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Minimal H_v-fields

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Abstract

Hyperstructures have applications in mathematics and in other sciences, which range from biology, hadronic physics, leptons, linguistics, sociology, to mention but a few. For this, the largest class of the hyperstructures, the H_v-structures, is used. They satisfy the weak axioms where the non-empty intersection replaces equality. The fundamental relations connect, by quotients, the H_v-structures with the classical ones. H_v-numbers are elements of H_v-field, and they are used in representation theory. We focus on minimal H_v-fields.

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1 Introduction

The class of hyperstructures called H_v*-structures* introduced in 1990 [Vougiouklis, 1991a], [Vougiouklis, 1994] by Vougiouklis, satisfy the *weak axioms* where the non-empty intersection replaces equality.

Algebraic hyperstructure $(H, \cdot)$ is a set H equipped with a *hyperoperation* (abbreviated: **hope**) $\cdot : H \times H \to P(H) - \{\varnothing\}$. We abbreviate by *WASS* the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ and by *COW* the *weak commutativity*: $xy \cap yx \neq \varnothing, \forall x, y \in H$. $(H, \cdot)$ is an H_v*-semigroup* if it is *WASS*, it is called H_v-**group** if it is reproductive H_v-semigroup, i.e., $xH = Hx = H, \forall x \in H.$

Motivation. The quotient of a group by an invariant subgroup, is a group. The quotient of a group by a subgroup is a hypergroup, Marty 1934. The quotient of a group by any partition (equivalence) is an H_v-group, Vougiouklis 1990.

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In an H_v-semigroup the *powers* are: $h^1 = \{h\}, h^2 = h \cdot h, ..., h^n = h \circ h \circ ... \circ h$, where $(\circ)$ is the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An $(H, \cdot)$ is called *cyclic of period s*, if there exists an element h, called *generator*, and the minimum s, such that $H = h^1 \cup h^2 ... \cup h^s$. Analogously the cyclicity for the infinite period is defined. If thereare h and s, the minimum one, such that $H = h^s$, then $(H, \cdot)$ is a *single-power cyclic of period s.*

$(R, +, \cdot)$ is called H_v-**ring** if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \varnothing,\ (x+y)z \cap (xz+yz) \neq \varnothing,\ \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ be an H_v-ring,a COW H_v-group $(M, +)$ is called H_v-**module** over R, if there is an external hope

$$\cdot : R \times M \to P(M) : (a, x) \to ax$$

such that $\forall a, b \in R$ and $\forall x, y \in M$ we have

$$a(x+y) \cap (ax+ay) \neq \varnothing,\ (a+b)x \cap (ax+bx) \neq \varnothing,\ (ab)x \cap a(bx) \neq \varnothing,$$

For more definitions and applications on H_v-structures one can see in books and papers as [Corsini, 1993],[Corsini and Leoreanu, 2003],[Davvaz, 2003],[Davvaz and Leoreanu, 2007],[Davvaz and Vougiouklis, 2018],[Vougiouklis, 1994],[Vougiouklis, 1995],[Vougiouklis, 1999b].

Let $(H, \cdot), (H, *)$ H_v-semigroups, the hope $(\cdot)$ is **smaller** than $(*)$, and $(*)$ **greater** than $(\cdot)$, iff there exists an automorphism

$$f \in Aut(H, *) \text{ such that } xy \subset f(x * y),\ \forall x, y \in H.$$

We write $\cdot \leq *$ and say that $(H, *)$ *contains* $(H, \cdot)$. If $(H, \cdot)$ is a structure then it is *basic structure* and $(H, *)$ is $H_b - structure$.

Minimal is called an H_v-group which contains no other H_v-group defined on the same set. We extend this definition to any H_v-structures with any more properties.

Theorem 1.1. ***(The Little Theorem)****. Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

The little theorem leads to a partial order on H_v-structures and to posets.

Let $(H, \cdot)$ be hypergroupoid. We *remove* $h \in H$, if we take the restriction of $(\cdot)$ in $H - \{h\}$. $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h by $\underline{h}$. $\underline{h} \in H$ merges with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ a class with representative $\underline{h}$.

M. Koskas in 1970, introduced in hypergroups the relation β^*, which connects hypergroups with groups and it is defined in H_v-groups as well. Vougiouklis [Vougiouklis, 1985], [Vougiouklis, 1988], [Vougiouklis, 1991a], [Vougiouklis, 1994], [Vougiouklis, 1995], [Vougiouklis, 2016] introduced the γ^* and ϵ^* relations, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named all these relations, *fundamental*.

Definition 1.1. *The* ***fundamental relations*** *β*, γ* and ϵ*, are defined, in H_v-groups, H_v-rings and H_v-vector space, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively.*

Remark: Let $(G, \cdot)$ be group and R a partition in G, then $(G/R, \cdot)$ is an H_v-group, therefore the quotient $(G/R, \cdot)/\beta$* is a group, the *fundamental* one.

