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KCD indices and coindices of graphs

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Abstract

The relationship between vertices of a graph is always an interesting fact, but the relations of vertices and edges also seeks attention. Motivation of the relationship between vertices and edges of a graph has helped to arise with a set of new degree based topological indices and coindices named KCD indices and coindices. These indices and coindices are elaborated by establishing a set of properties. More fascinating results of some graph operations using KCD indices are developed in this article.

Keywords: KCD indices, KCD coindices, graph operations.
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1 Introduction

Graph theory plays a vital role in the quantification of chemical structures through topological indices. Topological indices are molecular descriptors which characterize the topology of a graph through numerical parameters. Abundant number of topological indices are identified these days. Amongst these the first degree based topological indices are Zagreb indices [Gutman and Trinajstić, 1972]. Recently along with Zagreb indices Zagreb coindices is also gaining much attention for research. This has put forward versitile forms of Zagreb indices of graphs. The present work aims to establish some new form of topological indices of graphs.

This paper considers the graph to be simple, finite and undirected. The graph is denoted as $G = (V, E)$ with $|V(G)| = n$ as the vertex set and $|E(G)| = m$ as the edge set. The set of vertices are also referred to as the order of the graph G and the edge set as the size of the graph G. The edge connecting the two vertices u and v is denoted as $e = uv$. The degree of the vertex u in a graph G is denoted as $d_G(u)$ and defined as the number of edges of a graph G incident with the vertex u. The degree of edge $d_G(e)$ of a graph G is defined as $d_G(e) = d_G(u) + d_G(v) - 2$. The complement $\overline{G}$ of a graph G is one in which two vertices are adjacent if and only if they are not adjacent in G. For $\overline{G}$, $|V(\overline{G})| = n$, $|E(\overline{G})| = \overline{m} = \binom{n}{2} - m$ [Alwardi et al., 2018]. Also $uv \in E(\overline{G}) \iff uv \notin E(G)$. The degree of a vertex u in $\overline{G}$ is denoted as $d_{\overline{G}}(u)$ and defined as $d_{\overline{G}}(u) = n - 1 - d_G(u)$ [Alwardi et al., 2018]. The degree of edge of $\overline{G}$ is represented as $d_{\overline{G}}(e)$, defined as $d_{\overline{G}}(e) = d_{\overline{G}}(u) + d_{\overline{G}}(v) - 2$. For undefined terminologies refer [Harary, 1969].
The Zagreb indices were defined by Gutman and Trinajstić [Gutman and Trinajstić, 1972] as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 \tag{1}$$

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v) \, . \tag{2}$$

Here $M_1(G)$ refers first Zagreb index and $M_2(G)$ refers second Zagreb index.
First Zagreb index is also expressed as [Došlić, 2008, Došlic et al., 2011]

$$M_1(G) = \sum_{uv \in E(G)} \Big(d_G(u) + d_G(v) \Big). \tag{3}$$

For properties and information on Zagreb indices refer [Gutman and Das, 2004, Zhou and Gutman, 2005, Zhou, 2004].

Further, Zagreb coindices were introduced by Došlić [Došlić, 2008] as

$$\overline{M_1}(G) = \sum_{uv\notin E(G)} \Big(d_G(u)+d_G(v)\Big) \tag{4}$$

$$\overline{M_2}(G) = \sum_{uv\notin E(G)} d_G(u)d_G(v). \tag{5}$$

The detailed study on Zagreb coindices is reported in [Ashrafi et al., 2010, 2011], the association between Zagreb indices and coindices is encountered in [Das et al., 2012, Gutman et al., 2015].
Shirdel et al. [Shirdel et al., 2013] defined hyper Zagreb index as

$$HM(G) = \sum_{uv\in E(G)} \Big(d_G(u)+d_G(v)\Big)^2. \tag{6}$$

Further, hyper Zagreb coindex was introduced as

$$\overline{HM}(G) = \sum_{uv\notin E(G)} \Big(d_G(u)+d_G(v)\Big)^2. \tag{7}$$

These graph invariants were studied in [Pattabiraman and Vijayaragavan, 2017, Veylaki et al., 2016]. Relationship between hyper Zagreb index and coindex is established in [Gutman, 2017].

Now, we introduce a set of new degree-based topological indices and coindices named as *Karnatak College Dharwad* indices and coindices or KCD indices and coindices in short, which is dedicated to Karnatak College Dharwad as the college has completed hundred years of its service in education to the society in the year 2017. Further the research Supervisior and research scholar belong to the same college.
i.e., The first and second KCD indices of a graph G are respectively

$$KCD_1(G) = \sum_{e=uv\in E(G)} \Bigg(\Big(d_G(u)+d_G(v)\Big)+d_G(e)\Bigg) \tag{8}$$

$$KCD_2(G) = \sum_{e=uv\in E(G)} \Big(d_G(u)+d_G(v)\Big)d_G(e). \tag{9}$$

We proceed further to define KCD coindices as follows

$$\overline{KCD_1}(G) = \sum_{e=uv\notin E(G)} \Bigg(\Big(d_G(u)+d_G(v)\Big)+d_G(e)\Bigg) \tag{10}$$

$$\overline{KCD_2}(G) = \sum_{e=uv\notin E(G)} \Big(d_G(u)+d_G(v)\Big)d_G(e). \tag{11}$$

Here $\overline{KCD_1}(G)$ and $\overline{KCD_2}(G)$ are first and second KCD coindices of a graph G respectively.

