to which the final vertex v meets the initial vertex u: the *score of a path* is the product of the scores of the arcs that form the path.

- For every vertex v different from GO the score p(v) of v is the sum of the scores of all the paths that start from GO and arrive in v.

- For every level, the sum of the points of the vertices of level i is equal to 1.

The global properties determined will then be:

- A1=0,42*0,29+0,43*0,06+0,44*0,29+0,65*0,36=0,51;
- A2=0,28*0,29+0,42*0,06+0,15*0,29+0,26*0,36=0,24;
- A3=0,3*0,29+0,15*0,06+0,41*0,29+0,09*0,36=0,25.

We can conclude that, in order to achieve the objective, a new teaching strategy is preferred to a traditional one and direct communication and human contact are factors not to be overlooked. Therefore, teaching cannot be reduced to an online practice because the presence of the student is not attributable to a "*virtual presence*".

5 Mathematics and specific learning disorders

5.1 Background

Students with specific learning disorders (SLD) need a personalised learning plan that appropriately accommodates their difficulties. At the university a student with SLD attending the course of mathematical analysis has succeeded, thanks to the semester tutoring activity dedicated to the subject, to face his difficulties by passing the exam at the first useful date. The course has evolved through a personalised didactic strategy mostly based on the use of mind maps and peer comparison activities. Specific learning disorders SLD usually involve reading, writing and calculation skills. In Italy, two students with dyslexia out of three do not receive an adequate diagnosis of the disorder and therefore SLD are one of the main causes of school difficulties with important negative repercussions also in the personal sphere of the individual.

It is therefore clear the importance of a conscious environment able to respect the “*different*” way of learning of a student with SLD whose cognitive abilities and physical characteristics are in the norm. In fact, although they have an intelligence appropriate to their age, unlike their peers, these subjects learn at a slower pace because during the study they dissipate most of their energy to compensate for their disorders. Initiatives promoted by the MIUR, accompanied by individual schooling, support the right to study of students with SLD, which since the beginning of the year two thousand is also protected at the legislative level (MIUR law no.170/2010). To this end, the Regional School Offices are committed to promote the issue of detailed certifications that allow as much as possible, together with parents and the figures who follow the student in school activities, a Personalised Educational Plan that aims to achieve the same objectives of peers through compensatory tools and dispensative measures (MIUR, 2011).

5.2 SLDs in mathematics

Specific reading disorder

On the whole, this disorder leads to difficulties in reading, which is slow and incorrect, making it difficult to understand the text and to distinguish useful parts from those containing additional information (ICD F81.0, 2007). In mathematics, these deficits make even the simplest problems complex as the student may not be able to distinguish between hypotheses and data. In addition, to maximise learning, rather than studying theory from the book, we suggest the constant use of mind maps that allow you to correct and rework ideas slowly until the subject is mastered. This type of maps is the one that best suits the way of learning of students with specific reading disorders because, through images and colors, it aims to stimulate visual memory through the insertion of mental associations.

Specific writing disorder

The specific writing disorder is called dysgraphia if it affects writing and dysorthography if it affects spelling. The former involves a deficit of a motor nature and refers to the graphic aspects of handwriting, the latter is a text encoding disorder that therefore involves the linguistic component. Disgraphers therefore produce poorly readable texts (even by themselves) with words that are often misaligned and characterised by letters of different sizes, while disorthographers manifest errors such as inversion of syllables, arbitrary cuts of words and omissions of letters in words making the content unclear. In both cases the elaboration of a written text is a difficult and long process with serious repercussions also in the mathematical field. In fact, mathematics has its own language, characterised by symbols, signs and letters of the greek and latin alphabets: just think of the use of lowercase greek letters in the geometric field to indicate angles and uppercase latin letters to indicate vertices. In the set of symbols, the symbol of belonging "$\in$" may not be decoded, leading to confusion with "*E*" even though it does not denote, from a didactic point of view, a lack of understanding of the set meaning itself.

Inaccurate writing also causes errors in the resolution of algebraic expressions, for example by confusing the letter "s" and the number "5", or in the resolution of a linear system which requires many transcription steps. Another difficulty due to alignment is found in the case of powers where base and exponent are confused with a multi-digit number (34 instead of 3^4). Along this line, the teacher should take into account the content rather than the form, in the process of evaluating the written texts. Errors should not be penalised when the concept expressed is clear, whereas oral verifications should acquire more weight in the final evaluations. In order to deal with these problems, the teacher, in the process of evaluating the written tests, could take into account the content rather than the form, not penalising errors when the concept expressed is clear and giving more weight to oral verifications.