Theorem 1.2. *Let $(H, \cdot)$ be an H_v-group and denote by U the set of all finite products of elements of H. Define the relation β in H by: $x\beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then β* is the transitive closure of β.*

Analogous theorems are for H_v-rings, H_v-vector spaces and so on [Vougiouklis, 1994].

Theorem 1.3. *Let $(R, +, \cdot)$ be H_v-ring. Denote U all finite polynomials of elements of R. Define the relation γ in R by:*

$$x\gamma y \text{ iff } \{x, y\} \subset u \text{ where } u \in U.$$

Then the relation γ is the transitive closure of the relation γ.*

Proof. Let $\underline{\gamma}$ be the transitive closure of γ, and denote by $\underline{\gamma}(a)$ the class of a. First, we prove that the quotient set $R/\underline{\gamma}$ is a ring.

In $R/\underline{\gamma}$ the sum ($\oplus$) and the product ($\otimes$) are defined in the usual manner:

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \{\underline{\gamma}(c) : c \in \underline{\gamma}(a) + \underline{\gamma}(b)\},$$

$$\underline{\gamma}^*(a) \otimes \underline{\gamma}(b) = \{\underline{\gamma}(d) : d \in \underline{\gamma}^*(a) \cdot \underline{\gamma}(b)\}, \ \forall a, b \in R.$$

Take $a' \in \underline{\gamma}(a)$, $b' \in \underline{\gamma}(b)$. Then we have

$$a'\underline{\gamma}a \text{ iff } \exists x_1, ..., x_{m+1} \text{ with } x_1 = a', x_{m+1} = a \text{ and } u_1, ..., u_m \in U$$

$$\text{such that } \{x_i, x_{i+1}\} \subset u_i, i = 1, ..., m, \text{ and}$$

$$b'\underline{\gamma}b \text{ iff } \exists y_1, ..., y_{n+1} \text{ with } y_1 = b', y_{n+1} = b \text{ and } v_1, ..., v_n \in U$$

$$\text{such that } \{y_j, y_{j+1}\} \subset v_j, i = 1, ..., n.$$

From the above we obtain

$$\{x_i, x_{i+1}\} + y_1 \subset u_i + v_1,\ i = 1, ..., m-1,$$

$$x_{m+1} + \{y_j, y_{j+1}\} \subset u_m + v_j,\ j = 1, ..., n.$$

The sums

$$u_i + v_1 = t_i,\ i = 1, ...m-1 \text{ and } u_m + v_j = t_{im+j-1},\ j = 1, ..., n$$

are also polynomials, thus, $t_k \in U \forall k \in \{1, ..., m+n-1\}$.

Now, pick up $z_1, ..., z_{m+n}$ such that

$$zi \in x_i + y_1,\ i = 1, ..., n \text{ and } z_{m+j} \in x_{m+1} + y_{j+1},\ j = 1, ..., n,$$

therefore, using the above relations we obtain $\{z_k, z_{k+1}\} \subset t_k,\ k = 1, ..., m+n-1$.

Thus, every $z_1 \in x_1 + y_1 = a' + b'$ is $\underline{\gamma}$ equivalent to every $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$. So $\underline{\gamma}(a) \oplus \underline{\gamma}(b)$ is a singleton so we can write

$$\underline{\gamma}(a) \oplus \underline{\gamma}(b) = \underline{\gamma}(c), \forall c \in \underline{\gamma}(a) + \underline{\gamma}(b)$$

In a similar way we prove that

$$\underline{\gamma}(a) \otimes \underline{\gamma}(b) = \underline{\gamma}(d), \forall d \in \underline{\gamma}(a) \cdot \underline{\gamma}(b)$$

The WASS and the weak distributivity on R guarantee that associativity and distributivity are valid for $R/\gamma*$. Therefore $R/\gamma*$ is a ring.

Let σ be an equivalence relation in R such that R/σ is a ring and $\sigma(a)$ the class of a. Then $\sigma(a) \oplus \sigma(b)$ and $\sigma(a) \otimes \sigma(b)$ are singletons $\forall a, b \in R$, i.e.

$$\sigma(a) \oplus \sigma(b) = \sigma(c), \forall c \in \sigma(a) + \sigma(b),\ \sigma(a) \otimes \sigma(b) = \sigma(d), \forall d \in \sigma(a) \cdot \sigma(b).$$

Therefore we write, for every $a, b \in R$ and $A \subset \sigma(a)$, $B \subset \sigma(b)$,

$$\sigma(a) \oplus \sigma(b) = \sigma(a+b) = \sigma(A+B),\ \sigma(a) \otimes \sigma(b) = \sigma(ab) = \sigma(A \cdot B)$$

By induction, we extend these relations on finite sums and products. Thus, $\forall u \in U$, we have the relation $\sigma(x) = \sigma(u)\ \forall x \in u$. Consequently $x \in \gamma(a) => x \in \sigma(a), \forall x \in R$. But σ is transitively closed, so we obtain: $x \in \underline{\gamma}(x) => x \in \sigma(a)$. That $\underline{\gamma}$ is the smallest equivalence relation in R such that $R/\underline{\gamma}$ is a ring, i.e. $\underline{\gamma} = \gamma*$. □

An element is called **single** if its fundamental class is singleton [Vougiouklis, 1994].