The remaining paper is distributed as follows. Section 2 expresses the properties of first KCD indices and coindices of a graph and its complement. Section 3 concentrates on properties of second KCD indices and coindices of a graph and its complement, while Section 4 is devoted for the study of KCD indices of certain graph operations.
The following previously known results are considered for present investigation.

Theorem 1.1. *[Gutman et al., 2015] Let G be a graph with n vertices and m edges. Then,*

$$\begin{aligned} M_1(\overline{G}) &= M_1(G) + n(n-1)^2 - 4m(n-1) && (12)\\ \overline{M_1}(G) &= 2m(n-1) - M_1(G). && (13)\end{aligned}$$

Corollary 1.2. *[Gutman et al., 2015] Let G be any graph and $\overline{G}$ its complement. Then*

$$\overline{M_1}(G) = \overline{M_1}(\overline{G}). \tag{14}$$

Theorem 1.3. *[Gutman, 2017] Let G be a graph with n vertices and m edges. Then,*

$$\begin{aligned} \overline{HM}(G) &= 4m^2 + (n-2)M_1(G) - HM(G) && (15)\\ HM(\overline{G}) &= 2n(n-1)^3 - 12m(n-1)^2 + 4m^2 && (16)\\ &\quad +(5n-6)M_1(G) - HM(G) \\ \overline{HM}(\overline{G}) &= 4m(n-1)^2 + 4(n-1)M_1(G) + HM(G). && (17)\end{aligned}$$

2 Basic properties of first KCD indices and coindices

Theorem 2.1. *Let G be a graph with n vertices and m edges. Then,*

$$\begin{aligned} KCD_1(G) &= (4n-6)m - 4m(n-1) + 2M_1(G) && (18)\\ KCD_1(\overline{G}) &= 4n(\overline{m} - m) - 6\overline{m} + 4m + 2M_1(G) && (19)\\ \overline{KCD_1}(G) &= 4m(n-1) - 2\Big(\overline{m} + M_1(G)\Big) && (20)\\ \overline{KCD_1}(\overline{G}) &= (4n-6)m - 2M_1(G). && (21)\end{aligned}$$

Proof.
Proof of Eq. (18):

For any vertex u of G,

$$d_G(u) = n - 1 - d_{\overline{G}}(u). \tag{22}$$

and for any edge $e = uv$ of G,

$$d_G(e) = 2n - 4 - \Big(d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big). \tag{23}$$

Thus by Eqs. (8), (22) and (23), we have

$$\begin{aligned}
KCD_1(G) &= \sum_{e=uv \in E(G)} \Bigg(\Big(d_G(u) + d_G(v)\Big) + d_G(e)\Bigg) \\
&= \sum_{e=uv \notin E(\overline{G})} \Bigg(\Big(n - 1 - d_{\overline{G}}(u) + n - 1 - d_{\overline{G}}(v)\Big) \\
&\quad + \Big(2n - 4 - (d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big)\Bigg) \\
&= \sum_{e=uv \notin E(\overline{G})} \Bigg(4n - 6 - 2\Big(d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big)\Bigg) \\
&= (4n-6)m - 2 \sum_{e=uv \notin E(\overline{G})} \Big(d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big).
\end{aligned}$$

According To Eq. (4)

$$\overline{M_1}(\overline{G}) = \sum_{uv \notin E(\overline{G})} \Big(d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big).$$

Hence,

$$KCD_1(G) = (4n-6)m - 2\overline{M_1}(\overline{G}) \tag{24}$$

Substitution of Eqs. (13) and (14) in (24) results into Eq. (18).

Proof of Eq. (19):

For any vertex u of the complement $\overline{G}$,

$$d_{\overline{G}}(u) = n - 1 - d_G(u). \tag{25}$$

and for any edge $e = uv$ of the complement $\overline{G}$,

$$d_{\overline{G}}(e) = 2n - 4 - \Big(d_G(u) + d_G(v)\Big). \tag{26}$$

Bearing in mind Eqs. (8), (25) and (26), we get

$$\begin{aligned}
KCD_1(\overline{G}) &= \sum_{e=uv\in E(\overline{G})} \left(\Big(d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big) + d_{\overline{G}}(e)\right) \\
&= \sum_{e=uv\notin E(G)} \left(\Big(n - 1 - d_G(u) + n - 1 - d_G(v)\Big)\right. \\
&\quad \left. + \Big(2n - 4 - (d_G(u) + d_G(v)\Big)\right) \\
&= \sum_{e=uv\notin E(G)} \left(4n - 6 - 2\Big(d_G(u) + d_G(v)\Big)\right) \\
&= (4n-6)\overline{m} - 2\sum_{e=uv\notin E(G)} \Big(d_G(u) + d_G(v)\Big).
\end{aligned}$$

Thus by Eq. (4),

$$KCD_1(\overline{G}) = (4n-6)\overline{m} - 2\overline{M_1}(G) \tag{27}$$

Employing Eq. (13) in (27) generates Eq. (19).

