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Studies on A. Einstein, B. Podolsky and N. Rosen argument that "quantum mechanics is not a complete theory," II: Apparent confirmation of the EPR argument

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Abstract

In 1935, A. Einstein expressed his historical view, jointly with B. Podolsky and N. Rosen, that quantum mechanics could be "completed" into a form recovering classical determinism at least under limit conditions (*EPR argument*). In the preceding Paper I, we have outlined the novel methods underlying the "completion" of quantum mechanics into hadronic mechanics for the representation of extended, thus deformable particles within physical media. In this Paper II, we study the *isosymmetries* for interior dynamical systems; we confirm the 1998 apparent proof that interior dynamical systems admit a classical counterpart; we confirm the 2019 apparent proof that Einstein's determinism is progressively approached for extended particles in the interior of hadrons, nuclei and stars, while being fully verified in the interior of gravitational collapse; and we show for the first time that the recovering of Einstein's determinism in interior systems implies the apparent removal of quantum mechanical divergencies. In the subsequent Paper III, we present a number of illustrative examples and novel applications in mathematics, physics and chemistry.

Keywords: EPR argument, isomathematics, isomechanics.
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1. INTRODUCTION

1.1. The EPR argument.

As it is well known, Albert Einstein did not accept quantum mechanical uncertainties as being final, for which reason he made his famous quote "God does not play dice with the universe."

More particularly, Einstein believed that "quantum mechanics is not a complete theory," in the sense that it could be broadened into such a form to recover classical determinism at least under limit conditions.

Einstein communicated his views to B. Podolsky and N. Rosen and they jointly published in 1935 the historical paper [1] that became known as the *EPR argument.*

In view of the rather widespread belief that quantum mechanics is a final theory valid for all conceivable conditions existing in the universe, objections against the EPR argument have been voiced by numerous scholars, including by N. Bohr [2], J. S. Bell [3] [4], J. von Neumann [5] and others (see Ref. [6] for a review and comprehensive literature). The field became known as *local realism* and included the dismissal of the EPR argument based on claims that quantum axioms do not admit *hidden variables* λ [7] [8].

1.2. Outline of Paper I.

This paper, and the preceding Ref. [9] (hereinafter referred to as Paper I), are dedicated to the review and upgrade of decades of studies by mathematicians, physicists, and chemists (see Refs. [10] to [71] and papers quoted therein) on the apparent proof of the EPR argument via the "completion," also called *isotopic lifting,* of quantum mechanics into the axiom-preserving *hadronic mechanics* (see the 1995 monographs [29] [30] [31] and literature quoted therein).

More specifically, in Section I-1.1, we have outlined the EPR argument [1] jointly with representative objections [2] to [6].

In Section I-1.2, we have outlined the apparent proof by R. M. Santilli [10] (See also the detailed study in monograph [30], particularly Chapter 4 and Appendix 4C, page 166) that interior dynamical systems represented

Figure 1: *In this figure, we present a conceptual rendering of the tacit assumption underlying the objections against the EPR argument [2] - [6], namely, the representation of particles as being point-like because mandated by the Newton-Leibnitz differential calculus underlying quantum mechanics, namely, the representation of particles as isolated points in empty space. A first consequence is that, being dimensionless, particles can only be at a distance, with ensuing Einstein's argument on the need for superluminal interactions to explain quantum entanglement [1]. A second consequence is that, being at a distance, the sole possible interactions are of linear, local and potential type, under which assumptions the objections against the EPR argument are indeed valid.*

with hadronic mathematics and mechanics admit classical counterparts.

In the same Section I-1.2, we have outlined the apparent second proof by Santilli [11] that classical determinism is progressively approached in the interior of hadrons, nuclei, stars and gravitational collapse as predicted by Einstein.

In support of the plausibility of the EPR argument, in the subsequent Sections I-1.3 to I-1.7, we have outlined insufficiencies of quantum mechanics for time-irreversible processes, particle physics, nuclear physics, chemistry, and other fields. We have also provided various references indicating the apparent resolution of said insufficiencies by hadronic mechanics.

In Section I-2, we have outlined the *Lie-admissible covering of Lie's theory* [12] [13], with ensuing time-irreversible *Lie-admissible brach of hadronic mechanics,* also known as *genomechanics,* [12] [14] allowing studies on the compatibility of mechanics with thermodynamics, said compatibility being notoriously impossible for quantum mechanics.

Quantum mechanics and the objections against the EPR argument are formulated for time-reversal invariant systems of exterior dynamical systems. Therefore, in preparation for the proof of the EPR argument studied in Section 3, we have outlined and upgraded in Section I-3 the time-reversal invariant *Lie-isotopic subclass of Lie-admissible mathematics,* also known as *isomathematics,* [15] [18] which is used for the representation of time-reversible invariant interior dynamical systems.

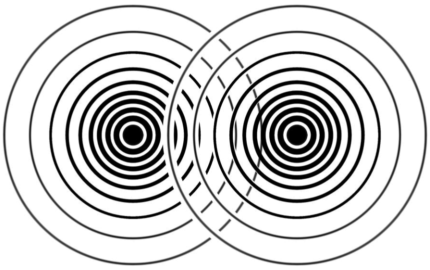

Figure 2: *A conceptual rendering of the main assumption of the apparent proofs [10] [11] of the EPR argument [1], is the representation of particles as extended, deformable and hyperdense in conditions of mutual overlapping/entanglement with ensuing continuous contact at a distance which eliminates the need for superluminal interactions to explain quantum entanglement. A first implication is the need, for consistency, of generalizing Newton-Leibnitz differential calculus from its historical form solely definable at isolated points, to a covering form definable in volumes [21]. Another implication is the emergence of contact, non-linear, non-local and non-potential interactions that, being not representable by Hamiltonians are Lagrangians, require a structural lifting of the Lie algebra of quantum mechanics under which the objections against the EPR argument are inapplicable (Section 3). Intriguingly, the "completions" here considered turned out to be of isotopic/axiom-preserving type, thus being fully admitted by quantum mechanical axioms, merely subjected to a realization broader than that of the Copenhagen school. The apparent proofs of the EPR argument [10] [11] become an unavoidable consequence of the indicated "completions" (Section 3).*

In the same Section I-3, we have devoted particular attention to the "completion" of conventional Hilbert spaces [19], numeric fields [20] and Newton-Leibnitz differential calculus [21] into forms defined on *volumes,* rather than points.

In the same Section I-1.3, we have provided particular attention to the main methods for the proofs of the EPR argument, namely, the *axiom-preserving, isotopic lifting of Lie's theory* [26], today known as the *Lie-Santilli isotheory* [38].

Finally, in Section I-4, we have outlined and upgraded the time-reversal invariant *isotopic branch of hadronic mechanics,* also known as *isomechanics* [30] which provides the dynamical foundations of the proofs of the EPR argument [10] [11].

1.3. Basic assumptions.

The most dominant aspects underlying the studies here considered are:

1) The validity of quantum mechanics for point-like particles in vacuum with ensuing linear, local and action-at-a-distance/potential interactions (*exterior dynamical problems*) occurring in atomic structures, particles in accelerators, crystals and numerous other systems in nature (Figure 1);

2) The "completion" of quantum mechanics into hadronic mechanics for the representation of extended, therefore deformable and hyperdense particles within physical media with ensuing, additional, non-linear, non-local and contact/non-potential interactions (*interior dynamical problems*), occurring in the structure of hadrons, nuclei and stars, with limit conditions occurring in the interior of gravitational collapse where the inapplicability (rather than the violation) of quantum mechanics is already accepted by the majority of serious scholars (Figure 2, 3).

The central assumption of these studies is the axiom-preserving lifting of the conventional associative product $ab = a \times b$ between *all* possible quantum mechanical quantities (numbers, functions, matrices, etc.) into the *isoproduct* [14] [26] (Section 3)

$$a \star b = a\,\hat{T}\,b, \tag{1}$$

where $\hat{T}$, called the *isotopic element,* is restricted to be positive-definite, $\hat{T} > 0$, but possesses otherwise an unrestricted functional dependence on all needed local variables.

Refs. [14] [26] constructed an axiom-preserving isotopy of the various branches of Lie's theory, resulting in a theory today known as the *Lie-Santilli isotheory* [38] (Section I-3.7) with isotopic lifting of lie algebras of the type [10]

$$[X_i \hat{,} X_j] = X_i \star X_j - X_j \star X_i = C_{ij}^k X_k. \;\; i, j = 1, 2, ..., N. \tag{2}$$

Following laborious efforts for the achievement of mathematical maturity, Ref. [10] applied the Lie-Santilli isotheory to the isotopy $\hat{SU}(2)$ of the $SU(2)$ spin with three-dimensional isoalgebras of type (2) and introduced the realization of hidden variables [7] [8] of the type

$$\hat{T} = Diag.(1/\lambda, \lambda), \;\; Det\hat{T} = 1. \tag{3}$$

Ref. [10], therefore establishing that, contrary to objections [2] to [6], *the abstract axioms of quantum mechanics do indeed admit explicit and concrete realizations of hidden variables.*

The proof in Ref. [10] that interior systems admit identical classical counterparts was consequential (Section 3).

