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Uniqueness of an entire function sharing fixed points with its derivatives

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Abstract

The uniqueness problems of an entire functions that share a nonzero finite value have been studied and many results on this topic have been obtained. In this paper we prove a uniqueness theorem for an entire function, which share a linear polynomial, in particular fixed points, with its higher order derivatives.
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1 Introduction, Definitions and Results

Let f be a non-constant meromorphic function in the open complex plane $\mathbb{C}$. A meromorphic function $a = a(z)$ is called a small function of f if $T(r, a) = S(r, f)$, where $T(r, f)$ is the Nevanlinna characteristic function of f and $S(r, f) = \circ\{T(r, f)\}$, as $r \to \infty$, possibly outside a set of finite linear measure.

Let f and g be two non-constant meromorphic functions and $a = a(z)$ be a polynomial. We say that f and g share a CM if $f - a$ and $g - a$ have the same zeros with same multiplicities. On the other hand, we say that f and g share a IM if $f - a$ and $g - a$ have the same zeros ignoring multiplicities. We express the CM sharing and IM sharing respectively by the notations $f = a \rightleftharpoons g = a$ and $f = a \Leftrightarrow g = a$.

Let $z_k (k = 1, 2, \ldots)$ be zeros of $f - a$ and t_k be the multiplicity of the zero z_k. If $z_k (k = 1, 2, \ldots)$ are also zeros of $g - a$ and the multiplicity of the zero z_k is at least t_k then we use the notation $f = a \to g = a$.

For standared definitions and notations of the distribution theory we refer the reader to Hayman [1964].

The problem of uniqueness of meromorphic functions sharing values with their derivatives is a special case of the uniqueness theory of meromorphic function. There are some results related to value sharing.

In the begining, Jank, Mues and Volkmann Jank et al. [1986] considered the situation that an entire function shares a nonzero value with its derivatives and they prove the following result.

Theorem A. *Jank et al. [1986]. Let f be a non-constant entire function and a be a non-zero finite value. If f, $f^{(1)}$ and $f^{(2)}$ share a CM, then $f \equiv f^{(1)}$.*

Following example shows that in Theorem A the second derivative cannot be replaced by any higher order derivatives.

Example 1.1. *Let $k(\geq 3)$ be an integer and $\omega(\neq 1)$ is a $(k-1)^{th}$ root of unity. We put $f = e^{\omega z} + \omega - 1$. Then f, $f^{(1)}$ and $f^{(k)}$ share the value ω CM, but $f \not\equiv f^{(1)}$.*

On the basis of this example, Zhong improved Theorem A by considering higher order derivetives in the following way.

Theorem B. *Let f be a non-constant entire function and a be a non-zero finite number. Also let $n(\geq 1)$ be a positive integer. If f and $f^{(1)}$ share the value a CM, and if $f^{(n)}(z) = f^{(n+1)}(z) = a$ whenever $f(z) = a$, then $f \equiv f^{(n)}$.*

In 2002, Chang and Fang [2002] extendeed Theorem A by considering shared fixed points.

Theorem C. *Chang and Fang [2002]. Let f be a non-constant entire function. If f, $f^{(1)}$ and $f^{(2)}$ share z CM, then $f \equiv f^{(1)}$.*

Later in 2003, Wang and Yi [2003] improved Theorem A and generalize Theorem B by considering higher order derivatives in the following way.

Theorem D. *Wang and Yi [2003]. Let f be a non-constant entire function and a be a non-zero finite constant. Also let m and n be positive integers satisfying $m > n$. If f and $f^{(1)}$ share the value a CM, and if $f^{(m)}(z) = f^{(n)}(z) = a$ whenever $f(z) = a$, then*

$$f(z) \quad = \quad Ae^{\lambda z} + a - \frac{a}{\lambda},$$

where $A(\neq 0)$ and λ are constants satisfying $\lambda^{n-1} = 1$ and $\lambda^{m-1} = 1$.

In this paper we improve Theorem D by considering the situation when a non-constant entire function f shares a linear plynomial $a(z) = \alpha z + \beta$, $\alpha(\neq 0)$ and β are constants, with higher order derivatives. The main result of the paper is the following theorem.

Theorem 1.1. *Let f be a non-constant entire function and $a(z) = \alpha z + \beta$ be a polynomial, where $\alpha(\neq 0)$ and β are constants. Also let m amd n be two positive integers satisfying $m > n > 1$. If*

$$f(z) = a(z) \rightleftharpoons f^{(1)}(z) = a(z)$$

and

$$f(z) = a(z) \to f^{(m)}(z) = f^{(n)}(z) = a(z),$$

then

$$f(z) = Ce^{z}$$

or

$$f(z) \quad = \quad Ce^{\lambda z} + a(z) - \frac{a(z)}{\lambda} + \frac{\alpha(1-\lambda)}{\lambda^2},$$

where C and λ are non-zero constants.