Proof of Eq. (20):

Using Eqs. (10), (22) and (23), we have

$$\begin{aligned}
\overline{KCD_1}(G) &= \sum_{e=uv\notin E(G)} \left(\Big(d_G(u) + d_G(v)\Big) + d_G(e)\right) \\
&= \sum_{e=uv\in E(\overline{G})} \left(\Big(n - 1 - d_{\overline{G}}(u) + n - 1 - d_{\overline{G}}(v)\Big)\right. \\
&\quad \left. + \Big(2n - 4 - (d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big)\right) \\
&= \sum_{e=uv\in E(\overline{G})} \left(4n - 6 - 2\Big(d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big)\right).
\end{aligned}$$

By Eq. (3)

$$M_1(\overline{G}) \;=\; \sum_{uv\in E(\overline{G})}\Big(d_{\overline{G}}(u)+d_{\overline{G}}(v)\Big).$$

Thus,

$$\overline{KCD_1}(G) \;=\; (4n-6)\overline{m}-2M_1(\overline{G}) \tag{28}$$

Substitution of Eq. (12) in (28) gives Eq. (20).

Proof of Eq. (21):

In view of Eq. (10), (25) and (26), we get

$$\begin{aligned}
\overline{KCD_1}(\overline{G}) &= \sum_{e=uv\notin E(\overline{G})}\Bigg(\Big(d_{\overline{G}}(u)+d_{\overline{G}}(v)\Big)+d_{\overline{G}}(e)\Bigg)\\
&= \sum_{e=uv\in E(G)}\Bigg(\Big(n-1-d_G(u)+n-1-d_G(v)\Big)\\
&\quad +\Big(2n-4-(d_G(u)+d_G(v)\Big)\Bigg)\\
&= \sum_{e=uv\in E(G)}\Bigg(4n-6-2\Big(d_G(u)+d_G(v)\Big)\Bigg)\\
&= (4n-6)m-2\sum_{e=uv\in E(G)}\Big(d_G(u)+d_G(v)\Big)
\end{aligned}$$

Considering Eq. (3) we directly arrive at Eq. (21).

□

3 Basic properties of second KCD indices and coindices

Theorem 3.1. *Let G be a graph with n vertices and m edges. Then,*

$$KCD_2(G) = HM(G) - 2M_1(G) \tag{29}$$

$$KCD_2(\overline{G}) = 4(n-1)\Big(\overline{m}(n-2) - m(2n-3)\Big) + 4m^2 + (5n-8)M_1(G) - HM(G) \tag{30}$$

$$\overline{KCD_2}(G) = 4(n-1)(n-2)\overline{m} - (4n-6)(n-1)\Big(n(n-1) - 4m\Big) + 2(n-1)^2\Big(n(n-1) - 6m\Big) + 4m^2 + nM_1(G) - HM(G) \tag{31}$$

$$\overline{KCD_2}(\overline{G}) = 4(n-1)(n-2)m - (4n-6)M_1(G) + HM(G). \tag{32}$$

Proof.

Proof of Eq. (29):

Considering Eqs. (9), (22) and (23), we have

$$\begin{aligned} KCD_2(G) &= \sum_{e=uv\in E(G)} \Big(d_G(u) + d_G(v)\Big)d_G(e) \\ &= \sum_{e=uv\notin E(\overline{G})} \Big(n-1-d_{\overline{G}}(u) + n-1-d_{\overline{G}}(v)\Big)\Big(2n-4-(d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big) \\ &= \sum_{e=uv\notin E(\overline{G})} 4(n-1)(n-2) - (4n-6) \sum_{e=uv\notin E(\overline{G})} \Big(d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big) \\ &\quad + \sum_{e=uv\notin E(\overline{G})} \Big(d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big)^2. \end{aligned}$$

By an analogous reasoning,

$$\overline{M_1}(\overline{G}) = \sum_{uv\notin E(\overline{G})} \Big(d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big) \;\; and \;\; \overline{HM}(\overline{G}) = \sum_{uv\notin E(\overline{G})} \Big(d_{\overline{G}}(u) + d_{\overline{G}}(v)\Big)^2.$$

Thus,

$$KCD_2(G) = 4m(n-1)(n-2) - (4n-6)\overline{M_1}(\overline{G}) + \overline{HM}(\overline{G}).$$

In view of Eq. (14)

$$KCD_2(G) = 4m(n-1)(n-2) - (4n-6)\overline{M_1}(G) + \overline{HM}(\overline{G}) \tag{33}$$

Taking into account Eqs. (13) and (17), Eq. (33) results into Eq. (29).

Proof of Eq. (30):

In view of Eqs. (9), (25) and (26), we get

$$\begin{aligned}
KCD_2(\overline{G}) &= \sum_{e=uv\in E(\overline{G})} \Big(d_{\overline{G}}(u)+d_{\overline{G}}(v)\Big)d_{\overline{G}}(e)\\
&= \sum_{e=uv\notin E(G)} \Big(n-1-d_G(u)+n-1-d_G(v)\Big)\Big(2n-4-(d_G(u)+d_G(v)\Big)\\
&= \sum_{e=uv\notin E(G)} 4(n-1)(n-2)-(4n-6)\sum_{e=uv\notin E(G)}\Big(d_G(u)+d_G(v)\Big)\\
&\quad+\sum_{e=uv\notin E(G)}\Big(d_G(u)+d_G(v)\Big)^2.
\end{aligned}$$

By Eqs. (4) and (7), it directly follows

$$KCD_2(\overline{G}) = 4\overline{m}(n-1)(n-2)-(4n-6)\overline{M_1}(G)+\overline{HM}(G) \qquad (34)$$

Application of Eqs. (13) and (15) to Eq. (34) yields Eq. (30).