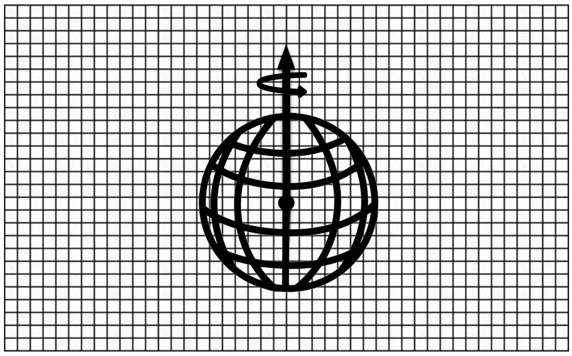

Figure 3: *A conceptual rendering of the central notion used for the study of the EPR argument, namely, a mathematically consistent representation invariant over time of extended, deformable and hyperdense particles in interior conditions, such as protons in the interior of a star, thus being under the most general known (non-singular) non-linear, non- local and non-potential interactions fully representable via the isotopic element of isoproduct (1).*

Isoproduct (1) also allows a direct and immediate representation of extended particles in conditions of mutual penetration with realizations of the type (Figure 3) [33]

$$\hat{T} = \Pi_{k=1,...,N} Diag.(\frac{1}{n_{1k}^2}, \frac{1}{n_{2k}^2}, \frac{1}{n_{3k}^2}, \frac{1}{n_{4k}^2}) e^{-\Gamma},$$
$$k = 1, 2, ..., N, \quad \mu = 1, 2, 3, 4, \tag{4}$$

where n_1^2, n_2^2, n_3^2, (called *characteristic quantities*) represent the deformable semi-axes of the particle normalized to the values $n_k^2 = 1,`k = 1, 2, 3$ for the sphere; n_4^2 represents the *density* of the particle considered normalized to the value $n_4 = 1$ for the vacuum; and Γ represents non-linear, non-local and non-Hamiltonian interactions caused by mutual penetrations/entanglement of particles.

The smaller than 1 absolute value of the isotopic element $\hat{T}$ occurring in all known applications [26]-[36]

$$| \hat{T} | \leq 1, \tag{5}$$

permitted Ref. **[?]** to show that *the standard deviations Δr and Δp appear to progressively tend to zero with the increase of the density of the medium, and appear to achieve full classical determinism in the interior of gravitational collapse, as originally conceived by Einstein.*

The initial construction of the isotopies of 20th century applied mathematics with isoproduct (1) defined over conventional numeric fields $F(n, \times, 1)$ [26] turned out to be inconsistent because the underlying time evolution is *non-unitary,* thus causing the lack of invariance over time of the traditional basic unit 1, with ensuing inapplicability over time of the entire field $F(n, \times, 1)$.

The above occurrence mandated the construction of *isofields* $\hat{F}(\hat{n}, \star, \hat{I})$ [20] [41](Section I-3.3) with basic *isounit*

$$\hat{I} = 1/\hat{T} > 0, \tag{6}$$

and *isonumbers* $\hat{n} = n\hat{I}$ equipped with isoproduct (1).

Ref. [20] essentially established that *the abstract axioms of a numeric field do not require that the multiplicative unit of the field be the trivial number* 1*, since said unit can be an arbitrary quantity with an unrestricted functional dependence on local variables, provided that said multiplicative unit is positive definite and the field is lifted into a compatible form.*

Despite all the above efforts, the ensuing isomathematics was still inapplicable to the proof of the EPR argument because it lacked the crucial *invariance over time,* namely, the prediction of the same interior dynamical systems under the same conditions but at different times.

The above occurrence forced the construction of the covering of the Newton-Leibnitz differential calculus into the covering *isodifferential isocalculus* [21] [44] (Section I-3.6) with basic *isodifferential* (Figure 2) [**?**]

$$\hat{d}\hat{r} = \hat{T}d[r\hat{I}(r, ...)] = dr + r\hat{T}d\hat{I}(r, ...), \tag{7}$$

and corresponding *isoderivative*

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}. \tag{8}$$

In essence, Ref. [21] established the inapplicability of the conventional differential calculus whenever the axioms of numeric fields admit multiplicative units with a dependence on the differentiation variable, with ensuing inapplicability of quantum mechanics, as well as of the objections against the EPR argument, for interior dynamical systems.

The "completion" of the differential calculus into an isotopic form compatible with basic isoproduct (1) finally allowed the achievement of invariance over time (Section I-3.9), thus signaling the achievement of maturity for the apparent proof of the EPR argument reviewed.

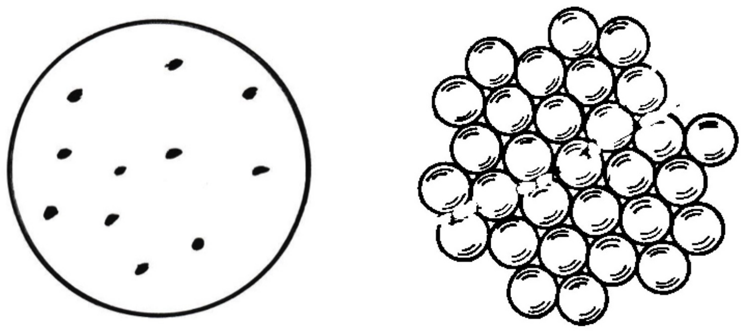

Figure 4: *In the l.h.s. of this picture, we present a conceptual rendering of the structure of nuclei as ideal spheres with isolated point-like particles in their interior. This view is an inevitable consequence of the elaboration of quantum mechanics via the conventional differential calculus, resulting in rather serious insufficiencies in nuclear physics outlined in Section I-1.5. In the r.h.s. of this picture, we present a conceptual rendering of the representation of nuclei as occurring in the physical reality, namely, as a collection of extended, therefore deformable charge distributions in condition of partial mutual penetration according to Eq. (4) of isomathematics and related isomechanics, thus permitting the resolution of at least some of the insufficiencies of quantum mechanics in nuclear physics reviewed in Section I-1.5.*

In Section 2 of this paper, we complete the methodological needs by outlining and upgrading the time-reversal invariant coverings of conventional spacetime symmetries, known as *isosymmetries,* for systems of extended particles in interior conditions; in Section 3, we review and upgrade the Lie-isotopic $SU(2)$-spin symmetry and related proofs [10] [11] of the EPR argument.

A few comments on terminologies appear to be recommendable.

The word "completion" is used in these studies to honor the memory of Albert Einstein and should not be intended to indicate "final" theories. In fact, isomathematics and isomechanics admit coverings of Lie-admissible character [12] (Section I-2) that, in turn, admits coverings of hyperstructural character [43], with additional coverings remaining possible in due time.

The terms "non-Hamiltonian interactions" are intended to indicate interactions that are not representable with a Hamiltonian, and are technically identified as interactions violating the integrability conditions for the existence of a Hamiltonian, namely, the *conditions of variational self-adjointness* [25].

When dealing with stable and isolated interior dynamical systems, the terms "non-conservative forces" are strictly referred to *internal* non-Hamiltonian exchanges verifying conditions (1-55) for the verification of the ten conventional total conservation laws for the total energy, momentum, angular momentum and the uniform motion of the center of mass.

The terms "physical media" refer to media composed by matter in its various states, and are often referred to as *hadronic media,* in the sense that the media are *not* composed by empty space, thus requiring the use of hadronic mathematics and mechanics for their quantitative treatment.

The terms "extended particles" refer to: the wavepacket of elementary particles such as the electron assumed to be of about $1\ fm = 10^{-15}\ cm$; extended charge distributions for protons and neutrons when members of a nuclear structure, also assumed to have a diameter of about $1\ fm$; and stable nuclei when considering the structure of stars. Due to its crucial significance for the structure of interior systems, a technical definition of the notion of "extended particles" will be given in Section 3 via the notion of *isoparticle* as isorepresentations of space-time isosymmetries.

2. ISOSYMMETRIES

2.1. Foreword.

In this section, we study the axiom-preserving "completion" (or isotopic lifting) of conventional space-time symmetries, known as *Lie-isotopic symmetries,* or *isosymmetries* for short, which provide the invariance of stable and isolated (thus time reversible) interior dynamical systems of extended particles at mutual distances smaller than their size as occurring, e.g., in nuclear structures (Figure 3).