2 Lemmas

In this section we state some necessary lemmas.

Lemma 2.1. *Ngoan and Ostrovskii [1965]. Let f be an entire function of order at most 1 and k be a positive integer, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = o(\log r),$$

as $r \to \infty$.

The above lemma motivates us to prove the following:

Lemma 2.2. *Let f be an entire function of finite order and k be a positive integer. Then for any small function $a(z)$ with respect to $f(z)$,*

$$m\left(r, \frac{f^{(k)}(z) - a^{(k)}(z)}{f(z) - a(z)}\right) = o(\log r),$$

as $r \to \infty$.

Proof. Let $g(z) = f(z) - a(z)$. Then

$$g^{(k)}(z) = f^{(k)}(z) - a^{(k)}(z).$$

Now by Lemma 2.1 and using above equality, we have

$$m\left(r, \frac{g^{(k)}(z)}{g(z)}\right) = o(\log r),$$

as $r \to \infty$. This implies

$$m\left(r, \frac{f^{(k)}(z) - a^{(k)}(z)}{f(z) - a(z)}\right) = o(\log r),$$

as $r \to \infty$. This proves the lemma. □

Lemma 2.3. *Clunie [1962]. Let f be a transcendental meromorphic solution of the equation*

$$f^n P(f) = Q(f),$$

where $P(f)$ and $Q(f)$ are polynomials in f and its derivatives with meromorphic coefficients a_j (say). If the total degree of $Q(f)$ is at most n, then

$$m(r, P(f)) \leq \sum_j m(r, a_j) + S(r, f).$$

Lemma 2.4. *Chen and Li [2014]. Let $a(z)$ be an entire function of finite order and $Q(z)$ be a non-constant polynomial. If f is an entire solution of the equation*

$$f^{(k)} - e^{Q(z)} f = a(z)$$

such that $\rho(f) > \rho(a)$, then $\rho(f) = \infty$.

We use this Lemma to prove the following one.

Lemma 2.5. *Let f be a non-constant entire function of finite order and $a(z) = \alpha z + \beta$ be a polynomial, where $\alpha(\neq 0)$ and β are constant. Also let k be a positive integer. If $f(z)$ and $f^{(k)}(z)$ share $a(z)$ CM, then*

$$\frac{f^{(k)}(z) - a(z)}{f(z) - a(z)} \equiv c, \tag{2.1}$$

for some nonzero constant c.

Proof. Since f has finite order and since $f(z)$ and $f^{(k)}(z)$ share $a(z)$ CM, it follows from the Hadamard factorization theorem that

$$\frac{f^{(k)}(z) - a(z)}{f(z) - a(z)} \equiv e^{Q(z)}, \tag{2.2}$$

where $Q(z)$ is a polynomial.

Suppose that $F(z) = f(z) - a(z)$. Then $F^{(k)}(z) = f^{(k)}(z)$.

From (2.2) and above equality, we have

$$F^{(k)}(z) - e^{Q(z)} F(z) = a(z).$$

If $Q(z)$ is non-constant, then from above equality and by Lemma 2.4, we get F has infinite order. Since f has finite order, this is impossible. Hence $Q(z)$ is a constant. Therefore from (2.2), we obtain (2.1) for a non-zero constant c. This proves the lemma. □

Lemma 2.6. *Let f be a transcendental entire function of finite order and $a(z) = \alpha z + \beta$ be a polynomial, where $\alpha(\neq 0)$ and β are constants. Also let m be a positive integer. If*

(i) $m\left(r, \frac{1}{f(z)-a(z)}\right) = S(r, f)$,

(ii) $f(z) = a(z) \rightleftharpoons f^{(1)}(z) = a(z)$

and

(iii) $f(z) = a(z) \to f^{(m)}(z) = a(z)$,

then

$$f(z) = Ce^{z},$$

where C is a non-zero constant.

Proof. Let

$$h(z) = \frac{f^{(1)}(z)-a(z)}{f(z)-a(z)}. \tag{2.3}$$

Since $f(z)$ and $f^{(1)}(z)$ share $a(z)$ CM, we see that $h(z)$ is an entire function.