General structures can be defined using fundamental structures. From 1990 there is the following [Vougiouklis, 1991a], [Vougiouklis, 1994]:

Definition 1.2. *An H_v-ring $(R, +, \cdot)$ is called H_v**-field** if R/γ^* is a field. An H_v-module over an H_v-field F, it is called H_v**-vector space**.*

The analogous to Theorem 1.3, on H_v-vector spaces, can be proved:

Theorem 1.4. *Let $(V, +)$ be H_v-vector space over the H_v-field F. Denote U the set of all expressions of finite hopes either on F and V or the external hope applied on finite sets of elements of F and V. Define the relation ϵ in V as follows: $x\epsilon y$ iff $\{x, y\} \subset u$ where $u \in U$. Then ϵ^* is the transitive closure of the relation ϵ.*

Definition 1.3. *Let $(L, +)$ be H_v-vector space over the H_v-field F, $\phi : F \to F/\gamma^*$ canonical; $\omega_F = \{x \in F : \phi(x) = 0\}$, the core, 0 is the zero of F/γ. Let ω_L be the core of $\phi' : L \to L/\epsilon^*$ and denote by 0 the zero of L/ϵ^*, as well. Take the bracket (commutator) hope:*

$$[,] : L \times L \to P(L) : (x, y) \to [x, y]$$

*then **L** is an H_v**-Lie algebra** over F if the following axioms are satisfied:*

(L1) *The bracket hope is bilinear, i.e.*
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \varnothing$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \varnothing$,
$\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$

(L2) $[x, x] \cap \omega_L \neq \varnothing, \ \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing, \ \forall x, y, z \in L$

Definition 1.4. *The H_v-semigroup $(H\cdot)$ is h/v**-group** if H/β^* is a group [Vougiouklis, 2003].*

The h/v-group is a generalization of H_v-group where a *reproductivity of classes* is valid: if $\sigma(x), \forall x \in H$ equivalence classes then $x\sigma(y) = \sigma(xy) = \sigma(x)y, \forall x, y \in H$. Similarly *$h/v$-rings*, *$h/v$-fields*, *$h/v$-vector spaces* etc, are defined.

The **uniting elements** method introduced by Corsini & Vougiouklis in 1989, is the following [Corsini and Vougiouklis, 1989]: Let G be a structure and a not valid property d, described by a set of equations. Take the partition in **G** for which put in the same class, all pairs of elements that causes the non-validity of d. The quotient by this partition G/d is an H_v-structure. Then, the quotient out G/d by β^*, is a stricter structure $(G/d)\beta^*$ for which d is valid.

Theorem 1.5. *Let $(R, +, \cdot)$ be a ring, and $F = \{f_1, ..., f_m, f_{m+1}, ..., f_{m+n}\}$ be a system of equations on **R** consisting of subsystems $F_m = \{f_1, ..., f_m\}$ and $F_n = \{f_{m+1}, ..., f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using F and F_m respectively, and σ_n the equivalence defined on F_n on the ring $\mathbf{R}_m = (\mathbf{R}/\sigma_m)/\gamma^*$. Then*

$$(\mathbf{R}/\sigma)/\gamma^* \cong (\mathbf{R}_m/\sigma_n)/\gamma^*$$

2 Large classes of hopes

A class of H_v-structures, introduced in [Vougiouklis, 1991b], [Vougiouklis, 1994], [Vougiouklis, 2014b], is the following:

Definition 2.1. *An H_v-structure is called **very thin** if there exists a pair $(a, b) \in H^2$ for which $ab = A$, with $cardA > 1$, and all the other products are singletons.*

From the very thin hopes the Attach Construction is obtained [Vougiouklis, 1999a], [Vougiouklis, 2014b], [Vougiouklis, 2017]: Let $(H, \cdot)$ be an H_v-semigroup and $v \notin H$. We extend the hope $(\cdot)$ into $\underline{H} = H \cup \{v\}$ by:

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

The $(\underline{H}, \cdot)$ is an H_v-group, where $(\underline{H}, \cdot)/\beta^* \cong \mathbf{Z}_2$ and v is a single.
Let $(H, \cdot)$ H_v-semigroup, and [x] the fundamental class of $\forall x \in H$. Unit class is [e] if

$$([e] \cdot [x]) \cap [x] \neq \varnothing \text{ and } ([x] \cdot [e]) \cap [x] \neq \varnothing, \forall x \in H,$$

and $\forall x \in H$, we call inverse class of [x], the class $[x]^{-1}$, if

$$([x] \cdot [x]^{-1}) \cap [e] \neq \varnothing \text{ and } ([x]^{-1} \cdot [x]) \cap [e] \neq \varnothing.$$

Enlarged hopes are the ones where a new element appears in one result. The useful cases are those h/v-structures with the same fundamental structure.