Lie-isotopic symmetries were first introduced by Santilli in the 1978 Harvard University paper [13] as a particular case of the broader *Lie-admissible symmetries* for irreversible, non-conservative systems [14]. Isosymmetries were then studied in various subsequent works quoted in this section.

The understanding of this section requires a knowledge of the *Lie-Santilli isotheory* (Section I-3.7), which was first formulated in monographs [25] [26] over the field of real numbers. Isosymmetries were then formulated in monographs [29] [30] with the full use of isomathematics, including the use isofields [20] [41] and the isodifferential calculus [21] [44] (see Refs. [38] [46] [47] [48] for works on the Lie-Santilli isotheory, and Ref. [45] for a general review with applications and experimental verifications).

The assumption at the foundation of isosymmetries is *the preservation of the abstract axioms of 20th century space-time symmetries, and the mere construction of their broadest possible realization permitted by isomathematics.*

Consequently, criticisms of isosymmetries and their novel implications

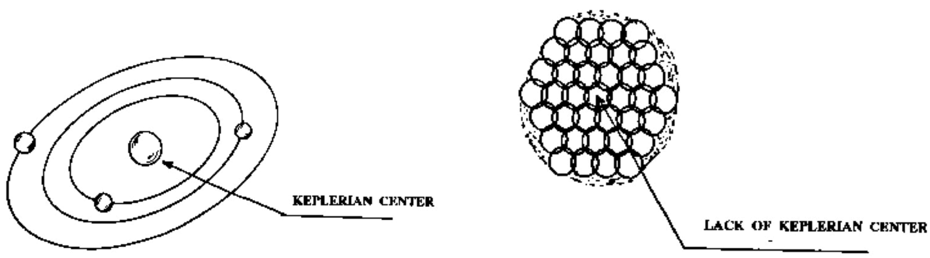

Figure 5: *The l.h.s. of this figure illustrates Keplerian systems for which space-time symmetries have been constructed, namely, exterior dynamical systems of point-like masses orbiting in vacuum around a heavier point-like mass known as the Keplerian center. The r.h.s. of this figure illustrates interior systems for which isosymmetries have been built, namely, systems of extended particles in conditions of mutual penetration without any Keplerian center.*

are *de facto* criticisms on 20th century space-time symmetries and their implications

2.2. Inapplicability of Lie symmetries for interior systems.

A rather widespread view of 20th century physics is the lack of any difference between exterior and interior dynamical systems on grounds that the latter can be reduced to their elementary constituents, by therefore recovering exterior conditions.

The above view was disproved by R. M. Santilli in his Ph. D. thesis (see the review in Ref. [30]) on numerous grounds, the first being the notorious incompatibility of quantum mechanics with thermodynamics whose resolution motivated the Lie-admissible generalization of Lie algebras and related physical theories [29] [30].

The absence of structural differences between exterior and interior systems was dismissed more directly with the following [26] [28]:

NO REDUCTION THEOREM 2.2.1: A classical dynamical system with non-conservative interior forces cannot be consistently reduced to a finite number of isolated particles all in conservative conditions and, vice- versa, the latter system cannot reproduce the former under the correspondence or other principles.

The first direct consequence of the above No Reduction Theorem is the "inapplicability" (rather than the "violation") for interior dynamical systems of conventional space-time symmetries that have been proved to be so effective for exterior dynamical systems.

Said inapplicability was also proved [*loc. cit.*] from the fact that the

Galileo and the Lorentz-Poincaré symmetries can only provide a non-relativistic and relativistic characterization, respectively, of *Keplerian systems*, namely, systems of point-like masses orbiting in vacuum around a heavier mass called the *Keplerian nucleus* [26].

However, interior dynamical systems do not admit a Keplerian structure because *nuclei have no nuclei* [33] and the same happens for hadrons, stars and gravitational collapse (Figure 5).

It is then possible to prove, e.g., via the imprimitivity theorem, that the lack of existence of a Keplerian structure implies the lack of exact validity of conventional space-time symmetries [26] [30].

On more technical grounds, Lie's theory is known to be solely applicable to exterior systems of point-like particles in vacuum with ensuing sole possible, linear, local and Hamiltonian interactions.

Experimental evidence on interior dynamical systems, e.g., on nuclear volumes compared to the volumes of individual nucleons, establishes that nuclei are composed of extended charge distributions in conditions of partial mutual penetration/entanglement with the ensuing existence of additional, non-linear, non-local and non-Hamiltonian interactions under which Lie's theory is inapplicable.

Hence, the transition of particles from exterior to interior conditions implies the inapplicability of the $SU(2)$-spin symmetry with consequential inapplicability of Bell's inequality [3] and other objections against the EPR argument [6] in favor of suitable covering vistas [10] [11].

In any case, the $SU(2)$ symmetry, while unquestionable effective for exterior dynamical systems, has been unable to provide a consistent representation of the spin of particles and nuclei, thus warranting the search for a suitable "completion."

2.3. The fundamental theorem on isosymmetries.

The construction of isosymmetries requires the full use of isomathematics with particular reference to the Lie-Santilli isotheory formulated on isospaces over isofield and elaborated via the isodifferential calculus (Section I-2.7).

Said construction can be done with the following theorem (for brevity, see the proof in Section 1.2 , Vol. I of Refs. [36]):

THEOREM 2.3.1: Let G be an N-dimensional Lie symmetry of the line element of a k-dimensional metric or pseudo-metric space $S(x, m, I)$ over a numeric field F

with coordinates x, metric m over a numeric field F with conventional unit I,

$$G: \quad x' = \Lambda(w)x, \quad y' = \Lambda(w)y, \quad x, y \in S,$$

$$(x' - y')^\dagger \Lambda^\dagger m \Lambda (x' - y') \equiv (x - y)^\dagger m (x - y), \tag{9}$$

$$\Lambda^\dagger(w) m \Lambda(w) \equiv m. \ \ w \in F.$$

Then, all infinitely possible (non-singular) Lie-Santilli isotopies $\hat{G}$ of G on isospace $\hat{S}(\hat{x}, \hat{M}, \hat{I})$ with isocoordinates

$$\hat{x} = xI, \tag{10}$$

isometric

$$\hat{M} = \hat{m}\hat{I} = (\hat{T}_i^k m_{kj})\hat{I}, \tag{11}$$

and isounit

$$\hat{I} = 1/\hat{T} > 0, \tag{12}$$

over an isofield $\hat{F}$ with isounit v$\hat{I}$ leave invariant the isoline element of the isospace $\hat{S}(\hat{x}, \hat{M}, \hat{I})$:

$$\hat{G}: \quad \hat{x}' = \hat{\Lambda}(\hat{w}) \star \hat{x}, \quad \hat{y}' = \hat{\Lambda}(\hat{w}) \star \hat{y}, \quad \hat{x}, \hat{y} \in \hat{S},$$

$$(\hat{x}' - \hat{y}')^\dagger \star \hat{\Lambda}^\dagger \star \hat{M} \star \hat{\Lambda} \star (\hat{x}' - \hat{y}') \equiv (x - y)^\dagger \hat{m} (x - y), \tag{13}$$

$$\hat{\Lambda}^\dagger(\hat{w}) \star \hat{M} \star \hat{\Lambda}(\hat{w}) \equiv \hat{M}.$$

All infinitely possible so constructed isosymmetries $\hat{G}$ are locally isomorphic to the original symmetry G.

The reader should note that, while a given Lie symmetry G is unique as well known, there can be an infinite number of covering isosymmetries $\hat{G}$ with generally different explicit forms o the isotransformations due to the infinite number of possible isotopic elements representing the infinitely different internal interactions of extended particles within physical media.

Note also that all possible isotopic images of a given Lie symmetry can be explicitly and uniquely constructed via the sole knowledge of the original Lie symmetry and of the isotopic element $\hat{T} > 0$, or of the isounit $\hat{I} = 1/\hat{T}$, which property shall be hereon tacitly assumed.

2.4. Isospaces and isogeometries.

As it is well known, the fundamental representation space of relativistic space-time symmetries is the conventional *Minkowski space* $M(x, \eta, I)$ formulated on the field of real numbers $\mathcal{R}$ with coordinates $x = (x^1, x^2, x^3, x^4 = ct)$, metric $\eta = Diag.(1, 1, 1, -1)$, unit $I = Diag(1, 1, 1, 1)$ and invariant

$$x^2 = (x^\mu \eta_{\mu\nu} x^\nu) I = \\ = (x_1^2 + x_2^2 + x_3^3 - c^2 t^2) I, \tag{14}$$

where the trivial multiplication by the conventional unit $I = Diag.(1, 1, 1, 1)$ is done for compatibility with isomathematics.