Now by Lemma 2.1, Lemma 2.2 and from the hypothesis of Lemma 2.6, we deduce that

$$\begin{aligned} T(r,h(z)) &= m(r,h(z)) \\ &= m\left(r,\frac{f^{(1)}(z)-a(z)}{f(z)-a(z)}\right) \\ &\leq m\left(r,\frac{f^{(1)}(z)-a^{(1)}(z)}{f(z)-a(z)}\right)+m\left(r,\frac{a^{(1)}(z)-a(z)}{f(z)-a(z)}\right)+\log 2 \\ &= S(r,f). \end{aligned} \tag{2.4}$$

We rewrite (2.3), as

$$\begin{aligned} f^{(1)}(z) &= h(z)f(z)+a(z)(1-h(z)) \\ &= \xi_1(z)f(z)+\eta_1(z), \end{aligned} \tag{2.5}$$

where $\xi_1(z)$ and $\eta_1(z)$ are defined by

$$\xi_1(z)=h(z),\ \ \eta_1(z)=a(z)(1-h(z)).$$

By (2.5), we have

$$\begin{aligned} f^{(2)}(z) &= \xi_1(z)f^{(1)}(z)+\xi_1^{(1)}(z)f(z)+\eta_1^{(1)}(z) \\ &= \xi_1(z)[\xi_1(z)f(z)+\eta_1(z)]+\xi_1^{(1)}(z)f(z)+\eta_1^{(1)}(z) \\ &= [\xi_1^{(1)}(z)+\xi_1(z)\xi_1(z)]f(z)+\eta_1^{(1)}(z)+\eta_1(z)\xi_1(z) \\ &= \xi_2(z)f(z)+\eta_2(z), \end{aligned}$$

where

$$\xi_2(z)=\xi_1^{(1)}(z)+\xi_1(z)\xi_1(z)\ \ and\ \ \eta_2(z)=\eta_1^{(1)}(z)+\eta_1(z)\xi_1(z).$$

Now from above equality and using (2.5), we get

$$\begin{aligned} f^{(3)}(z) &= \xi_2(z)f^{(1)}(z)+\xi_2^{(1)}(z)f(z)+\eta_2^{(1)}(z) \\ &= [\xi_2^{(1)}(z)+\xi_1(z)\xi_2(z)]f(z)+\eta_2^{(1)}(z)+\eta_1(z)\xi_2(z) \\ &= \xi_3(z)f(z)+\eta_3(z), \end{aligned}$$

where

$$\xi_3(z) = \xi_2^{(1)}(z) + \xi_1(z)\xi_2(z) \;\; and \;\; \eta_3(z) = \eta_2^{(1)}(z) + \eta_1(z)\xi_2(z).$$

Similarly,

$$f^{(k)}(z) = \xi_k(z)f(z) + \eta_k(z), \tag{2.6}$$

where

$$\xi_{k+1}(z) = \xi_k^{(1)}(z) + \xi_1(z)\xi_k(z) \tag{2.7}$$

and

$$\eta_{k+1}(z) = \eta_k^{(1)}(z) + \eta_1(z)\xi_k(z). \tag{2.8}$$

Puting $k = 1$ in (2.7), we have

$$\begin{aligned} \xi_2(z) &= \xi_1^{(1)}(z) + \xi_1(z)\xi_1(z) \\ &= h^2(z) + h^{(1)}(z). \end{aligned}$$

Again puting $k = 2$ in (2.7), we get

$$\begin{aligned} \xi_3(z) &= \xi_2^{(1)}(z) + \xi_1(z)\xi_2(z) \\ &= \left[h^2(z) + h^{(1)}(z)\right]^{(1)} + h(z)[h^2(z) + h^{(1)}(z)] \\ &= h^3(z) + h^{(2)}(z) + 3h(z)h^{(1)}(z). \end{aligned}$$

Similarly,

$$\xi_4(z) = h^4(z) + h^{(3)}(z) + 4h(z)h^{(2)}(z) + 3\left[2h^2(z) + h^{(1)}(z)\right] h^{(1)}(z).$$

Hence using mathematical induction, one can easily check

$$\xi_k(z) = h^k(z) + P_{k-1}(z, h(z)), \tag{2.9}$$

where $P_{k-1}(z, h(z))$ is a polynomial such that total degree $deg P_{k-1}(z, h(z)) \leq k-1$ in $h(z)$ and its derivatives, and all coefficients in $P_{k-1}(z, h(z))$ are constants.