Proof of Eq. (31):

Using Eqs. (11), (22) and (23), we have

$$\begin{aligned}
\overline{KCD_2}(G) &= \sum_{e=uv\notin E(G)} \Big(d_G(u)+d_G(v)\Big)d_G(e)\\
&= \sum_{e=uv\in E(\overline{G})} \Big(n-1-d_{\overline{G}}(u)+n-1-d_{\overline{G}}(v)\Big)\\
&\quad\Big(2n-4-(d_{\overline{G}}(u)+d_{\overline{G}}(v)\Big)\\
&= \sum_{e=uv\in E(\overline{G})} 4(n-1)(n-2)-(4n-6)\sum_{e=uv\in E(\overline{G})}\Big(d_{\overline{G}}(u)+d_{\overline{G}}(v)\Big)\\
&\quad+\sum_{e=uv\in E(\overline{G})}\Big(d_{\overline{G}}(u)+d_{\overline{G}}(v)\Big)^2.
\end{aligned}$$

By reasoning,

$$M_1(\overline{G}) = \sum_{uv\in E(\overline{G})}\Big(d_{\overline{G}}(u)+d_{\overline{G}}(v)\Big) \;\; and \;\; HM(\overline{G}) = \sum_{uv\in E(\overline{G})}\Big(d_{\overline{G}}(u)+d_{\overline{G}}(v)\Big)^2.$$

Hence

$$\overline{KCD_2}(G) \;=\; 4\overline{m}(n-1)(n-2)-(4n-6)M_1(\overline{G})+HM(\overline{G}) \qquad (35)$$

Substituting Eqs. (12) and (16) in Eq. (35), simple calculation yields Eq. (31).

Proof of Eq. (32):

With the help of Eqs. (11), (25) and (26), we get

$$\begin{aligned}
\overline{KCD_2}(\overline{G}) &= \sum_{e=uv\notin E(\overline{G})} \Big(d_{\overline{G}}(u)+d_{\overline{G}}(v)\Big)d_{\overline{G}}(e) \\
&= \sum_{e=uv\in E(G)} \Big(n-1-d_G(u)+n-1-d_G(v)\Big)\Big(2n-4-(d_G(u)+d_G(v)\Big) \\
&= \sum_{e=uv\in E(G)} 4(n-1)(n-2)-(4n-6)\sum_{e=uv\in E(G)}\Big(d_G(u)+d_G(v)\Big) \\
&\quad + \sum_{e=uv\in E(G)}\Big(d_G(u)+d_G(v)\Big)^2
\end{aligned}$$

Eq. (32) immediately follows.

□

4 KCD indices of some graph operations

In this section, we study the graph operations using KCD indices. The well-known graph operations sum(join), cartesian product and composition of graphs are considered. All operations considered under the context are binary, with finite and simple graphs G and H. For the graphs G and H vertex and edge sets are denoted by $V(G)$ and $V(H)$, $E(G)$ and $E(H)$ respectively. The detailed information on sum(join) of graphs is refered in[Khalifeh et al., 2008a], cartesian product of graphs studied in[Khalifeh et al., 2008b] and composition of graphs is reported in [Imrich and Klavzar, 2000, Khalifeh et al., 2008a]. We refer [Khalifeh et al., 2009] for detailed information about graph operations.

Sum(join):

The sum(join) $G+H$ of two graphs G and H with disjoint vertex sets $|V(G)|$ and $|V(H)|$ is the graph on the vertex set $V(G)\cup V(H)$ and the edge set $E(G)\cup E(H)\cup\{uv : u\in V(G) \;\; and \;\; v\in V(H)\}$. For the graph $G+H$, $|V(G+H)| = |V(G)|+V(H)|$, $|E(G+H)| = |E(G)|+|E(H)|+|V(G)||V(H)|$, the degree of any vertex $u \in G+H$ is

$$d_{G+H}(u) = \begin{cases} d_G(u) + |V(H)| & u \in V(G) \\ d_H(u) + |V(G)| & u \in V(H). \end{cases}$$

Theorem 4.1. *Let G and H be graphs. Then*

$$\begin{aligned} KCD_1(G+H) &= 2\Bigg(M_1(G) + M_1(H) + |E(H)|\Big(4|V(G)| - 1\Big) \\ &\quad +|E(G)|\Big(4|V(H)| - 1\Big) \\ &\quad +|V(G)||V(H)|\Big(|V(G)| + |V(H)| - 1\Big)\Bigg). \end{aligned}$$

Proof:
By definition of sum(join) $G+H$ of two graphs G, H and Eq. (8), we have

$$KCD_1(G+H) = \sum_{e=uv \in E(G+H)} \Bigg(\Big(d_{G+H}(u) + d_{G+H}(v)\Big) + d_{G+H}(e)\Bigg).$$

Since,

$$d_{G+H}(e) = d_{G+H}(u) + d_{G+H}(v) - 2.$$

$$KCD_1(G+H) = 2 \sum_{e=uv \in E(G+H)} \Big(d_{G+H}(u) + d_{G+H}(v) - 1\Big).$$

$$\begin{aligned} KCD_1(G+H) &= 2 \sum_{e=uv \in E(H)} \Big(d_{G+H}(u) + d_{G+H}(v) - 1\Big) \\ &\quad +2 \sum_{e=uv \in E(G)} \Big(d_{G+H}(u) + d_{G+H}(v) - 1\Big) \\ &\quad +2 \sum_{e=uv \in \{uv: u \in V(G), v \in V(H)\}} \Big(d_{G+H}(u) + d_{G+H}(v) - 1\Big). \end{aligned} \tag{36}$$