The fundamental isospaces of space-time isosymmetries are given by the infinite family of *iso-Minkowski isospaces,* also called *Minkowski-Santilli isospaces,* $\hat{M}(\hat{x}, \hat{\Omega}, \hat{I})$ formulated on the isofield of isoreal isonumbers $\hat{\mathcal{R}}$. (Section I-3.9), which isospaces were first introduced by R. M. Santilli in Ref. [23] of 1983 and then treated in details in works [29] [30].

Iso-Minkowskian isospaces are characterized by *space-time isocoordinates* $\hat{x} = x\hat{I}$; *isounit* $\hat{I} = 1/\hat{T}$, *isometric*

$$\hat{\Gamma} = (\hat{T}_\mu^\rho \eta_{\rho\nu}) \hat{I}, \tag{15}$$

(where one should note the necessary structure of an isomatrix [29]), positive-definite *isotopic element* (4) representing a system of extended particles in interior dynamical conditions with a restricted functional dependence on local quantities such as coordinates x, momenta p, energy E, frequency ν, density α, temperature τ, pressure pi, wavefunction ψ, etc., under the conditions

$$n_\mu = n_\mu(x, p, E, \nu, \alpha, \tau, \pi, \psi, \partial\psi, ...) > 0, \ \mu = 1, 2, 3, 4, \tag{16}$$

$$\Gamma(x, p, E, \nu, \alpha, \tau, \pi, \psi, \partial\psi, ...) \geq 0, \tag{17}$$

$$\hat{T} = e^{-\Gamma} \ \ll \ 1. \tag{18}$$

Iso-Minkowskian isospaces are characterized by the infinite family of isoinvariants (I-28) with isotopic element (4) that, for the case of one single extended particle can be written

$$\hat{x}^{\hat{2}} = \hat{x}^\mu \star \hat{\Omega}_{\mu\nu} \star \hat{x}^\nu = (x^\mu \hat{\eta}_{\mu\nu} x^\nu) \hat{I} = \\ = (\tfrac{x_1^2}{n_1^2} + \tfrac{x_2^2}{n_2^2} + \tfrac{x_3^2}{n_3^2} - t^2 \tfrac{c^2}{n_4^2}) \hat{I}, \tag{19}$$

where the exponential $exp{-}\Gamma$ has been absorbed in the characteristic quantities n_μ, and the final multiplication by the isounit is necessary for the isoinvariant to be an isoscalar, namely, an element of the isoreal isofield [20] (Section I-3-5).

The following aspects treated in Paper I are important for the understanding of the apparent proof of the EPR argument:

1. The characteristic quantities n_1^2, n_2^2, n_3^2, admit the first interpretation as representing the deformable semi-axes of elementary or composite particles normalized to the values for the sphere $n_1^2 = n_2^2, = n_3^2 = 1,$.

2. The characteristic quantity n_4^2 admit the first interpretation as representing the *density* of the hadronic medium normalized to the value $n_4 = 1$ for the vacuum.

3. The function $\Gamma \geq 0$ provides an invariant representation (Section I-3-9) of all non-linear, non-local and non-Hamiltonian interactions.

4. Property (18) is verified for all applications of isosymmetries to date [10] to [68].

5. The correct elaboration of iso-Minkowskian isospaces requires the use of the *isospherical and isohyperbolic isocoordinates* (see Refs. [29] [30]).

6. Isoinvariant (19) provides a unified representation of both exterior and interior gravitational problems. In fact, K. Schwartzchild wrote in 1916 *two* important papers, the first paper [49] on the *exterior gravitational problem* which became world famous for its initiation of gravitational singularities, and the second paper [50] in the *interior gravitational problem* which has been vastly ignored, except rare studies (such as that in Section 23.2, page 609, Ref. [52]). Such an oblivion is essentially due to the fact that Schwartzchild's second paper is not aligned with the widespread tendency of reducing masses to point-like constituents, in which case all differences between exterior and interior gravitational problems disappear to the detriment of the depth of the gravitational analysis. Readers should keep in mind the full parallelism between exterior and interior dynamical problems for *particles and gravitation.*

7. The *exterior gravitational interpretation* of isoinvariant (19) is given by the following identical representation of Schwartzchild's exterior metric [50]

$$\hat{T}_{kk} = \frac{1}{1 - \frac{2M}{r}}, \quad \hat{T}_{44} = 1 - \frac{2M}{r}. \tag{20}$$

The corresponding *interior gravitational representation* is given by the following isotopy of Schwartzchild exterior metric

$$\hat{T}_{kk} = \frac{1}{(1 - \frac{2M}{r})n_k^2}, \quad \hat{T}_{44} = (1 - \frac{2M}{r})/n_4^2. \tag{21}$$

In view of the arbitrariness of the functional dependence of the characteristic quantities n_μ, it is easy to prove that Schwartzchild's interior metric [50] is a particular case of the much broader class of interior gravitational models (21).

8. The geometry of the iso-Minkowskian isospaces, first presented by Santilli in Ref. [24] under the name of *iso-Minkowskian isogeometry,* contains the machinery of the Riemannian geometry (due to the dependence of the isometric $\hat{\eta}$ on the local coordinates x), although such a machinery is formulated for consistency over isofields [20] and elaborated via the isodifferential isocalculus [21] (Section I-3.5). Hence, *the isominkowskian isogeometry can unify exterior and interior problems for both particles and gravitation.*

9. Recall that iso-Minkowskian isospaces are locally isomorphic to the conventional Minkowski space (Refs. [23] [24] and Theorem 2.3.1). Therefore, *the iso-Minkowskian isogeometry has a null curvature.* This is due to the fact that, under isotopic lifting, the conventional Minkowski metric $\eta = Diag.(1, 1, 1, -1)$ is lifted into a coordinate-dependent isometric $\hat{T}(x)\eta = \hat{\eta}(x)$ which is *identical* to any given Riemannian metric

$$\eta \rightarrow \hat{\eta}(x) = \hat{T}(x)\eta = g(x). \tag{22}$$

Jointly, the original unit of the Minkowski space $\hat{I} = Diag.(1, 1, 1, 1)$ is lifted by the *inverse* amount

$$I > 0 \rightarrow \hat{I}(x) = 1/\hat{T}(x) > 0, \tag{23}$$

resulting in no actual curvature. The above features have suggested the introduction of the new notion of *isoflat isospace,* referred to an isospace that has null curvature when formulated on isofields, while recovering conventional curvature when formulated on conventional fields. Readers should be aware that the achievement of the universal symmetry of (non–singular) Riemannian line elements studied in the next sections are due precisely to the isoflatness of the iso-Minkowski isospace since no such symmetry is possible for a convemtional Riemannian space, as well known.

Recall that the fundamental representation space of symmetries in 3-space dimensions is the conventional Euclidean space $E(r, \times, I$ with coordinates $r = (x^1, x^2, x^3)$, metric $\delta = Diag.(1, 1, 1)$ and unit $I = Diag.(1, 1, 1)$ on the conventional field of real numbers.

Similarly, the fundamental representation space of isosymmetries in 3-dimensions is the *iso-Euclidean isospace* $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$, also called *Euclid-Santilli isospace* (Refs. [14] [26] [29] and Section I-3.5) which is the space component of the iso-Minkowskian isospace. As such, the iso-Euclidean isospace is hereon tacitly assumed to be known.

2.5. Lorentz-Poincaré-Santilli isosymmetries.

2.5.1. Main references. Following, and only following the construction of the isotopies of Lie's theory, Santilli conducted systematic studies on the isotopies of the various aspects of the Lorentz-Poincaré symmetry for the achievement of the universal invariance of spacetime isoinvariant (19), including:

1) The classical isotopies $\hat{SO}(3.1)$ of the Lorentz symmetry $SO(3.1)$ [53];

2) The operator isotopies $\hat{SO}(3.1)$ of the Lorentz symmetry $SO(3.1)$ [54];

3) The isotopies $\hat{SO}(3)$ of the rotational symmetry $SO(3)$ [55] [56] [57];

4) The isotopies $\hat{SU}(2)$ of the $SU(2)$ spin symmetry [10] [58];

5) The isotopies $\hat{P}(3.1)$ of the Poincaré symmetry $P(3.1)$ [59] [60], which included the universal symmetry of (non-singular) Riemannian line elements;

6) The isotopies $\hat{\mathcal{P}}(3.1)$ of the spinorial covering $\mathcal{P}(3.1)$ of the Poincaré symmetry [61] [62];

7) The isotopies $\hat{M}(3.1)$ of the Minkowskian geometry $M(3.1)$ [24].