Construction 2.1. *(a) Let $(H, \cdot)$ be an H_v-semigroup, $v \notin H$. We extend $(\cdot)$ into $\underline{H} = H \cup \{v\}$ by:*

$$x \cdot v = v \cdot x = v, \forall x \in H, \text{ and } v \cdot v = H.$$

*The $(\underline{H}, \cdot)$ is an h/v-group, called **attach**, where $(\underline{H}, \cdot)/\beta^* \cong Z_2$ and v is single. Scalars and units of $(H, \cdot)$ are scalars and units in $(\underline{H}, \cdot)$. If $(H, \cdot)$ is COW then $(\underline{H}, \cdot)$ is COW.*

(b) $(H, \cdot)$ Hv-semigroup, $v \notin H$, $(\underline{H}, \cdot)$ its attached h/v-group. Take $0 \notin H$ and define in $\underline{H}_\circ = H \cup \{v, 0\}$ two hopes:

$$\textit{hypersum}(+) : 0 + 0 = x + v = v + x = 0,$$

$$0 + v = v + 0 = x + y = v, 0 + x = x + 0 = v + v = H, \forall x, y \in H$$

$$\textit{hyperproduct}(\cdot) : \textit{ remains the same as in } \underline{H} \textit{ moreover}$$

$$0 \cdot 0 = v \cdot x = x \cdot 0 = 0, \forall x \in \underline{H}$$

Then $(\underline{H}_o, +, \cdot)$ is h/v-field with $(\underline{H}_o, +, \cdot)/\gamma^* \cong \mathbf{Z}_3$. $(+)$ is associative, $(\cdot)$ is WASS and weak distributive to $(+)$. 0 is zero absorbing in $(+)$. $(\underline{H}_o, +, \cdot)$ is the **attached h/v-field** of $(H, \cdot)$.

Definition 2.2. *[Vougiouklis, 2008], [Vougiouklis, 2016] Let $(G, \cdot)$ be groupoid and $f : G \to G$ be a map. We define a hope (∂), called theta-hope, we write ∂-hope, on G as follows*

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \ \ \forall x, y \in G.$$

If $(\cdot)$ is commutative then ∂ is commutative. If $(\cdot)$ is COW, then ∂ is COW.

If $(G, \cdot)$ is groupoid and $f : G \to P(G) - \{\varnothing\}$ be multivalued map. We define the ∂-hope on G as follows

$$x\partial y = (f(x) \cdot y) \cup (x \cdot f(y)), \ \ \forall x, y \in G$$

.

Motivation for the ∂-hope is the derivative where only the product of functions is used.

Basic property: if $(G, \cdot)$ is semigroup then $\forall f$, the ∂-hope is WASS.

Examples (a) In integers $(\mathbf{Z}, +, \cdot)$ fix $n \neq 0$, a natural number. Consider the map f such that $f(0) = n$ and $f(x) = x, \forall x \in \mathbf{Z} - \{0\}$. Then $(\mathbf{Z}, \partial_+, \partial_\cdot)$, where ∂_+ and $\partial_\cdot$ are the ∂-hopes refereed to the addition and the multiplication respectively, is an H_v-near-ring, with

$$(\mathbf{Z}, \partial_+, \partial_\cdot)/\gamma^* \cong \mathbf{Z}_n.$$

(b) In $(\mathbf{Z}, +, \cdot)$ with $n \neq 0$, take f such that $f(n) = 0$ and $f(x) = x, \forall x \in \mathbf{Z} - \{n\}$. Then $(\mathbf{Z}, \partial_+, \partial_\cdot)$ is an H_v-ring, moreover, $(\mathbf{Z}, \partial_+, \partial_\cdot)/\gamma^* \cong \mathbf{Z}_n$.

Special case of the above is for n=p, prime, then $(\mathbf{Z}, \partial_+, \partial_\cdot)$ is an H_v-field.

Combining the uniting elements procedure with the enlarging theory or the ∂-theory, we can obtain analogous results [Vougiouklis, 1999a], [Vougiouklis, 2014b], [Vougiouklis, 2017].

Theorem 2.1. *In the ring $(\mathbf{Z}_n, +, \cdot)$, with $n = ms$ we enlarge the multiplication only in the product of the elements $0 \cdot m$ by setting $0 \otimes m = \{0, m\}$ and the rest results remain the same. Then*

$$(\mathbf{Z}_n, +, \otimes)/\gamma^* \cong (\mathbf{Z}_m, +, \cdot)$$

Remark that we can enlarge other products as well, for example $2 \cdot m$ by setting $2 \otimes m = \{2, m + 2\}$, then the result remains the same. In this case 0 and 1 are scalars.

Corolary 2.1. *In the ring $(\mathbf{Z}_n, +, \cdot)$, with $n = ps$, where p is prime , we enlarge only the product $0 \cdot p$ by $0 \oplus p = \{0, p\}$ and the rest results remain the same. Then $(\mathbf{Z}_n, +, \oplus)$ is a very thin H_v-field.*

Now we focus on **Very Thin minimal H_v-fields** obtained by a classical field.