Observe that,

$$\begin{aligned} \sum_{e=uv \in E(H)} \Big(d_{G+H}(u) + d_{G+H}(v) - 1\Big) &= \sum_{e=uv \in E(H)} \Big(d_H(u) + |V(G)| \\ &\quad +d_H(v) + |V(G)| - 1\Big) \\ &= \sum_{e=uv \in E(H)} \Big(d_H(u) + d_H(v) + 2|V(G)| - 1\Big). \end{aligned}$$

Thus,

$$\sum_{e=uv\in E(H)}\Big(d_{G+H}(u)+d_{G+H}(v)-1\Big) = M_1(H)+2|V(G)||E(H)| -|E(H)|. \quad (37)$$

Similarly,

$$\sum_{e=uv\in E(G)}\Big(d_{G+H}(u)+d_{G+H}(v)-1\Big) = M_1(G)+2|V(H)||E(G)| -|E(G)|. \quad (38)$$

In the same way,

$$\sum_{u\in V(G),v\in V(H)}\Big(d_{G+H}(u)+d_{G+H}(v)-1\Big) = 2|V(H)||E(G)|+|V(H)|^2|V(G)| +2|E(H)||V(G)|+|V(G)|^2|V(H)|-|V(G)||V(H)|. \quad (39)$$

Substituting Eqs. (37), (38) and (39) in Eq. (36) completes the proof.

□

Theorem 4.2. *Let G and H be graphs. Then*

$$KCD_2(G+H) = HM(G)+HM(H)+\Big(5|V(H)|-2\Big)M_1(G)+\Big(5|V(G)|-2\Big)M_1(H) +8\Bigg(|V(G)||E(H)|\Big(|V(G)|-1\Big)+|V(H)||E(G)|\Big(|V(H)|-1\Big)+|E(G)||E(H)|\Bigg) +|V(G)||V(H)|\Bigg(\Big(|V(G)|+|V(H)|\Big)^2+4\Big(|E(G)|+|E(H)|\Big)-2\Big(|V(G)|+|V(H)|\Big)\Bigg).$$

Proof.
With the knowledge of sum(join) $G+H$ of two graphs G, H and Eq. (9), we have

$$KCD_2(G+H) = \sum_{e=uv\in E(G+H)}\Big(d_{G+H}(u)+d_{G+H}(v)\Big)d_{G+H}(e).$$

As,

$$d_{G+H}(e) = d_{G+H}(u)+d_{G+H}(v)-2.$$

This implies,

$$\begin{aligned}
KCD_2(G+H) &= \sum_{e=uv\in E(G+H)} \Big(d_{G+H}(u)+d_{G+H}(v)\Big)^2 - 2\Big(d_{G+H}(u)+d_{G+H}(v)\Big)\\
&= \sum_{e=uv\in E(H)} \Big(d_{G+H}(u)+d_{G+H}(v)\Big)^2 - 2\Big(d_{G+H}(u)+d_{G+H}(v)\Big)\\
&\quad + \sum_{e=uv\in E(G)} \Big(d_{G+H}(u)+d_{G+H}(v)\Big)^2 - 2\Big(d_{G+H}(u)+d_{G+H}(v)\Big)\\
&\quad + \sum_{e=uv\in \{uv:u\in V(G),v\in V(H)\}} \Big(d_{G+H}(u)+d_{G+H}(v)\Big)^2\\
&\quad -2\Big(d_{G+H}(u)+d_{G+H}(v)\Big).
\end{aligned}$$

It follows that,

$$\begin{aligned}
\sum_{e=uv\in E(H)} \Big(d_{G+H}(u)+d_{G+H}(v)\Big)^2 - 2\Big(d_{G+H}(u)+d_{G+H}(v)\Big) &= \sum_{e=uv\in E(H)} \Bigg(\Big(d_H(u)\\
+|V(G)|+d_H(v)+|V(G)|\Big)^2 - 2\Big(d_H(u)+|V(G)|+d_H(v)+|V(G)|\Big)\Bigg)
\end{aligned}$$

$$\begin{aligned}
= \sum_{e=uv\in E(H)} \Bigg(\Big(d_H(u)+d_H(v)\Big)^2 + 4|V(G)|^2 + 4|V(G)|\Big(d_H(u)+d_H(v)\Big) - 2\Big(d_H(u)\\
+d_H(v)\Big) - 4|V(G)|\Bigg).
\end{aligned}$$

$$\begin{aligned}
\sum_{e=uv\in E(H)} \Big(d_{G+H}(u)+d_{G+H}(v)\Big)^2 - 2\Big(d_{G+H}(u)+d_{G+H}(v)\Big) &= HM(H)\\
+4|V(G)|^2|E(H)| + 4|V(G)|M_1(H) - 2M_1(H) - 4|V(G)||E(H)|. \qquad (40)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{e=uv\in E(G)} \Big(d_{G+H}(u)+d_{G+H}(v)\Big)^2 - 2\Big(d_{G+H}(u)+d_{G+H}(v)\Big) &= HM(G)\\
+4|V(H)|^2|E(G)| + 4|V(H)|M_1(G) - 2M_1(G) - 4|V(H)||E(G)|. \qquad (41)
\end{aligned}$$