Specific disturbance of arithmetic activities

The specific disturbance of arithmetic activities disorder implies a deficit that concerns the mastery of fundamental calculation skills which generally can be divided into two different profiles according to the type of error. The first profile is defined as "*Weakness in the cognitive structuring of numerical cognition components*" and summarises both the difficulty in comparing/quantifying the elements of one or more sets and the reduced counting capacity. This type of disorder is inspired by Butterworth's studies (1999; 2005): he hypothesised the existence of a "*mathematical brain*" specialised in classifying the world in terms of numbers. The second profile is renamed with "*Difficulty in acquiring calculation procedures and algorithms*" and includes the following three cases (Temple, 1991): dyslexia for digits indicating an incorrect reading/writing of the numbers (the student sees the number 3 and pronounces 6); procedural dyscalculia indicating the difficulty in the choice, application and maintenance of procedures leading to errors in borrowing, carry-over or sticking; dyscalculia for arithmetic facts which leads to confusion between the rules of rapid access with the consequent compromise of the acquisition of numerical facts within the calculation system. For both profiles examined the tools to compensate could be tables, diagrams, calculators and an extensive collection of procedural examples.

Objective

Learning disorders are, therefore varied and create different deficits that require different compensatory instruments and dispensation measures. For this reason, it is important to experiment the different strategies for teaching in order to identify a scheme which, although it can never be universal, can be taken as a canvas and then refined according to the personal limits and objectives of the student. The aim is, therefore to encourage learning with the aim of making the student as autonomous as possible, also increasing the level of self-esteem and personal gratification. Below is the strategy used for a student with specific reading and writing disorders during the tutoring activity of the mathematical analysis course 1.

The case study

The student, after presenting his certificate at the beginning of the course, immediately showed interest in possible remedial activities. Although the student was initially autonomous, after almost a month, the first difficulties began to emerge. This situation prompted the student to make constant use of the tutoring hours in which a personalised teaching strategy was constructed. The construction of such a strategy was obtained through the procedure previously shown: alternatives, objectives and criteria were modified considering the specific difficulties. The method proved to be successful: indeed, the student was able to acquire his study methodology, which proved to be effective.

Stuff and methods

From a didactical point of view, it is particularly interesting that problems related to short-term memory are a common feature of the different SLDs. Baddeley and Hitch (1974) extended this concept, in the field of cognitive psychology, to the "*working memory*" which specifically indicates that set of notions necessary for written and oral productions that remain short in the student's mind. If this capacity is reduced, then temporary archiving and the first management/manipulation of data will be compromised with the following consequences: difficulty in taking notes, difficulty in maintaining attention, need for longer periods. To cope with this situation, it is necessary to reduce the information load by giving priority to fundamental concepts and using support tools that favour direct observation and experimentation.
Some indications for the didactic strategy are:

- Use of schemes and concept maps;
- Dispense with reading aloud and mnemonic study;
- Privileging learning from experience and laboratory teaching;
- Encourage students to self-assess their learning processes;
- Encourage peer tutoring and promote collaborative learning;
- Guarantee longer times for written tests and study;
- Take an encouraging attitude to improve self-esteem;
- Evaluate according to progress and difficulties;
- Use of calculator and digital devices.

In our case, the implementation of the didactic strategy took place through afternoon meetings, lasting two hours, usually held after the lesson held by the teacher in the same morning. At first, the meeting was based on the study of the last topics explained by the teacher and already at this stage it was evident to what extent the characteristic features of dysgraphia hindered learning: the bare and confused notes, characterised by large empty spaces, were practically impossible to read and to study. In order to try to provide constructivist learning and to avoid the use of the book, since reading was slow and incorrect due to dyslexia, the theory was flanked by practice or questions to answer. Once the new subject was finished, work was done on the previous ones.