A general presentation is available in the 1995 monographs [29] [30] with the full use of isomathematics, including isofields and isodifferential calculus, with upgrades in the 2008 monographs [36].

The resulting infinite family of isosymmetries $\hat{SO}(3.1)$ are known as the *Lorentz-Santilli (LS) isosymmetries* while the broader isosymmetries $\hat{P}(3.1)$ and $\hat{\mathcal{P}}(3.1)$ are known as *Lorentz-Poincaré-Santilli isosymmetries* (see Refs. [37] [42] [45] and papers quoted therein).

Experimental verifications of LPS isosymmetries for interior dynamical systems are available in monographs [31] and in Section 3 of the more recent review [63].

In inspecting the subsequent sections, the reader should be aware of the "direct universality" of the LPS isosymmetries for the considered infinite family of interior dynamical systems [64], including the treatment of exterior and interior, particle and gravitational problems (Section 4).

2.5.2. Basic definitions. As it is well known, the conventional Lorentz-Poincaré (LP) symmetry is the symmetry of line element (14) which we rewrite in the form

$$(x-y)^2 = (x^\mu - y^\mu)\eta_{\mu\nu}(x^\nu - y^\nu)I =$$

$$= [(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2 - (t_1-t_2)^2c^2)]\, I, \qquad (24)$$

$$\eta = Diag.(1,1,1,-c^2),, \quad I = Diag.(1,1,1,1),$$

where the exponential component $exp{-}\Gamma$ is again embedded for simplicity in the characteristic quantities $n\mu^2$.

The LPS isosymmetry is the universal symmetry of the isoline element (19) in the iso-Minkowski isospace $\hat{M}(\hat{x},\hat{\Omega},\hat{I})$ over the isoreal isonumbers $\hat{\mathcal{R}}$ rewritten in the form

$$(\hat{x}-\hat{y})^{\hat{2}} = \Big[(\hat{x}^\mu - \hat{y}^\mu)\star\hat{\Omega}_{\mu\nu}\star(\hat{x}^\nu - \hat{y}^\nu)\Big] =$$

$$= [x^\mu - y^\mu)\hat{\eta}_{\mu\nu}(x^\nu - y^\nu)]\,\hat{I} =$$

$$= \Big[\tfrac{(x_1-y_1)^2}{n_1^2} + \tfrac{(x_2-y_2)^2}{n_2^2} + \tfrac{(x_3-y_3)^2}{n_3^2} - (t_1-t_2)^2\tfrac{c^2}{n_4^2}\Big]\,\hat{I}, \qquad (25)$$

$$\hat{\eta} = \hat{T}\eta, \quad \hat{T} = Diag((\tfrac{1}{n_1^2}, \tfrac{1}{n_2^2}, \tfrac{1}{n_3^2}, \tfrac{1}{n_4^2}),$$

$$n_\mu = n_\mu(x, v, a, E, d, \omega, \tau, \psi, \partial\psi, ...) > O, \quad \hat{I} = 1/\hat{T} > 0.$$

2.5.3. Isotransformations. By following Theorem 2.3.1, the isotransformations of the LPS isosymmetries can be written

$$\hat{x}' = \hat{\Lambda}(\hat{w})\star\hat{x}, \qquad (26)$$

where $\hat{\Lambda}(\hat{w}) = \Lambda(\hat{w})\hat{I}$, resulting in generally *non-linear* isotransformations, including isotranslations of the type

$$\hat{x}' = \hat{x} + \hat{A}(\hat{x}, ...), \qquad (27)$$

verifying the following property

$$\hat{\Lambda}^\dagger\star\hat{\eta}\star\hat{\Lambda} = \Lambda\hat{\eta}\Lambda^\dagger. \qquad (28)$$

Under the *condition of isomodularity*

$$\hat{D}et\;(\hat{\Lambda}) = +\hat{I}, \qquad (29)$$

we have the *isoconnected LS isosymmetries* $\hat{SO}^0(3.1)$ and the *isoconnected LPS isosymmetries* $\hat{P}^0(3.1)$.

Consider the conventional generators of the Poincaré symmetry

$$(J_k) = (J_{\mu\nu}),\ P_\mu, k = 1, 2, 3, 4, 5, 6,\ \mu, \nu = 1, 2, 3, 4. \tag{30}$$

By keeping in mind isoexponentiation (I-16), the isotransformations of $\hat{SO}^0(3.1)$ can be written [60]

$$\begin{aligned}\hat{x}' &= (\hat{e}^{iJ_k w_k}) \star \hat{x} \star (\hat{e}^{-iJ_k w_k}) = \\ &= \left[(e^{iJ_k \hat{T} w_k}) x (e^{-i w_k \hat{T} J_k})\right] \hat{I},\end{aligned} \tag{31}$$

and the isotranslations $\hat{A}(3.1)$ can be written

$$\begin{aligned}\hat{x}' &= (\hat{e}^{iP_\mu a_\mu}) \star \hat{x} \star (\hat{e}^{-iP_\mu a_\mu}) = \\ &= \left[(e^{iP_\mu \hat{T} a_\mu}) x (e^{-i a_\mu \hat{T} P_\mu})\right] \hat{I}.\end{aligned} \tag{32}$$

It is evident that the above isotransformations do constitute Lie-Santilli isogroups according to Theorem I-2.7.3.

2.5.4. Isocommutation rules. As recalled earlier, the total quantities of an isolated, stable, interior system must be conserved for consistency.

In order to represent this evidence, the Lie-Santilli isotheory was constructed [26] in such a way to preserve conventional generators, because they represent total conservation laws, and isotopically lift their product.

By expanding the preceding finite isotransforms in terms of the isounit, the *LPS isoalgebra* $\hat{so}^0(3.1)$ is characterized by the conventional generators of the LP algebra and the isocommutation rules [30] [60] (here written in their projection on conventional spaces over conventional fields)

$$\begin{aligned}&[J_{\mu\nu} \hat{,} J_{\alpha\beta}] = \\ &= \imath(\hat{\eta}_{\nu\alpha} J_{\beta\mu} - \hat{\eta}_{\mu\alpha} J_{\beta\nu} - \hat{\eta}_{\nu\beta} J_{\alpha\mu} + \hat{\eta}_{\mu\beta} J_{\alpha\nu}),\end{aligned} \tag{33}$$

$$[J_{\mu\nu} \hat{,} P_\alpha] = i(\hat{\eta}_{\mu\alpha} P_\nu - \hat{\eta}_{\nu\alpha} P_\mu)$$

$$[P_\mu \hat{,} P_\nu] = 0, \tag{34}$$

$$\hat{\eta}_{\mu\nu} = \hat{T}\eta = (\hat{T}_\mu^\rho \eta_{\rho\nu}). \tag{35}$$

where one should note the appearance of the *structure functions* $\hat{\eta}(x, p, E, \nu, \alpha, \tau, \psi, ...)$, rather than the traditional structure constants (Theorem I-2.7.2).

The presence of structure functions $\hat{\eta}$ in isocommutation rules (33)-(35), Theorem I-3.7.2 and the analysis of Section I-3.8 imply the following important property (Section I-3.8):

LEMMA 2.5.1: LPS isosymmetries cannot be derived via non-unitary transformations of the conventional LP symmetry.

Despite the above non-equivalence, the property $\hat{T} > 0$, the topological structure $(+1, +1, +1, -1)$ of the isometric $\hat{\eta} = \hat{T}\eta$ and Theorem 2.3.1 imply that:

LEMMA 2.5.2. *All LPS isosymmetries are locally isomorphic to the conventional LP symmetry.*

Recall from Section I-1 that an important limitation of quantum mechanics for the study of the EPR argument is the inability to achieve a consistent and effective treatment of non-linear interactions that are expected in the structure of hadrons, nuclei and stars. In Section I-4.12, we have shown that the isotopic "completion" of quantum mechanics into hadronic mechanics does indeed allow a consistent and effective treatment of non-linear interactions via their embedding in the isotopic element $\hat{T}$.

Due to the unrestricted functional dependence of the isotopic element $\hat{T}$ and, therefore, of the isometric $\hat{\eta} = \hat{T}\eta$, it is easy to see that the LPS isosymmetries are indeed non-linear as a necessary condition to provide the invariance of non-linear dynamical equations.