Now putting $k = 1$ in (2.8), we have

$$\begin{aligned} \eta_2(z) &= \eta_1^{(1)}(z) + \eta_1(z)\xi_1(z) \\ &= [a(z)(1 - h(z))]^{(1)} + a(z)(1 - h(z))h(z) \\ &= -a(z)h^2(z) - a(z)h^{(1)}(z) + (a(z) - \alpha)h(z) + \alpha. \end{aligned}$$

Again putting $k = 2$ in (2.8), we get

$$\begin{aligned}
\eta_3(z) &= \eta_2^{(1)}(z) + \eta_1(z)\xi_2(z) \\
&= \left[-a(z)h^2(z) - a(z)h^{(1)}(z) + (a(z)-\alpha)h(z) + \alpha\right]^{(1)} \\
&\quad + a(z)(1-h(z))(h^2(z) + h^{(1)}(z)) \\
&= -a(z)h^3(z) - a(z)h^{(2)}(z) + \left[2a(z) - 3a(z)h(z) - 2\alpha\right] h^{(1)}(z) \\
&\quad + (a(z)-\alpha)h^2(z) + \alpha h(z).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&= -a(z)h^4(z) - a(z)h^{(3)}(z) + \left[3a(z) - 4a(z)h(z) - 3\alpha\right] h^{(2)}(z) \\
&\quad + \left[5a(z)h(z) - 5\alpha h(z) - 6a(z)h^2(z) - 3a(z)h^{(1)}(z) + 3\alpha\right] h^{(1)}(z) \\
&\quad + (a(z)-\alpha)h^3(z) + \alpha h^2(z).
\end{aligned}$$

Like the previous one, it can be easily verified that

$$\eta_k(z) = -a(z)h^k(z) + Q_{k-1}(z, h(z)), \tag{2.10}$$

where $Q_{k-1}(z, h(z))$ is a polynomial such that total degree $degQ_{k-1}(z, h(z)) \leq k-1$ in $h(z)$ and its derivatives, and all coefficients in $Q_{k-1}(z, h(z))$ are either constants or polynomial $a(z)$.

From (2.4) and (2.9), for $k = 1, 2, \cdots$, we have

$$\begin{aligned}
T(r, \xi_k(z)) &= T(r, h^k(z) + P_{k-1}(z, h(z))) \\
&\leq T(r, h^k(z)) + T(r, P_{k-1}(z, h(z))) + \log 2 \\
&= S(r, f).
\end{aligned}$$

Similarly,

$$T(r, \eta_k(z)) = S(r, f).$$

From hypothesis of Lemma 2.6, we have

$$\begin{aligned}
N\left(r, \frac{1}{f(z)-a(z)}\right) &= T(r, f(z)) - m\left(r, \frac{1}{f(z)-a(z)}\right) + O(1) \\
&= T(r, f(z)) + S(r, f), \qquad (2.11)
\end{aligned}$$

which implies that $f(z) - a(z)$ must have zeros.

Let z_j be a zero of $f(z) - a(z)$ with multiplicity $\delta(j)$. Since $f(z) = a(z) \rightarrow f^{(m)}(z) = a(z)$, we see that z_j is also a zero of $f^{(m)}(z) - a(z)$ with multiplicity at least $\delta(j)$. Hence $f(z_j) = a(z_j)$ and $f^{(m)}(z_j) = a(z_j)$.

It follows from (2.6) that, for $k = m$,

$$f^{(m)}(z) = \xi_m(z)f(z) + \eta_m(z) \tag{2.12}$$

and then

$$a(z_j) \quad = \quad a(z_j)\xi_m(z_j) + \eta_m(z_j).$$

Now we shall prove that,

$$a(z) \equiv a(z)\xi_m(z) + \eta_m(z). \tag{2.13}$$

Otherwise,

$$a(z)\xi_m(z) + \eta_m(z) - a(z) \not\equiv 0.$$

From (2.12), we have

$$a(z)\xi_m(z) + \eta_m(z) - a(z) \quad = \quad (f^{(m)}(z) - a(z)) - \xi_m(z)(f(z) - a(z)).$$

By the reasoning as mentioned above, we deduce that z_j is a zero of $(f^{(m)}(z) - a(z)) - \xi_m(z)(f(z) - a(z))$, that is, a zero of $a(z)\xi_m(z) + \eta_m(z) - a(z)$ with multiplicity at least $\delta(j)$. It follows from this and the fact that $\xi_m(z)$ and $\eta_m(z)$ are small functions of $f(z)$,

$$\begin{aligned} N\left(r, \frac{1}{f(z) - a(z)}\right) &\leq N\left(r, \frac{1}{a(z)\xi_m(z) + \eta_m(z) - a(z)}\right) \\ &\leq T\left(r, \frac{1}{a(z)\xi_m(z) + \eta_m(z) - a(z)}\right) \\ &= S(r, f), \end{aligned}$$

which contradicts (2.11). Thus

$$a(z) \quad \equiv \quad a(z)\xi_m(z) + \eta_m(z),$$

which is (2.13).