In the same way

$$\begin{aligned}\sum_{u\in V(G),v\in V(H)} \Big(d_{G+H}(u)+d_{G+H}(v)\Big)^2-2\Big(d_{G+H}(u)+d_{G+H}(v)\Big) &= M_1(G)|V(H)| \\ +M_1(H)|V(G)|+8|E(G)||E(H)|+|V(G)||V(H)|\Big(|V(G)|+|V(H)|\Big)^2 \\ +4|E(G)||V(H)|^2+4|E(G)||V(G)||V(H)|+4|E(H)||V(G)||V(H)| \\ +4|E(H)||V(G)|^2-4|E(G)||V(H)|-4|E(H)||V(G)| \\ -2|V(G)||V(H)|\Big(|V(G)|+|V(H)|\Big).\end{aligned} \tag{42}$$

Finally, the summaton of Eqs. (40), (41) and (42) gives the desired result.

□

Cartesian Product:

The cartesian product $G\times H$ of two graphs G and H has the vertex set $V(G\times H)=V(G)\times V(H)$ and $e=(a,x)(b,y)$ is an edge of $G\times H$ if $a=b$ and $xy\in E(H)$, or $ab\in E(H)$ and $x=y$. For the graph $G\times H$, $|V(G\times H)|=|V(G)|V(H)|$, $|E(G\times H)|=|E(G)||V(H)|+|V(G)||E(H)|$, The degree of any vertex $(a,x)\in G\times H$ is $d_{G\times H}((a,x))=d_G(a)+d_H(x)$.

Theorem 4.3. *Let G and H be graphs. Then*

$$\begin{aligned}KCD_1(G\times H) &= 2\Bigg(|V(G)|M_1(H)+|V(H)|M_1(G)+8|E(G)||E(H)|- \\ &\qquad \Big(|V(G)||E(H)|+|V(H)||E(G)|\Big)\Bigg).\end{aligned}$$

Proof.
In the view of definition of cartesian product $G\times H$ of two graphs G, H and Eq. (8), we have

$$KCD_1(G\times H) = \sum_{e=(a,x)(b,y)\in E(G\times H)} \Bigg(\Big(d_{G\times H}((a,x))+d_{G\times H}((b,y))\Big)+d_{G\times H}((e))\Bigg).$$

It is known that,

$$d_{G\times H}((e)) = d_{G\times H}((a,x))+d_{G\times H}((b,y))-2.$$

Thus,

$$
\begin{aligned}
KCD_1(G\times H) &= 2\sum_{e=(a,x)(b,y)\in E(G\times H)}\Big(d_{G\times H}((a,x))+d_{G\times H}((b,y))-1\Big)\\
&= 2\sum_{a\in V(G)}\sum_{xy\in E(H)}\Big(d_G(a)+d_H(x)+d_G(a)+d_H(y)-1\Big)\\
&\quad +2\sum_{x\in V(H)}\sum_{ab\in E(G)}\Big(d_H(x)+d_G(a)+d_H(x)+d_G(b)-1\Big)\\
&= 2\sum_{a\in V(G)}\sum_{xy\in E(H)}\left(2d_G(a)+\Big(d_H(x)+d_H(y)\Big)-1\right)\\
&\quad +2\sum_{x\in V(H)}\sum_{ab\in E(G)}\left(2d_H(x)+\Big(d_G(a)+d_G(b)\Big)-1\right)
\end{aligned}
$$

By simple reasoning we straightforwardly obtain the required result.

□

Theorem 4.4. *Let G and H be graphs. Then*

$$
\begin{aligned}
KCD_2(G\times H) &= |V(G)|HM(H)+|V(H)|HM(G)+\Big(12|E(H)|-2|V(H)|\Big)M_1(G)\\
&\quad +\Big(12|E(G)|-2|V(G)|\Big)M_1(H)-16|E(G)||E(H)|.
\end{aligned}
$$

Proof.
Taking into account the definition of cartesian product $G\times H$ of two graphs G and H, start with Eq. (9) as

$$
KCD_2(G\times H) = \sum_{e=(a,x)(b,y)\in E(G\times H)}\Big(d_{G\times H}((a,x))+d_{G\times H}((b,y))\Big)d_{G\times H}((e)).
$$