Note that *isolinear isomomenta $\hat{P}_\mu$ isocommute on isospaces over isofields, but they do not commute on conventional spaces over conventional fields,* Eqs. (35), thus confirming that *the LPS isosymmetry is isolinear, that is, linear on isospaces over isofields but generally non-linear in their projection on conventional spaces over conventional fields.*

This important property can be illustrated by recalling the isolinear isomomentum (I-79) on a Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ with isostates $\hat{\psi} >$ over the isocomplex isonumbers $\hat{\mathcal{C}}$

$$\hat{P}_\mu \star |\hat{\psi} >= -i\hat{I}\partial_\mu|\hat{\psi} > . \tag{36}$$

Isocommutators (35) on $\hat{\mathcal{H}}$ over $\hat{\mathcal{C}}$ can then be explicitly written

$$[\hat{P}_\mu,\hat{P}_\nu]\star|\hat{\psi}>=(\hat{P}_\mu\star\hat{P}_\nu-\hat{P}_\nu\star\hat{P}_\mu)\star|\hat{\psi}>=$$
$$=(-i\hat{I}\partial_\mu)\hat{T}(-i\hat{I}\partial_\nu)-(-i\hat{I}\partial_\nu)\hat{T}(-i\hat{I}\partial_\mu)\hat{T}|\hat{\psi}>= \tag{37}$$
$$=(i\hat{I}\partial_\mu\partial_\nu-i\hat{I}\partial_\nu\partial_\mu)|\hat{\psi}>=0.$$

By contrast, the projection of the same isocommutators (35) on a conventional Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$ no longer commutes,

$$[\hat{P}_\mu,\hat{P}_\nu]|\hat{\psi}>=(\hat{P}_\mu\hat{P}_\nu-\hat{P}_\nu\hat{P}_\mu)|\hat{\psi}>=$$
$$=(-i\hat{I}\partial_\mu)(-i\hat{I}\partial_\nu)-(-i\hat{I}\partial_\nu)(-i\hat{I}\partial_\mu)|\hat{\psi}>\neq 0. \tag{38}$$

because, in general, $\partial_\mu\hat{I}\neq\partial_\nu\hat{I}$, and this proves the isolinear character of the isomomentum.

Besides a direct relevance for the structure of hadrons, nuclei and stars, the above isolinearity has important implications, such as a new consistent operator form of gravitation, a new grand unification and other advances [35].

The presence of the structure functions in the isocommutation rules, the capability to provide the invariance under non-linear interactions and other features and applications outlined in Section 4 illustrate the non-triviality of the Lie-Santilli isotheory.

2.5.5. Iso-Casimir Isoinvariants. The simple direct use of isocommutation rules (33)-(35) establishes that the *iso-Casimir-isoinvariants* of $\hat{p}^0 0(3.1)$ are given by [60]

$$\hat{C}_1=\hat{I}((t,r,p,E,\mu,\tau,\psi,\partial\psi,...)>0,$$
$$\hat{C}_2=\hat{P}^{\hat{2}}=\hat{P}_\mu\star\hat{P}^\mu=(\hat{\eta}_{\mu\nu}P^\mu P^\nu)\hat{I}=$$
$$=(\textstyle\sum_{k=1,2,3}\frac{1}{n_k^2}P_k^2-\frac{c^2}{n_4^2}p_4^2)\hat{I}, \tag{39}$$
$$\hat{C}_3=\hat{W}^{\hat{2}}=\hat{W}_\mu\star\hat{W}^\mu,\quad \hat{W}=W\hat{I},$$
$$\hat{W}_\mu=\hat{\epsilon}_{\mu\alpha\beta\rho}\star J^{\alpha\beta}\star P^\rho,$$

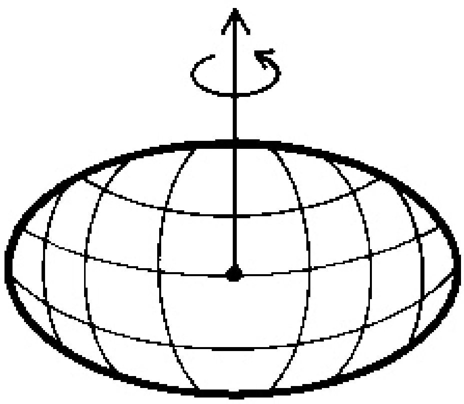

Figure 6: *It was generally believed in the 20th century physics that the rotational symmetry is broken for ellipsoids. Santilli isorotational isosymmetry has restored the exact character of the rotational symmetry for all possible (topology preserving) deformations of the sphere [30].*

and they are at the foundation of classical and operator *relativistic isomechanics* (Section I-4) with deep implications for structure models of interior dynamical systems [31].

2.5.6. Isorotations. By using isotransforms (32), the explicit form of the isorotations $\hat{SO}(3)$, first derived in Refs. [55] [56], can be written in the isoplane $(\hat{x},^1, \hat{x}^2)$ of iso-Euclidean isospaces $\hat{E}(\hat{x}, \hat{\Delta}, \hat{I})$ over the isoreals $\hat{\mathcal{R}}$, here formulated for simplicity in their projection on the conventional Euclidean space (see Ref. [30] for the general case)

$$\begin{aligned} x^{1'} &= x^1 \cos[\theta(n_1 n_2)^{-1}] - x^2 \tfrac{n_1^2}{n_2^2} \sin[\theta(n_1 n_2)^{-1}], \\ x^{2'} &= x^1 \tfrac{n_2^2}{n_1^2} \sin[\theta(n_1 n_2)^{-1}] + x^2 \cos[\theta(n_1 n_2)^{-1}]. \end{aligned} \tag{40}$$

It was generally believed in the 20th century that the $SO(3)$ symmetry is broken for ellipsoid deformations of the sphere. By contrast, as shown by isotransforms (40) the $\hat{SO}(3)$ isosymmetry achieves the invariance of ellipsoids (Figure 6). But $SO(3)$ and $\hat{SO}(3)$ are locally isomorphic (Theorem 2.3.1). We therefore have the following property [55] [56]:

LEMMA 2.5.3: The Lie-Santilli $\hat{SO}(3)$ isosymmetry restores the exact character of the rotational symmetry for all ellipsoid deformations of the sphere.

This property is due to the fact that the mutation of the semiaxes of the sphere occur jointly with the *inverse*, mutation of the related units, thus maintaining the perfect spherical shape in isospaces over isofields

$$Radius \quad 1_k \to 1/n_k^2, \;\; Unit \quad 1_k \to n_k^2. \tag{41}$$

Note the crucial role of isonumbers for the reconstruction of the exact rotational symmetry because said reconstruction occurs thanks to the isoinvariant by the isounit.

2.5.7. Lorentz-Santilli isotransforms. The infinite family of isoconnected Lorentz-Santilli (LS) isotransforms $\hat{SO}^0(3.1)$ on iso-Minkowskian isospaces $\hat{M}(\hat{x}, \hat{\Omega}, \hat{I})$ over the isoreals $\hat{\mathcal{R}}$, first derived by in Ref. [53] of 1983, can be written in the $(\hat{x}^3, \hat{x}^4)$-isoplane in their projection in the conventional l Minkowski space $M(x, \eta, I)$, as follows (see Ref. [30] for the general case):

$$\begin{aligned} x^{1'} &= x^1, \;\; x^{2'} = x^2, \\ x^{3'} &= \hat{\gamma}(x^3 - \hat{\beta}\tfrac{n_3}{n_4}x^4), \\ x^{4'} &= \hat{\gamma}(x^4 - \hat{\beta}\tfrac{n_4}{n_3}x^3), \end{aligned} \tag{42}$$

where

$$\hat{\beta} = \frac{v_3/n_3}{c/n_4}, \;\; \hat{\gamma} = \frac{1}{\sqrt{1 - \hat{\beta}^2}}. \tag{43}$$

A significant aspect of Ref. [53] is the solution of the *historical Lorentz problem,* namely, the invariance of locally varying speeds of light within physical media

$$C = \frac{c}{n_4}. \tag{44}$$

In fact, Lorentz first attempted the invariance of the speed of light $C = c/n_4$, but had to restrict his study to the invariance of the constant speed of light in vacuum c, due to insurmountable technical difficulties. Santilli has shown that Lorentz's difficulties were due to the use of Lie's theory, because, under the use of the covering Lie-Santilli isotheory, the invariance of $C = c/n_4$ was achieved in two pages of the 1983 letter [53].

A second significant aspect of Ref. [53] is the achievement of the first invariant formulation of extended, thus deformable and hyperdense particles, as stated beginning with the title of the quoted paper.