Now by induction we prove that

$$\eta_{k+1}(z) + a(z)\xi_{k+1}(z) \quad = \quad (a(z) - \alpha)h^k(z) + R_{k-1}(z, h(z)), \tag{2.14}$$

where $R_{k-1}(z, h(z))$ is a polynomial such that $degR_{k-1}(z, h(z)) \leq k - 1$ in $h(z)$ and its derivatives, and all the coefficients in $R_{k-1}(z, h(z))$ are constants or polynomial $a(z)$.

Firstly, from (2.7), (2.8) and for $k=1$, we have

$$\begin{aligned}
\eta_2(z)+a(z)\xi_2(z) &= \eta_1^{(1)}(z)+\eta_1(z)\xi_1(z)+a(z)\left[\xi_1^{(1)}(z)+\xi_1(z)\xi_1(z)\right]\\
&= [a(z)(1-h(z))]^{(1)}+a(z)(1-h(z))h(z)+a(z)h^{(1)}(z)\\
&\quad +a(z)h^2(z)\\
&= a(z)(-h^{(1)}(z))+\alpha(1-h(z))+a(z)h(z)-a(z)h^2(z)\\
&\quad +a(z)h^{(1)}(z)+a(z)h^2(z)\\
&= (a(z)-\alpha)h(z)+\alpha.
\end{aligned}$$

Secondly, we suppose that the following equation holds

$$\eta_k(z)+a(z)\xi_k(z) = (a(z)-\alpha)h^{k-1}(z)+R_{k-2}(z,h(z)).$$

Now, by (2.7)–(2.10), we deduce that

$$\begin{aligned}
\eta_{k+1}(z)+a(z)\xi_{k+1}(z) &= \eta_k^{(1)}(z)+\eta_1(z)\xi_k(z)+a(z)(\xi_k^{(1)}(z)+\xi_1(z)\xi_k(z))\\
&= \left[-a(z)h^k(z)+Q_{k-1}(z,h(z))\right]^{(1)}+a(z)(1-h(z))\xi_k(z)\\
&\quad +a(z)\left[h^k(z)+P_{k-1}(z,h(z))\right]^{(1)}+a(z)h(z)\xi_k(z)\\
&= -ka(z)h^{k-1}(z)-\alpha h^k(z)+[Q_{k-1}(z,h(z))]^{(1)}+a(z)\xi_k(z)\\
&\quad -a(z)h(z)\xi_k(z)+ka(z)h^{k-1}(z)+a(z)\left[P_{k-1}(z,h(z))\right]^{(1)}\\
&\quad +a(z)h(z)\xi_k(z)\\
&= a(z)\left[h^k(z)+(P_{k-1}(z,h(z)))\right]-\alpha h^k(z)+[Q_{k-1}(z,h(z))]^{(1)}\\
&\quad +a(z)\left[P_{k-1}(z,h(z))\right]^{(1)}\\
&= (a(z)-\alpha)h^k(z)+R_{k-1}(z,h(z)),
\end{aligned}$$

which proves (2.14).

From (2.13) and (2.14), we obtain

$$(a(z)-\alpha)h^{m-1}(z)+R_{k-2}(z,h(z))\equiv a(z). \tag{2.15}$$

Clearly, $R_{k-2}(z,h(z))\not\equiv a(z)$. Othewise, from (2.3), (2.15) and the hypothesis of Lemma 2.6, we have a contradiction. Hence by Lemma 2.3 and from (2.15), we can deduce that $h(z)$ must be constant.

From (2.7) and $\xi_1(z)=h(z)$, we have

$$\xi_2(z)=h^2(z),\ \ \xi_3(z)=h^3(z),\ \ \xi_4(z)=h^4(z).$$

Similarly,

$$\xi_k(z)=h^k(z),\ \ for\ \ k=1,2,\ldots. \tag{2.16}$$

Now, from (2.8) and $\eta_1(z) = a(z)(1 - h(z))$, we get

$$\begin{aligned} \eta_2(z) &= (1 - h(z))(\alpha + a(z)h(z)), \\ \eta_3(z) &= (1 - h(z))(\alpha + a(z)h(z))h(z), \\ \eta_4(z) &= (1 - h(z))(\alpha + a(z)h(z))h^2(z). \end{aligned}$$