Since

$$
d_{G\times H}((e)) = d_{G\times H}((a,x))+d_{G\times H}((b,y))-2.
$$

We have

$$
\begin{aligned}
KCD_2(G\times H) &= \sum_{e=(a,x)(b,y)\in E(G\times H)} \Bigg(\Big(d_{G\times H}((a,x))+d_{G\times H}((b,y))\Big)^2 \\
&\quad -2\Big(d_{G\times H}((a,x))+d_{G\times H}((b,y))\Big)\Bigg) \\
&= \sum_{a\in V(G)}\sum_{xy\in E(H)} \Bigg(\Big(d_G(a)+d_H(x)+d_G(a)+d_H(y)\Big)^2 \\
&\quad -2\Big(d_G(a)+d_H(x)+d_G(a)+d_H(y)\Big)\Bigg) + \sum_{x\in V(H)}\sum_{ab\in E(G)} \Bigg(\Big(d_H(x) \\
&\quad +d_G(a)+d_H(x)+d_G(b)\Big)^2 - 2\Big(d_H(x)+d_G(a)+d_H(x)+d_G(b)\Big)\Bigg) \\
&= \sum_{a\in V(G)}\sum_{xy\in E(H)} \Bigg(\Big(2d_G(a)+d_H(x)+d_H(y)\Big)^2 - 2\Big(2d_G(a)+d_H(x) \\
&\quad +d_H(y)\Big)\Bigg) + \sum_{x\in V(H)}\sum_{ab\in E(G)} \Bigg(\Big(2d_H(x)+d_G(a)+d_G(b)\Big)^2 \\
&\quad -2\Big(2d_H(x)+d_G(a)+d_G(b)\Big)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
KCD_2(G\times H) &= \sum_{a\in V(G)}\sum_{xy\in E(H)} \Bigg(4\Big(d_G(a)\Big)^2 + \Big(d_H(x)+d_H(y)\Big)^2 \\
&\quad +4d_G(a)\Big(d_H(x)+d_H(y)\Big) - 2\Big(2d_G(a)+\Big(d_H(x)+d_H(y)\Big)\Big)\Bigg) \\
&\quad + \sum_{x\in V(H)}\sum_{ab\in E(G)} \Bigg(4\Big(d_H(x)\Big)^2 + \Big(d_G(a)+d_G(b)\Big)^2 \\
&\quad +4d_H(x)\Big(d_G(a)+d_G(b)\Big) - 2\Big(2d_H(x)+\Big(d_G(a)+d_G(b)\Big)\Big)\Bigg)
\end{aligned}
$$

and the required result immediately follows.

□

Composition:

The composition $G[H]$ of two graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$, edge sets E(G) and E(H) is the graph with vertex set $V(G) \times V(H)$ and (a,x) is adjacent to (b,y) whenever a is adjacent to b, or $a = b$ and x is adjacent to y. For the graph $G[H]$, $|V(G[H])| = |V(G)||V(H)|$, $|E(G[H])| = |E(G)||V(H)|^2 + |E(H)||V(G)|$, The degree of any vertex $(a, x) \in G[H]$ is $d_{G[H]}((a, x)) = |V(H)|d_G(a) + d_H(x)$.

Theorem 4.5. *Let G and H be graphs. Then*

$$\begin{aligned} KCD_1(G[H]) &= 2\Big(|V(H)|^3 M_1(G) + |V(G)|M_1(H) + 8|V(H)||E(G)||E(H)| \\ &\quad -|V(H)|^2|E(G)| - |E(H)||V(G)|\Big). \end{aligned}$$

Proof.
Using the definition of composition $G[H]$ of two graphs G, H and Eq. (8), we have

$$KCD_1(G[H]) = \sum_{e=(a,x)(b,y)\in E(G[H])} \Bigg(\Big(d_{G[H]}((a,x)) + d_{G[H]}((b,y))\Big) + d_{G[H]}((e))\Bigg).$$

But

$$d_{G[H]}((e)) = d_{G[H]}((a,x)) + d_{G[H]}((b,y)) - 2.$$

This implies,

$$KCD_1(G[H]) = 2 \sum_{e=(a,x)(b,y)\in E(G[H])} \Big(d_{G[H]}((a,x)) + d_{G[H]}((b,y)) - 1\Big).$$

$$\begin{aligned} KCD_1(G[H]) = {} & 2 \sum_{x\in V(H)} \sum_{y\in V(H)} \sum_{ab\in E(G)} \Big(|V(H)|d_G(a) + d_H(x) + |V(H)|d_G(b) \\ & +d_H(y) - 1\Big) + 2 \sum_{a\in V(G)} \sum_{xy\in E(H)} \Big(|V(H)|d_G(a) + d_H(x) \\ & +|V(H)|d_G(a) + d_H(y) - 1\Big). \end{aligned} \tag{43}$$

We start with

$$\sum_{x\in V(H)}\sum_{y\in V(H)}\sum_{ab\in E(G)}\Big(|V(H)|d_G(a)+d_H(x)+|V(H)|d_G(b)+d_H(y)-1\Big)=$$
$$\sum_{x\in V(H)}\sum_{y\in V(H)}\sum_{ab\in E(G)}\left(|V(H)|\Big(d_G(a)+d_G(b)\Big)+\Big(d_H(x)+d_H(y)\Big)-1\right)$$

Thus,

$$\begin{aligned}\sum_{x\in V(H)}\sum_{y\in V(H)}\sum_{ab\in E(G)}\Big(|V(H)|d_G(a)+d_H(x)+|V(H)|d_G(b)+d_H(y)-1\Big)&=\\ |V(H)|^3M_1(G)+4|V(H)||E(G)||E(H)|-|V(H)|^2|E(G)|.\end{aligned} \tag{44}$$

Similarly,

$$\begin{aligned}\sum_{a\in V(G)}\sum_{xy\in E(H)}\Big(|V(H)|d_G(a)+d_H(x)+|V(H)|d_G(a)+d_H(y)-1\Big)&=\\ 4|V(H)||E(G)||E(H)|+|V(G)|M_1(H)-|V(G)||E(H)|.\end{aligned} \tag{45}$$

Substituting Eqs. (44) and (45) in Eq. (43) generates the desired result.