It was generally believed in the 20th century that the Lorentz symmetry $SO^0(3.1)$ is broken for locally varying speed of light within physical media represented with the wiggly circle of Figure 7. Ref. [53] proved that the isosymmetry $\hat{SO}^0(3.1)$ achieves the invariance of $C = c/n_4$. But $SO^0(3.1)$

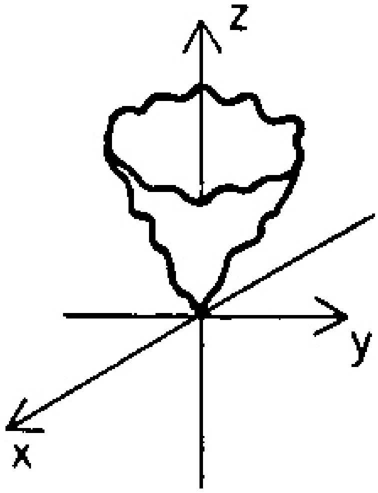

Figure 7: *It was generally believed in the 20th century physics that the Lorentz symmetry is broken for locally varying speeds of light within physical media (here represented with a wiggly light cone). The Lorentz-Santilli isosymmetry has restored the exact validity of the Lorentz symmetry for interior dynamical problems [53] [30].*

and $\hat{SO}^0(3.1)$ are locally isomorphic, thus restoring the exact character of the abstract axioms of the Lorentz for all possible values $C = c/n_4$. We therefore have the following important property [30]:

LEMMA 2.3.5: The Lie-Santilli $\hat{SO}^0(3.1)$ *isosymmetry restores the exact validity of Lorentz's axioms for locally varying speeds of light.*

This property is due to the reconstruction of the exact light cone on the iso-Minkowskian isospace over isofields with maximal causal value c, called the *light isocone,*

$$\hat{x}^{\hat{2}} = \hat{x}_3^2 + \hat{x}_4^2 = 0, \tag{45}$$

while its projection on the conventional Minkowski space over conventional fields represents a locally varying speed

$$\hat{x}^2 = (\frac{x_3^2}{n_3^2} - t^2\frac{c^2}{n^4})\hat{I} = 0. \tag{46}$$

This property is due to the fact that the mutation of the $\hat{x}^3$ and $\hat{x}^4$ isocoordinates occurs jointly with the *inverse* mutation of the corresponding isounits, by therefore preserving the original perfect light cone with c as the maximal causal speed (see the 1966 monograph [30] for details)

$$\begin{aligned} x^3 \ \rightarrow \ \tfrac{x^3}{n_3}, \quad & I_3 = 1 \ \rightarrow \ \hat{I}_3 = n_3 \\ x^4 = tc \ \rightarrow \tfrac{x^4}{n_4} = t\tfrac{c}{n_4}, \quad & I_4 = 1 \ \rightarrow \ \hat{I}_4 = n_4. \end{aligned} \tag{47}$$

Another significant aspect of Ref. [53] is the achievement of the first known invariance of non-linear, non-local and non-Hamiltonian interactions thanks to their embedding in the characteristic n-quantities of the isoinvariant (25).

2.5.8. Isotranslations. In view of their non-linearity, isotranslations in four parameters a_μ can be written in their projection in the conventional Minkowski space [30]

$$x'^\mu = x^\mu + A^\mu(a, x, \ldots), \tag{48}$$

and can be written via a power series expansion of the general expression

$$A^\mu = a^\mu(n_\mu^{-2} + a^\alpha[n_\mu^{-2}\hat{;}P_\alpha]/1! + \ldots), \tag{49}$$

The understanding of the isotopic completion of 20th century spacetime symmetries requires the knowledge that, when properly written on iso-Minkowskian isospace over isofields, isotranslations recover their conventional form . [30].

2.5.9. Isodilatations. Santilli introduced in Ref. [60] a novel one-dimensional isoinvariance denoted $\hat{D}$ which is given by the dilatation of the isometric caused by its multiplication by as parameter w, while the isounit is jointly subjected to the *inverse* dilatation

$$\begin{aligned} \hat{\Omega} = \hat{\eta}\hat{I} \ \rightarrow \ \hat{w} \star \hat{\Omega} = w\hat{\eta}\hat{I}' \\ \hat{I} \ \rightarrow \ \hat{I}' = \tfrac{1}{w}\hat{I}, \end{aligned} \tag{50}$$

under which isoinvariant (25) remain manifestly unchanged.

In essence, the new symmetry originates from the fact that, for mathematical consistency, isoinvariants must be elements of t isofields, thus having structure (25), namely, isoinvariants must be given by a conventional invariant multiplied by the isounit.

Ref. [60] showed that, by writing conventional invariants with the multiplication, in this case, by the trivial unit 1, the new dilatation symmetry persists for conventional space-time symmetries,

$$\eta \rightarrow \eta' = w\eta, \quad 1 \ \rightarrow \ 1' = \frac{1}{w}1. \tag{51}$$

The above properties imply the following:

LEMMA 2.5.5: The conventional Lorentz-Poincaré symmetry is eleven-dimensional with structure

$$P^0(3.1) = so^0(3.1) \times A(3.1) \times D, \tag{52}$$

and, consequently, the Lorentz-Poincaré-Santilli isosymmetry is also eleven-dimensional with the structure

$$\hat{P}^0(3.1) = \hat{so}^0(3.1) \star \hat{A}(3.1) \star \hat{D}. \tag{53}$$

The above seemingly trivial property has permitted Santilli the study of a new grand unification of electroweak and gravitational interactions based on the embedding of gravitation in the isotopic degree of freedom of the theory [35].

2.5.10. Isoinversions. The isotopic "completion" of conventional inversions has been studied in details in Refs. [30] and consists of the *isotime isoinversions*

$$\hat{\tau}\hat{t} = (\tau\hat{t})\hat{I} \tag{54}$$

plus the *isospace isoinversions*

$$\hat{\pi}\hat{r} = (\pi\hat{r})\hat{I} \tag{55}$$

where τ and π are conventional time and space inversions, respectively.

Despite their simplicity, Santilli has shown in Ref. [30] that *not only continuous, but also discrete space-time symmetries can be reconstructed as being exact on isospaces over isofields when assumed to be broken on conventional spaces over conventional fields.*

2.5.11. Isospinorial LPS isosymmetry. Recall that the spinorial covering $\mathcal{P}^0(3.1)$ of the connected component of the LP symmetry $P^0(3.1)$ is constructed via the use of the Dirac gamma matrices. In fact, the conventional generators are realized via suitable combination of Dirac gamma matrices.

By following the same historical pattern, Santilli proposed in the 1995 communication [61] of the *Joint Institute for Nuclear Research,* Dubna, Russia (see also the subsequent paper [62]) the following eleven-dimensional isotopic "completion" $\hat{\mathcal{P}}^0(3.1)$ of $\mathcal{P}^0(3.1)$

$$\hat{\mathcal{P}}(3.1) = \hat{SL}(2.\hat{C}) \star \hat{\mathcal{A}}(3.1) \star \hat{\mathcal{D}}, \tag{56}$$

with realization of the generators in terms of the Dirac-Santilli isogamma isomatrices $\hat{\Gamma}_\mu = \hat{\gamma}_\mu \hat{I}$, Eqs. (I-89),

$$\hat{SL}(2.\hat{C}): \quad \hat{R}_k = \tfrac{1}{2}\epsilon_{kij}\hat{\Gamma}_i \star \hat{\Gamma}_j, \quad \hat{S}_k = \tfrac{1}{2}\hat{\Gamma}_k \star \hat{\Gamma}_4,$$

$$\hat{\mathcal{A}}(3.1): \quad \hat{P}_\mu, \tag{57}$$

$$k = 1, 2, 3, 4, 5, 6, \quad \mu = 1, 2, 3, 4.$$

The verification by the above isogenerators of isocommutation rules (33)-(35) is an instructive exercise for the interested reader. The proof that the Dirac-Santilli isoequations (I-88) transform isocovariantly under $\hat{\mathcal{P}}^0(3.1)$ is equally instructive.

2.5.12. Galilean isosymmetries.

As it is well known, the Galileo symmetry $G(3.1)$ characterizes the non-relativistic motion of point particles in vacuum, with consequential absence of resistive or non-potential forces (see the vertical line of Figure 8).

The isotopies of the Galileo symmetry are intended to characterize the non-relativistic motion of extended particles within physical media, by therefore experiencing resistive non-potential forces (see the wiggly line of Figure 8).

The resulting infinite family of isosymmteries $\hat{G}(3.1)$ are here called *Galilean isosymmetries* to stress the preservation of the basic axioms of the Galileo symmetry and the mere construction of the broadest possible realizations permitted by isomathematics.