In order to prevent the pupil from distracting himself, while following me with interest, I often drew his attention by naming him and repeating the concept in different ways. Difficulties began to come up from the first topics when, in the numerical set exercises, there was a lot of confusion between the parentheses used to describe the intervals. For example:

- interval from 'a' to 'b' not including the extremes,]a, b[;

- interval from 'a' to 'b' including the extremes, [a, b];

To cope with this difficulty, with a bit of imagination, I made him imagine drawing two arms 'embracing' the number if the number had to be present in the interval or, otherwise, they refused to do so. In this way, together with the use of graphic representations, the pupil acquired the competence to distinguish between the two types of parentheses and rarely found confusion until the end of the course. The mind maps, which we built together, were of great help especially in solving equations and inequalities with absolute values. Initially, the student's difficulty consisted in not being able to "*visualize*" the writing of the systems that came out of the procedure. With the use of a map, similar to the one below, he was able, after several lessons, to carry out the simplest exercises correctly but still presenting difficulties for the more complex ones. The result was satisfactory, however, because I believe that these difficulties were linked not so much to a lack of understanding of the absolute value function, but rather to a lack of ability to concentrate for so long on the same procedure. Example of a map for the absolute value function:

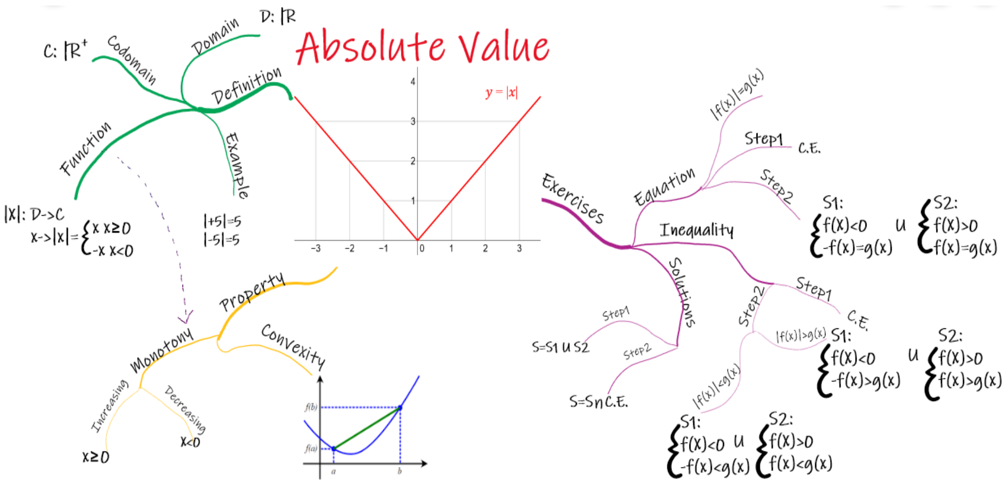

Figure 5.1 Example of a map built during the activity.

Towards the end of the course, as the exam date was approaching, other students also started to attend tutoring lessons on a regular basis. For this motive, the last topics of the course, derived and function study, were addressed through a collective study during which the student with specific learning disorders discussed with his classmates to the point of realising, under my guidance, his mistakes. During the lessons we made extensive use of the calculator, reducing the material to be memorised as much as possible. For example, to study the domain of a function, initially the student made extensive use of the tables summarising the domains of elementary functions but, subsequently, using the calculator (we get "*ERR*" if the function is not applied to an elementary of the domain) he learned what this procedure really represented by obtaining, on his own, the conditions for simple functions such as roots and logarithms.

Results

The boy achieved the minimum objectives, and at the end of the course, he passed the exam. The fundamental concepts were acquired and therefore, the test was approached independently. The improvements were gradual: as the tutoring lessons increased, the student was able to follow more and more. The shared study experience with peers was certainly helpful as, through peer comparison, he gained such confidence that he could guide his peers in difficulty during the exercises. In this regard, it is important to stress that the objectives achieved are not only didactic in nature but also personal: the ability to compare oneself with peers while respecting the ideas of others has increased the level of self-esteem and gratification.

6 Conclusion

The purpose of this document is, therefore, to underline that by using Saaty's hierarchical structure it is possible to choose the didactic strategies that best suit the various situations. In fact, all students and in particular students with SLD can maximise their learning if followed correctly. In our case, the didactic plan allowed for peer learning. This strategy, having been applied to only one student, cannot be generalised because each case requires different objectives and tools depending on the disturbance and the problems that this entails. To conclude, we stress that the affective dimension has played a fundamental role: a serene and stimulating environment has been a necessary condition to motivate students to improve themselves. However, the suggested indications are a good start to deal with different cases.

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