Theorem 2.2. *In a field $(F, +, \cdot)$, we enlarge only in the product of the special elements a and b, by setting $a \otimes b = \{ab, c\}$, where $c \neq ab$, and the rest results remain the same. Then we obtain the degenerate, minimal very thin, H_v-field*

$$(F, +, \otimes)/\gamma^* \cong \{0\}$$

Thus, there is no non-degenerate H_v-field obtained by a field by enlarging any product.

Proof. Take any $x \in F - \{0\}$, then from $a \otimes b = \{ab, c\}$ we obtain

$(a \otimes b) - ab = \{0, c - ab\}$ and then $(x(c - ab)^{-1}) \otimes ((a \otimes b) - ab) = \{0, x\}$

thus, $0\gamma x, x \in F - \{0\}$. Which means that every x is in the same fundamental class with the element 0. Thus, $(F, +, \otimes)/\gamma^* \cong \{0\}$.

□

Theorem 2.3. *In a field $(F, +, \cdot)$, we enlarge only in the sum of the special elements a and b, by setting $a \oplus b = \{a + b, c\}$, where $c \neq a + b$, and the rest results remain the same. Then we obtain the degenerate, minimal very thin, H_v-field $(F, +, \oplus)/\gamma^* \cong \{0\}$. Thus, there is no non-degenerate H_v-field obtained by a field by enlarging any sum.*

Proof. Take any $x \in F - \{0\}$, then from $a \oplus b = \{a + b, c\}$ we obtain

$(a \oplus b) - a + b = \{0, c - a + b\}$ and then $(x(c - a + b)^{-1}) \cdot ((a \oplus b) - a + b) = \{0, x\}$

thus, $0\gamma x, x \in F - \{0\}$. Which means that every x is in the same fundamental class with the element 0. Thus,

$$(\mathbf{F}, +, \oplus)/\gamma^* \cong \{0\}.$$

□

The above two theorems state that there is no non-degenerate H_v-field obtained by a field by enlarging any sum or product.

Hopes defined on classical structures are the following [Corsini, 1993], [Corsini and Leoreanu, 2003], [Vougiouklis, 1987], [Vougiouklis, 1994] :

Definition 2.3. *Let $(G, \cdot)$ be groupoid then for every $P \subset G$, $P \neq \varnothing$, we define the following hopes called* ***P-hopes****: $\forall x, y \in G$*

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \ \ \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G, \underline{P})$,$(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called ***P-****hyperstructures. If $(G, \cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is a semihypergroup.*

3 Representations and Applications

H_v-structures used in Representation (abbr. rep) Theory of H_v-groups can be achieved by generalized permutations [Vougiouklis, 1992] or by H_v-matrices [Vougiouklis, 1985], [Vougiouklis, 1994], [Vougiouklis, 1999b].

$\mathbf{H_v}$**-matrix** is called a matrix if has entries from an H_v-ring. The hyperproduct of H_v-matrices (a_{ij}) and (b_{ij}), of type $m \times n$ and $n \times r$, respectively, is defined in the usual manner, and it is a set of $m \times r$ H_v-matrices. The sum of products of elements of the H_v-ring is the *n-ary circle hope* on the hyper-sum.

The problem of the H_v-matrix (or h/v-group) reps is the following:

Definition 3.1. *Let $(H, \cdot)$ be H_v-group. Find an H_v-ring $(R, +, \cdot)$, a set M_R=$\{(a_{ij})|a_{ij}{\in}R\}$, and a map $\boldsymbol{T} : H \to \boldsymbol{M}_R : h \mapsto T(h)$ such that*

$$T(h_1h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H.$$

T is an $\mathbf{H_v}$***-matrix rep****. If $T(h_1h_2) \subset T(h_1)T(h_2), \forall h_1, h_2 \in H$, then $\boldsymbol{T}$ is an inclusion rep. If $T(h_1h_2) = T(h_1)T(h_2), \forall h_1, h_2 \in H$, then $\boldsymbol{T}$ is a good rep. If T is a good rep and one to one then it is a faithful rep.*

The rep problem is simplified in cases such as if the h/v-rings have scalars 0 and 1. The main theorem of the theory of reps is the following:

Theorem 3.1. *A necessary condition in order to have an inclusion rep T of an h/v-group $(H, \cdot)$ by $n \times n$ h/v-matrices over the h/v-ring $(R, +, \cdot)$ is the following: $\forall \beta^*(x)$, $x \in H$ there must exist elements $a_{ij} \in H, i, j \in \{1, ..., n\}$ such that*

$$T(\beta^*(a)) \subset \{A = (a'_{ij})|a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, ..., n\}\}$$

The inclusion rep $T : H \to M_R : a \mapsto T(a) = (a_{ij})$ induces an homomorphic T^* of H/β^* on R/γ^* by $T^*(\beta^*(a)) = [\gamma^*(a_{ij})], \beta^*(a)H/\beta^*$, where $\gamma^*(a_{ij})R/\gamma^*$ is the ij entry of $T^*(\beta^*(a))$. T^* is called ***fundamental induced rep*** *of T.*

In reps we need small H_v-fields with results of few elements.