□

Theorem 4.6. *Let G and H be graphs. Then*

$$\begin{aligned}KCD_2(G[H]) &= |V(H)|^4HM(G)+|V(G)|HM(H)\\ &\quad+2|V(H)|^2M_1(G)\Big(6|E(H)|-|V(H)|\Big)\\ &\quad+2M_1(H)\Big(5|V(H)||E(G)|-|V(G)|\Big)\\ &\quad+8|E(G)||E(H)|\Big(|E(H)|-2|V(H)|\Big).\end{aligned}$$

Proof.
In view of definition of composition $G[H]$ of two graphs G, H and Eq. (9), we start with

$$KCD_2(G[H]) = \sum_{e=(a,x)(b,y)\in E(G[H])}\Big(d_{G[H]}((a,x))+d_{G[H]}((b,y))\Big)d_{G[H]}((e)).$$

It is known that,

$$d_{G[H]}((e)) = d_{G[H]}((a,x))+d_{G[H]}((b,y))-2.$$

We get,

$$KCD_2(G[H]) = \sum_{e=(a,x)(b,y)\in E(G[H])} \Bigg(\Big(d_{G[H]}((a,x)) + d_{G[H]}((b,y))\Big)^2 - 2\Big(d_{G[H]}((a,x)) + d_{G[H]}((b,y))\Big)\Bigg).$$

$$\begin{aligned} KCD_2(G[H]) = & \sum_{x\in V(H)}\sum_{y\in V(H)}\sum_{ab\in E(G)} \Bigg(\Big(|V(H)|d_G(a) + d_H(x) + |V(H)|d_G(b) + d_H(y)\Big)^2 \\ & -2\Big(|V(H)|d_G(a) + d_H(x) + |V(H)|d_G(b) + d_H(y)\Big)\Bigg) \\ & + \sum_{a\in V(G)}\sum_{xy\in E(H)} \Bigg(\Big(|V(H)|d_G(a) + d_H(x) + |V(H)|d_G(a) + d_H(y)\Big)^2 \\ & -2\Big(|V(H)|d_G(a) + d_H(x) + |V(H)|d_G(a) + d_H(y)\Big)\Bigg). \end{aligned}$$

Thus,

$$\begin{aligned} KCD_2(G[H]) = & \sum_{x\in V(H)}\sum_{y\in V(H)}\sum_{ab\in E(G)} \Bigg(\Big(|V(H)|\Big(d_G(a) + d_G(b)\Big) + d_H(x) + d_H(y)\Big)^2 \\ & -2\Big(|V(H)|\Big(d_G(a) + d_G(b)\Big) + d_H(x) + d_H(y)\Big)\Bigg) \\ & + \sum_{a\in V(G)}\sum_{xy\in E(H)} \Bigg(\Big(2|V(H)|d_G(a) + d_H(x) + d_H(y)\Big)^2 \\ & -2\Big(2|V(H)|d_G(a) + d_H(x) + d_H(y)\Big)\Bigg). \end{aligned}$$

It follows that,

$$\sum_{x\in V(H)}\sum_{y\in V(H)}\sum_{ab\in E(G)}\Bigg(\Big(|V(H)|\Big(d_G(a)+d_G(b)\Big)+d_H(x)+d_H(y)\Big)^2 -2\Big(|V(H)|\Big(d_G(a)+d_G(b)\Big)+d_H(x)+d_H(y)\Big)\Bigg)$$

$$=\sum_{x\in V(H)}\sum_{y\in V(H)}\sum_{ab\in E(G)}\Bigg(|V(H)|^2\Big(d_G(a)+d_G(b)\Big)^2+\Big(d_H(x)+d_H(y)\Big)^2 +2|V(H)|\Big(d_G(a)+d_G(b)\Big)\Big(d_H(x)+d_H(y)\Big)-2|V(H)|\Big(d_G(a)+d_G(b)\Big) -2|V(H)|d_H(x)-2|V(H)|d_H(y)\Bigg)$$

Hence,

$$\sum_{x\in V(H)}\sum_{y\in V(H)}\sum_{ab\in E(G)}\Bigg(\Big(|V(H)|\Big(d_G(a)+d_G(b)\Big)+d_H(x)+d_H(y)\Big)^2 -2\Big(|V(H)|\Big(d_G(a)+d_G(b)\Big)+d_H(x)+d_H(y)\Big)\Bigg)=|V(H)|^4HM(G) +2|V(H)||E(G)|M_1(H)+8|E(H)|^2|E(G)|+8|V(H)|^2|E(H)|M_1(G) -2|V(H)|^3M_1(G)-8|E(H)||E(G)||V(H)|. \tag{46}$$

Similarly,

$$\sum_{a\in V(G)}\sum_{xy\in E(H)}\Bigg(\Big(2|V(H)|d_G(a)+d_H(x)+d_H(y)\Big)^2-2\Big(2|V(H)|d_G(a) +d_H(x)+d_H(y)\Big)\Bigg)=4|V(H)|^2|E(H)|M_1(G)+|V(G)|HM(H) +8|V(H)||E(G)|M_1(H)-8|V(H)||E(G)||E(H)|-2|V(G)|M_1(H). \tag{47}$$

Finally, summation of Eqs. (46) and (47) gives the required result.

□

5 Conclusion

In this paper, we have introduced few new degree based topological indices and coindices named KCD indices and coindices. A set of properties of these indices and coindices are obtained. Finally, some graph operations are studied using KCD indices. These results have scope for further development using remaining graph operations.

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