Similarly,

$$\eta_k(z) = (1 - h(z))(\alpha + a(z)h(z))h^{k-2}(z), \ \ for \ \ k = 2, 3, \dots. \quad (2.17)$$

From (2.13), (2.16) and (2.17), we have

$$\begin{aligned} a(z) &\equiv a(z)h^m(z) + (1 - h(z))(\alpha + a(z)h(z))h^{m-2}(z) \\ &\equiv h^{m-2}(z)\left[a(z)h^2(z) + \alpha(1 - h(z)) + a(z)h(z) - a(z)h^2(z)\right] \\ &\equiv h^{m-2}(z)\left[a(z)h(z) + \alpha(1 - h(z))\right], \end{aligned}$$

which implies that $h(z) = 1$.

Hence from (2.3) and $h(z) = 1$, we can obtain

$$f^{(1)}(z) = f(z).$$

This implies

$$f(z) = Ce^z.$$

where $C(\neq 0)$ ia a constant. This proof the Lemma 2.6.

□

3 Proof of the theorem 1.1

First we verify that $f(z)$ cannot be a polynomial. Let $f(z)$ be a polynomial of degree 1. Suppose that $f(z) = A_1 z + B_1$, where $A_1(\neq 0)$ and B_1 are constants. Then $f^{(1)}(z) = A_1$, $f^{(m)}(z) \equiv 0 \equiv f^{(n)}(z)$. Now $\frac{\beta - B_1}{A_1 - \alpha}$ is the only zero of $f(z) - a(z)$, $\frac{A_1 - \beta}{\alpha}$ is the only zero of $f^{(1)}(z) - a(z)$ and $-\frac{\beta}{\alpha}$ is the only zero of $f^{(m)}(z) - a(z)$. Since $f(z)$ and $f^{(1)}(z)$ share polynomial $a(z)$ CM and the zeros of $f(z) - a(z)$ are the zeros of $f^{(m)}(z) - a(z)$, we have $\frac{A_1 - \beta}{\alpha} = -\frac{\beta}{\alpha}$ and so $A_1 = 0$, a contradiction.

Now let $f(z)$ be a polynomial of degree greater than 1. Suppose that deg$(f(z)) = p$. Then deg$(f(z) - a(z)) = p$ and deg$(f^{(1)}(z) - a(z)) = p - 1$, it contradicts the fact that $f(z)$ and $f^{(1)}(z)$ share polynomial $a(z)$ CM.

Hence $f(z)$ is a transcendental entire function. Thus $T(r, a(z)) = S(r, f)$.

To prove the theorem let us consider two functions defined as follows.

$$\Phi(z)=\frac{(a(z)-a^{(1)}(z))f^{(m)}(z)-a(z)(f^{(1)}(z)-a^{(1)}(z))}{f(z)-a(z)} \tag{3.1}$$

and

$$\Psi(z)=\frac{(a(z)-a^{(1)}(z))f^{(n)}(z)-a(z)(f^{(1)}(z)-a^{(1)}(z))}{f(z)-a(z)}. \tag{3.2}$$

Then $\Phi(z)\not\equiv\Psi(z)$.

We know from the hypothesis of Theorem 1.1 that $\Phi(z)$ and $\Psi(z)$ are entire functions. Then, by Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
T(r,\Phi(z)) &= m(r,\Phi(z))\\
&= m\left(r,\frac{(a(z)-a^{(1)}(z))f^{(m)}(z)-a(z)(f^{(1)}(z)-a^{(1)}(z))}{f(z)-a(z)}\right)\\
&\leq m\left(r,(a(z)-a^{(1)}(z))\frac{f^{(m)}(z)}{f(z)-a(z)}\right)+m\left(r,a(z)\frac{(f^{(1)}(z)-a^{(1)}(z))}{f(z)-a(z)}\right)\\
&\quad+\log 2\\
&= S(r,f).
\end{aligned}$$

Similarly,

$$T(r,\Psi(z))=S(r,f).$$

We shall the following three cases.