An important hope on non-square matrices is defined [Vougiouklis, 2009], [Vougiouklis and Vougiouklis, 2005]:

Definition 3.2. *Let $A = (a_{ij}) \in M_{m\times n}$ and $s, t \in N$, such that $1 \leq s \leq m$, $1 \leq t \leq n$. Define a mod-like map $\underline{st}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{st} = (\underline{a}_{ij})$ with entries the sets*

$$\underline{a}_{ij} = \{a_{i+\kappa s, j+\lambda t} | 1 \leq i \leq s, 1 \leq j \leq t \text{ and } \kappa, \lambda \in N, i + \kappa s \leq m, j + \lambda t \leq n\}.$$

*The map $\underline{st} : M_{m\times n} M_{s\times t} : A \to A\underline{st}(\underline{a}_{ij})$, is called **helix-projection** of type $\underline{st}$. $A\underline{st}$ is a set of $s \times t$-matrices $X = (x_{ij})$ such that $x_{ij} \in a_{ij}, \forall i, j$. Obviously $A\underline{mn} = A$.*

Let$A = (a_{ij}) \in M_{m\times n}$ and $s, t \in N$, $1 \leq s \leq m$, $1 \leq t \leq n$. We apply the helix-projection first on the columns and then on the rows and the result is the same: $(A\underline{sn})\underline{st} = (A\underline{mt})\underline{st} = A\underline{st}$.

Definition 3.3. *Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{u\times v}$ be matrices. Denote s=min(m,u), t=min(n,u), then we define the **helix-sum** by*

$$\oplus : M_{m\times n} M_{u\times v} P(M_{s\times t}) :$$

$$(A, B) \to A \oplus B = A\underline{st} + B\underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{s\times t},$$

where $(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) | a_{ij} \in \underline{a}_{ij}$ and $b_{ij} \in \underline{b}_{ij}\}$. Denote s=min(n,u), then we define the helix-product by

$$\otimes : M_{m\times n} M_{u\times v} P(M_{s\times t}) :$$

$$(A, B) \to A \otimes B = A\underline{ms} + B\underline{sv} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{m\times v},$$

where $(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = \sum (a_{ij} + b_{ij}) | a_{ij} \in \underline{a}_{ij}$ and $b_{ij} \in \underline{b}_{ij}\}$..

Remark. The definition of the Lie-bracket is immediate, therefore the helix-Lie Algebra is defined, as well.

Last decades H_v-structures have applications in mathematics and in other sciences. Applications range from biology and hadronic physics or leptons to mention but a few. The hyperstructure theory is related to fuzzy one; consequently, can be widely applicable in industry and production, too [Corsini and Leoreanu, 2003], [Davvaz and Leoreanu, 2007], [Davvaz et al., 2015], [Davvaz and Vougiouklis, 2018], [Santilli and Vougiouklis, 1996], [Vougiouklis, 2014a], [Vougiouklis, 2020], [Vougiouklis and Kambaki-Vougioukli, 2013].

An application, which combines H_v-hyperstructures and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis **(V & V bar)** [Vougiouklis and Kambaki-Vougioukli, 2013]. They suggest the following:

Definition 3.4. *In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:*

0 ________________ 1

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question.

The use of V $\&$ V bar bar instead of a Likert scale has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm.

The Lie-Santilli theory on isotopies was born to solve Hadronic Mechanics problems. Santilli proposed a 'lifting' of the n-dimensional trivial unit matrix into an appropriate new matrix. The original theory is reconstructed to admit the new matrix as left and right unit. The **isofields** needed in this theory correspond into the hyperstructures called e-hyperfields, introduced by [Santilli and Vougiouklis, 1996, Davvaz et al., 2015].

Definition 3.5. *A hyperstructure $(H, \cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse x^{-1}, i.e. $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$.*

Definition 3.6. *A hyperstructure $(F, +, \cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called **e-hyperfield** if the following axioms are valid: $(F, +)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$, there exist a multiplicative scalar unit 1, i.e. $1 \cdot x = x \cdot 1 = x, \forall x \in F$, and for all $x \in F$ there exists a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.*

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we say that we have a *strong e-hyperfield*.

Definition 3.7. *The Main e-Construction. Given a group $(G, \cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ as follows:*

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, ... \in G - \{e\}$$

g_1, g_2,... are not necessarily the same for each pair (x,y). Then $(G, \otimes)$ becomes an H_v-group, actually is an H_b-group which contains the $(G, \cdot)$. The H_v-group $(G, \otimes)$ is an e-hypergroup.

Example. Consider the quaternions $\mathbf{Q} = \{1, -1, i, -i, j, -j, k, -k\}$ with $i^2 = j^2 = -1, ij = -ji = k$ and denote $\underline{i} = \{i, -i\}, \underline{j} = \{j, -j\}, \underline{k} = \{k, -k\}$. We define a lot of hopes $(*)$ by enlarging few products. For example, $(-1) * k = \underline{k}, k * i = \underline{j}$ and $i * j = \underline{k}$. Then $(Q, *)$ is strong e-hypergroup.

A **generalization of P-hopes** used in Santilli's isotheory, is [Davvaz et al., 2015], [Vougiouklis, 2016] : Let $(G, \cdot)$ be abelian group, $P \subset G$ with $\#P < 1$. We define the hope $\times_p$ as follows:

$$x \times_p y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \text{if } x \neq e \text{ and } c \neq e \\ x \cdot y & \text{if } x = e \text{ and } y = e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G, \times_p)$ is abelian H_v-group.

4 Small hypernumbers. Minimal h/v-fields

The small non-degenerate h/v-fields on $(\mathbf{Z}_n, +, \cdot)$ in iso-theory, satisfy the following:

1. very thin minimal,
2. COW (non-commutative),
3. they have the elements 0 and 1, scalars,
4. if an element has inverse element, this is unique.

Therefore, we cannot enlarge the result if it is 1 and we cannot put 1 in enlargement.

Theorem 4.1. *[Vougiouklis, 2017] All multiplicative h/v-fields defined on $(\mathbf{Z}_4, +, \cdot)$, with non-degenerate fundamental field, satisfying the above 4 conditions, are the following isomorphic cases: The only product which is set is $2 \otimes 3 = \{0, 2\}$ or $3 \otimes 2 = \{0, 2\}$. Fundamental classes: [0]=0,2, [1]=1,3 and we have*

$$(\mathbf{Z}_4, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot)$$

Example. Denote E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries. Take the 2×2 upper triangular h/v-matrices on the above h/v-field $(\mathbf{Z}_4, +, \otimes)$ of the case that only 2⊗3={0,2} is a hyperproduct:

$I = E_{11} + E_{22}, a = E_{11} + E_{12} + E_{22}, b = E_{11} + 2E_{12} + E_{22}, c = E_{11} + 3E_{12} + E_{22},$

$d = E_{11} + 3E_{22}, e = E_{11} + E_{12} + 3E_{22}, f = E_{11} + 2E_{12} + 3E_{22}, g = E_{11} + 3E_{12} + 3E_{22},$

then, we obtain for X={I,a,b,c,d,e,f,g}, that $(X,\otimes)$ is non-COW, H_v-group where the fundamental classes are $\underline{a}=\{a,c\},\underline{d}=\{d,f\},\underline{e}=\{e,g\}$ and the fundamental group is isomorphic to $(\mathbf{Z}_2\times\mathbf{Z}_2,+)$. There is only one unit and every element has unique double inverse. Only f has one more right inverse element d, since $f\otimes d=\{I,b\}$. $(X,\otimes)$ is not cyclic.

Theorem 4.2. *All multiplicative h/v-fields on $(\mathbf{Z}_6,+,\cdot)$, with non-degenerate fundamental field, satisfying the above 4 conditions, are the following isomorphic cases: We have the only one hyperproduct,*

(I) $2\otimes3=\{0,3\},2\otimes4=\{2,5\},3\otimes4=\{0,3\},3\otimes5=\{0,3\},4\otimes5=\{2,5\}$. *The fundamental classes are [0]=0,3, [1]=1,4, [2]=2,5 and we have*

$$(\mathbf{Z}_6,+,\otimes)/\gamma^*\cong(\mathbf{Z}_3,+,\cdot).$$

(II) $2\otimes3=\{0,2\}$ *or* $2\otimes3=\{0,4\}$, $2\otimes4=\{0,2\}$ *or* $\{2,4\},2\otimes5=\{0,4\}$ *or* $2\otimes5=\{2,4\}$, $3\otimes4=\{0,2\}$ *or* $\{0,4\},3\otimes5=\{3,5\},4\otimes5=\{0,2\}$ *or* $\{2,4\}$. *In all these cases the fundamental classes are [0]=0,2,4, [1]=1,3,5 and we have*

$$(\mathbf{Z}_6,+,\otimes)/\gamma^*\cong(\mathbf{Z}_2,+,\cdot).$$

Example. In the h/v-field $(\mathbf{Z}_6,+,\otimes)$ where only the hyperproduct is $2\otimes4=\{2,5\}$ take the h/v-matrices of type $\underline{i}=E_{11}+iE_{12}+4E_{22}$, where i=0,1,...,5, then the multiplicative table of the hyperproduct of those h/v-matrices is

$\otimes$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{1}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{2}$	$\underline{2}$	$\underline{0,3}$	$\underline{1,4}$	$\underline{2,5}$	$\underline{0,3}$	$\underline{1,4}$
$\underline{3}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{4}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{5}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$

The fundamental classes are $(0)=\underline{0,3},(1)=\underline{1,4},(2)=\underline{2,5}$ and the fundamental group is isomorphic to $(\mathbf{Z}_3,+)$. The $(\mathbf{Z}_6,\otimes)$ is h/v-group which is cyclic where 2 and 4 are generators of period 4.