Case 1. First we suppose that $\Phi(z)\not\equiv 0$. Then by (3.1), we have

$$f(z)=a(z)+\frac{1}{\Phi(z)}\{(a(z)-a^{(1)}(z))f^{(m)}(z)-a(z)(f^{(1)}(z)-a^{(1)}(z))\}. \tag{3.3}$$

From (3.1) and (3.2), we get

$$f^{(1)}(z) = \frac{(a(z)-a^{(1)}(z))}{a(z)(\Phi(z)-\Psi(z))}(\Phi(z)f^{(n)}(z)-\Psi(z)f^{(m)}(z))+a^{(1)}(z).$$

Therefore

$$\begin{aligned}
f^{(1)}(z)-a(z) &= \frac{(a(z)-a^{(1)}(z))}{a(z)(\Phi(z)-\Psi(z))}(\Phi(z)f^{(n)}(z)-\Psi(z)f^{(m)}(z))\\
&\quad+a^{(1)}(z)-a(z).
\end{aligned} \tag{3.4}$$

First we suppose that $m > n > 2$. Then from (3.4), we get

$$\begin{aligned}\frac{1}{f^{(1)}(z)-a(z)} &= \frac{1}{a(z)(\Phi(z)-\Psi(z))}\frac{(\Phi(z)f^{(n)}(z)-\Psi(z)f^{(m)}(z))}{f^{(1)}(z)-a(z)}\\ &\quad+\frac{1}{a^{(1)}(z)-a(z)}. \end{aligned}\tag{3.5}$$

Using Lemma 2.1 and from (3.5), we have

$$m\left(r,\frac{1}{f^{(1)}(z)-a(z)}\right)=S(r,f). \tag{3.6}$$

Next we suppose $m > n = 2$. Then from (3.4), we get

$$\begin{aligned}(f^{(1)}(z)-a(z))(\Phi(z)-\Psi(z))a(z) &= \gamma(z)+(a(z)-a^{(1)}(z))\Phi(z)(f^{(2)}(z)-a^{(1)}(z))\\ &\quad-(a(z)-a^{(1)}(z))\Psi(z)f^{(m)}(z),\end{aligned}\tag{3.7}$$

where

$$\gamma(z)=(a(z)-a^{(1)}(z))(\Phi(z)a^{(1)}(z)-(\Phi(z)-\Psi(z))a(z)).$$

Clearly $\gamma(z)\not\equiv 0$.
If $\gamma(z)\equiv 0$, then

$$\Phi(z)\equiv\frac{a(z)}{a(z)-a^{(1)}(z)}\Psi(z),$$

which is a contradiction because $\Phi(z)$ and $\Psi(z)$ are entire functions and $\Psi(z)\neq 0$ when $a(z)-a^{(1)}(z)=0$.

Now from (3.7) we get

$$\begin{aligned}\frac{1}{f^{(1)}(z)-a(z)} &= \frac{(\Phi(z)-\Psi(z))a(z)}{\gamma(z)}-\frac{a(z)-a^{(1)}(z)}{\gamma(z)}\Phi(z)\frac{f^{(2)}(z)-a^{(1)}(z)}{f^{(1)}(z)-a(z)}\\ &\quad+\frac{a(z)-a^{(1)}(z)}{\gamma(z)}\Psi(z)\frac{f^{(m)}(z)}{f^{(1)}(z)-a(z)}.\end{aligned}\tag{3.8}$$

Again using Lemma 2.1 and from (3.8), we get

$$m\left(r,\frac{1}{f^{(1)}(z)-a(z)}\right)=S(r,f),$$

which is (3.6).

Since $\Phi(z) \not\equiv 0$, it follows from (3.3) and Lemma 2.1 that

$$
\begin{aligned}
T(r, f(z)) &= m(r, f(z)) \\
&= m\left(r, a(z) + \frac{1}{\Phi(z)}\{(a(z) - a^{(1)}(z))f^{(m)}(z) - a(z)(f^{(1)}(z) - a^{(1)}(z))\}\right) \\
&= m\left(r, a(z) + \frac{(a(z) - \alpha)f^{(m)}(z) - a(z)f^{(1)}(z) + a(z)\alpha}{\Phi(z)}\right) \\
&\leq m(r, a(z)) + m\left(r, \frac{(a(z) - \alpha)f^{(m)}(z) - a(z)f^{(1)}(z)}{\Phi(z)}\right) + m\left(r, \frac{\alpha a(z)}{\Phi(z)}\right) \\
&\quad + \log 3 \\
&= m\left(r, a(z)f^{(1)}(z)\frac{\frac{(a(z)-\alpha)}{a(z)}\frac{f^{(m)}(z)}{f^{(1)}(z)} - 1}{\Phi(z)}\right) + S(r, f) \\
&\leq m\left(r, \frac{\frac{(a(z)-\alpha)}{a(z)}\frac{f^{(m)}(z)}{f^{(1)}(z)} - 1}{\Phi(z)}\right) + m(r, f^{(1)}(z)) + S(r, f) \\
&\leq m\left(r, \frac{f^{(m)}(z)}{f^{(1)}(z)} - 1\right) + m(r, f^{(1)}(z)) + S(r, f) \\
&= T(r, f^{(1)}(z)) + S(r, f). \qquad (3.9)
\end{aligned}
$$

Applying Lemma 2.1, We can easily see that

$$
\begin{aligned}
T(r, f^{(1)}(z)) &= m(r, f^{(1)}(z)) \\
&= m\left(r, \frac{f^{(1)}(z)}{f(z)} \cdot f(z)\right) \\
&\leq m\left(r, \frac{f^{(1)}(z)}{f(z)}\right) + m(r, f(z)) \\
&= m(r, f(z)) + S(r, f) \\
&\leq T(r, f(z)) + S(r, f). \qquad (3.10)
\end{aligned}
$$

Combining (3.9) and (3.10), we have

$$T(r, f^{(1)}(z)) = T(r, f(z)) + S(r, f). \qquad (3.11)$$

Since $f(z)$ and $f^{(1)}(z)$ share $a(z)$ CM, by using (3.6) and (3.11) together with the First Fundamental Theorem, we obtain

$$\begin{aligned} m\left(r, \frac{1}{f(z)-a(z)}\right) &= T(r,f(z)) - N\left(r, \frac{1}{f(z)-a(z)}\right) + O(1) \\ &= T(r,f^{(1)}(z)) - N\left(r, \frac{1}{f^{(1)}(z)-a(z)}\right) + S(r,f) \\ &= m\left(r, \frac{1}{f^{(1)}(z)-a(z)}\right) + N\left(r, \frac{1}{f^{(1)}(z)-a(z)}\right) \\ &\quad -N\left(r, \frac{1}{f^{(1)}(z)-a(z)}\right) + S(r,f) \\ &= m\left(r, \frac{1}{f^{(1)}(z)-a(z)}\right) + S(r,f) \\ &= S(r,f). \end{aligned}$$

Hence by Lemma 2.6, we have

$$f(z) = Ce^z,$$

where $C(\neq 0)$ is a constant.

Case 2. Now we suppose that $\Psi(z) \not\equiv 0$. Then following the similar arguments of Case-1 and using Lemma 2.6, we have

$$f(z) = Ce^z.$$

where $C(\neq 0)$ is a constant.

Case 3. Finally we suppose that $\Phi(z) \equiv 0$ and $\Psi(z) \equiv 0$. Then from (3.1) and (3.2), we get

$$(a(z) - a^{(1)}(z))f^{(m)}(z) - a(z)(f^{(1)}(z) - a^{(1)}(z)) \equiv 0 \tag{3.12}$$

and

$$(a(z) - a^{(1)}(z))f^{(n)}(z) - a(z)(f^{(1)}(z) - a^{(1)}(z)) \equiv 0. \tag{3.13}$$

Now subtracting (3.13) from (3.12), we have

$$(a(z) - a^{(1)}(z))(f^{(m)}(z) - f^{(n)}(z)) \equiv 0.$$

Since $a(z) \not\equiv a^{(1)}(z)$, we get

$$f^{(m)}(z) \equiv f^{(n)}(z).$$

Solving this we have

$$f(z) = p_0 + p_1 e^{t_1 z} + p_2 e^{t_2 z} + \cdots + p_{m-n} e^{t_{m-n} z},$$

where $t_1, t_2, \cdots, t_{m-n}$ are distinct $(m-n)^{th}$ roots of unity and $p_0, p_1, p_2, \cdots$ p_{m-n} are constants.

Since $f(z)$ and $f^{(1)}(z)$ share $a(z)$ CM, applying Lemma 2.5, we get

$$\frac{f^{(1)}(z) - a(z)}{f(z) - a(z)} = \lambda,$$

for some nonzero constant λ.

Solving above equality, we obtain

$$f(z) = Ce^{\lambda z} + a(z) - \frac{a(z)}{\lambda} + \frac{\alpha(1-\lambda)}{\lambda^2},$$

where $C(\neq 0)$ is a constant. This completes the proof of Theorem 1.1.

4 Conclusions

After the above discussion we arrive at the conclusion that if an entire function and its first derivative share a linear polynomial with counting multiplicity and it partially shares the linear polynomial with its two higher order derivatives then the funtion is either one of the following two forms.
(i) $f(z) = Ce^z$,
(ii) $f(z) = Ce^{\lambda z} + a(z) - \frac{a(z)}{\lambda} + \frac{\alpha(1-\lambda)}{\lambda^2}$,
where C and λ are non-zero constants.

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