Example. Consider the h/v-field $(\mathbf{Z}_10,+,\otimes)$ where only $3\times8=\{4,9\}$ is a hyperproduct. Let us take the h/v-matrix

$$A=3E_{11}+E_{22}+2E_{33}+6E_{12}+2E_{13}+9E_{23}$$

Then from the above formulas we obtain that the set of inverse h/v-matrices is

$$A^{-1}=[2]E_{11}+[1]E_{22}+[3]E_{33}+[3]E_{12}+[2]E_{13}+[3]E_{23}$$

So, for example, if we take the h/v-matrix

$$A^{-1} = 7E_{11} + 6E_{22} + 8E_{33} + 8E_{12} + 2E_{13} + 3E_{23}$$

we obtain that

$$A \cdot A^{-1} = E_{11} + E_{22} + E_{33} + \{0, 5\}E_{12} + 5E_{23}$$

therefore, it contains a unit h/v-matrix.

Theorem 4.3. *All multiplicative h/v-fields defined on* $(\mathbf{Z}_9, +, \cdot)$*, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases: We have the only one hyperproduct,* $2 \otimes 3 = 0, 6$ *or* $3, 6, 2 \otimes 4 = 2, 8$ *or* $5, 8, 2 \otimes 6 = 0, 3$ *or* $3, 6, 2 \otimes 7 = 2, 5$ *or* $5, 8, 2 \otimes 8 = 1, 7$ *or* $4, 7, 3 \otimes 4 = 0, 3$ *or* $3, 6, 3 \otimes 5 = 0, 6$ *or* $3, 6, 3 \otimes 6 = 0, 3$ *or* $0, 6, 3 \otimes 7 = 0, 3$ *or* $3, 6, 3 \otimes 8 = 0, 6$ *or* $3, 6, 4 \otimes 5 = 2, 5$ *or* $2, 8, 4 \otimes 6 = 0, 6$ *or* $3, 6, 4 \otimes 8 = 2, 5$ *or* $5, 8, 5 \otimes 6 = 0, 3$ *or* $3, 6, 5 \otimes 7 = 2, 8$ *or* $5, 8, 5 \otimes 8 = 1, 4$ *or* $4, 7, 6 \otimes 7 = 0, 6$ *or* $3, 6, 6 \otimes 8 = 0, 3$ *or* $3, 6, 7 \otimes 8 = 2, 5$ *or* $2, 8$*. In all the above cases the fundamental classes are*

$[0] = \{0, 3, 6\}, [1] = \{1, 4, 7\}, [2] = \{2, 5, 8\}, and we have (\mathbf{Z}_9, +, \otimes)/\gamma^* \cong (\mathbf{Z}_3, +, \cdot).$

Theorem 4.4. *All multiplicative h/v-fields on* $(\mathbf{Z}_{10}, +, \cdot)$*, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:*

(I) We have the only one hyperproduct,
$2 \otimes 4 = \{3, 8\}, 2 \otimes 5 = \{0, 5\}, 2 \otimes 6 = \{2, 7\}, 2 \otimes 7 = \{4, 9\},$
$2 \otimes 9 = \{3, 8\}, 3 \otimes 4 = \{2, 7\}, 3 \otimes 5 = \{0, 5\}, 3 \otimes 6 = \{3, 8\}, 3 \otimes 8 = \{4, 9\},$
$3 \otimes 9 = \{2, 7\}, 4 \otimes 5 = \{0, 5\}, 4 \otimes 6 = \{4, 9\},$
$4 \otimes 7 = \{3, 8\}, 4 \otimes 8 = \{2, 7\}, 5 \otimes 6 = \{0, 5\}, 5 \otimes 7 = \{0, 5\},$
$5 \otimes 8 = \{0, 5\}, 5 \otimes 9 = \{0, 5\}, 6 \otimes 7 = \{2, 7\}, 6 \otimes 8 = \{3, 8\},$
$6 \otimes 9 = \{4, 9\}, 7 \otimes 9 = \{3, 8\}, 8 \otimes 9 = \{2, 7\}.$

In all the above cases the fundamental classes are $[0] = \{0, 3, 6\}, [1] = \{1, 4, 7\}, [2] = \{2, 5, 8\}$,
and we have

$$(\mathbf{Z}_9, +, \otimes)/\gamma^* \cong (\mathbf{Z}_3, +, \cdot).$$

(II) The cases with classes $[0] = \{0, 2, 4, 6, 8\}$ *and* $[1] = \{1, 3, 5, 7, 9\}$*, and fundamental field*

$$(\mathbf{Z}_{10}, +, \otimes)/\gamma^* \cong (\mathbf{Z}_2, +, \cdot).$$

are described as follows: In the multiplicative table only the results above the diagonal, we enlarge each of the products by putting one element of the same class of the results. We do not enlarge setting 1, and we cannot enlarge only the $3 \otimes 7 = 1$*. The number of those h/v-fields is 103.*

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