The Lie-isotopic lifting of the Galileo symmetry were introduced by Santilli in the 1978 paper [12] as a particular case of the covering *Lie-Admissible symmetries,* also called *genosymmetries,* which are intended fo characterize the *time rate of variation of physical quantities.*

The first direct study of Galilean isosymmetries was done in Section 5.3, pages 225 on, of the 1981 monograph [26] formulated over conventional fields. These isotopies were then systematically studies and upgraded in the two 1991 volumes [27] [28]. The formulation of Galilean isosymmetries with the full use of isomathemaics was done in the 1995 monographs [29] [30] with a final study presented in Ref. [32].

The above studies attracted the attention of Abdus Salam, founder and president of the *International Center for Theoretical Physics* (ICTP), Trieste, Italy, who invited Santilli in 1991 to deliver at his Center a series of lectures

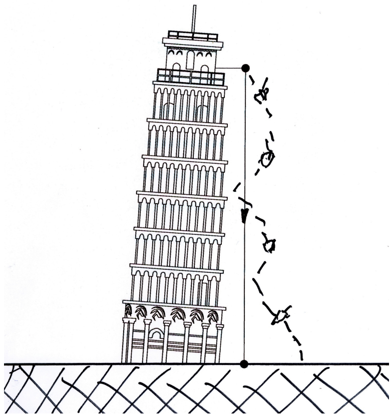

Figure 8: *This figure presents a conceptual rendering of the free fall of point-masses in vacuum studied by Galileo (represented with a straight line), and the free fall of extended masses experiencing resistive forces from our atmosphere studied by Santilli (represented with a wiggling line) [26] [30] [37]. It is symptomatic to note that the achievement of the symmetry for extended masses required the construction of a covering of the mathematics used for the point masses with particular reference to the generalization of Newton-Leibnitz differential calculus, from its historical formulation for isolated point, to a covering formulation for volumes [21].*

in the isotopies of the Galileo symmetry and relativity, said invitation being apparently the last by Salam prior to his death.

During his visit at the ICTP, Santilli wrote papers [65] through [71]. The notes from Santilli's lectures were collected by A. K. Aringazin, A. Jannussis, F. Lopez, M. Nishioka and B. Vel-janosky and published in volume [37] of 1992.

This work is primarily intended for relativistic isosymmetries. Additionally, all primary applications require relativistic treatments. Therefore, we regret to be unable to review Galilean isosymmetries to prevent a prohibitive length.

Nevertheless, the reader should be aware that an introductory knowledge of the Galilean isosymmetries is suggested, e.g., from the reading of the ICTP papers [65] to [71].

3. APPARENT PROOFS OF THE EPR ARGUMENT

3.1. Foreword.

As it is well known, the conventional Pauli matrices $\sigma_k,\ k = 1, 2, 3$, are the fundamental (also called adjoint), irreducible unitary representation of the $SU(2)$-spin symmetry and play a crucial role for the objections against the

EPR argument [2] - [6] .

In this section, we review the isotopic "completion'" of Pauli's matrices into isomatrices

$$\hat{\Sigma}_k = \hat{\sigma}_k \hat{I} \tag{58}$$

which constitute the isofundamental, isoirreducible, isounitary isorepresentation of the Lie-Santilli $\hat{SU}(2)$ isosymmetry and play a crucial role in the apparent proof of the EPR argument for extended particles within physical media studied later on in this section.

By recalling that the $SU(2)$ symmetry characterizes the spin of point-particles in vacuum, the "completed $\hat{SU}(2)$ isosymmetry is intended to characterize the spin of extended particles within hyperdense media called *hadronic spin,* such as the spin of an electron in the core of a star.

The isotopic "completion" of Pauli's matrices was introduced by Santilli in 1993 while visiting the *Joint Institute for Nuclear Research,* Dubna, Russia [58]. Said "completion" was presented systematically in Refs. [29] [30], used for the apparent proofs of the EPR argument [10] [11], and they are nowadays known as the *Pauli-Santilli isomatrices* [45].

In particular, the preceding studies have shown that, unlike the case for the $SU(2)$ symmetry, the isotopic $\hat{SU}(2)$ isosymmetry admits an explicit and concrete realization of hidden variables λ [3] [4] via realizations of the isotopic element of type $\hat{T} = Diag.(1.\lambda, \lambda)$ Eq. (3).

In this section, we shall review the construction of $\hat{SU}(2)$ isosymmetry and of Pauli-Santilli isomatrices of regular and irregular type with hidden variables. We shall then use the methods acquired in this and in the preceding paper [9], for the proof that *interior dynamical systems represented via isomathematics and isomechanics appear to admit identical classical counterparts* [10] (Section 3.7), and to *progressively approach the classical EPR determinism [1] in the structure of hadrons, nuclei and stars, while achieving the EPR determinism in the interior of gravitational collapse* [11] (Section 3.8).

A first understanding of this section requires a knowledge of the Lie-Santilli isotheory (Section I-2.7) [26] [30] [38] [46] [47] [49]. A technical understanding of this section requires a technical knowledge of hadronic mechanics [29]- [31].

3.2. Pauli matrices.

As it is well known (see, e.g., Ref. [72]), the carrier space of the two-dimensional group of special unitary transformations $SU(2)$ is the two-dimensional complex Euclidean space $E(z, \delta, I)$ with coordinates $z = (z_1, z_2)$, metric $\delta = Diag.(1, 1)$ and unit $I = Diag/(1, 1)$.

The two-dimensional, fundamental (also called adjoint), irreducible, unitary representation of the special unitary Lie algebra $su(2)$ of the $SU(2)$-spin symmetry is given by the celebrated *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i1 \\ i1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{59}$$

with commutations rules

$$[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i = i2\epsilon_{ijk}\sigma_k, \tag{60}$$

and eigenvalues on a Hilbert space $calH$ over the field of complex numbers $\mathcal{C}$ with basis $|b>$

$$\begin{gathered} \sigma^2|b>= (\sigma_1\sigma_1 + \sigma_2\sigma_2 + \sigma_3\sigma_3)|b>= 3|b>, \\ \sigma_3|b>= \pm 1|b>. \end{gathered} \tag{61}$$

Among the various properties of Pauli's matrices, we should recall their uniqueness in the sense that their expression is invariant under the class of equivalence admitted by quantum mechanics, that under unitary transformation.

We should also recall that Pauli's matrices are also fundamental for the structure of Dirac's equation, Eq. (I-9) since they appear in the very definition of Dirac's gamma matrices, Eqs. (I-89).

3.3. Regular Pauli-Santilli isomatrices.

By following Ref. [58], the carrier isospace of the two-dimensional Lie-Santilli isogroup of isospecial isounitary isotransformations $\hat{SU}(2)$ is the isocomplex iso-Euclidean isospace $\hat{E}(\hat{z}, \hat{\Delta}, \hat{I})$ with isocoordinates

$$\hat{z} = z\hat{I} = (z_1, z_2) = (z_1, z_2)\hat{I}; \tag{62}$$

isounit and isotopic element

$$\hat{I} = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix} = 1/\hat{T} > o, \tag{63}$$

$$\hat{T} = \begin{pmatrix} n_1^{-2} & 0 \\ 0 & n_2^{-2}; \end{pmatrix} \tag{64}$$

isometric

$$\hat{\Delta} = \hat{\delta}\hat{I} = (\hat{T}_i^k\delta_{ki})\hat{I} = \begin{pmatrix} n_1^{-2} & 0 \\ 0 & n_2^{-2} \end{pmatrix}\hat{I}; \tag{65}$$

positive-definite characteristic quantities n_k with unrestricted functional dependence on the variables for interior dynamical problems

$$n_k = n_k(z, \bar{z}, E, \mu, \alpha, \tau, \psi, \partial\psi, ...) > 0, \quad k = 1, 2; \tag{66}$$

and basic isoinvariant

$$\begin{gathered} \hat{z}_i \star \hat{\Delta}_{ij} \star \hat{\bar{z}}_j = (z_i \hat{\delta}_{ij} z_j)\hat{I} = \\ = (\tfrac{z_1 \bar{z}_1}{n_1^2} + \tfrac{z_2 \bar{z}_2}{n_2^2})\hat{I}. \end{gathered} \tag{67}$$

By also following Refs. [29] [58], the isogroup of regular, isospecial, isounitary, isotransformations $\hat{SU}(2)$ leaving invariant isoline element (67), is characterized by the isotransforms

$$\hat{z}' = \hat{U}(\hat{\theta}) \star z = \hat{U}(\hat{\theta})\hat{T}\hat{z}, \tag{68}$$

verifying the following conditions [30]:

1. Isounitarity

$$\hat{U}(\hat{\theta}) \star \hat{U}^{\dagger}(\hat{\theta}) = \hat{U}^{\dagger}(\hat{\theta}) \star \hat{U}(\hat{\theta}) = \hat{I}; \tag{69}$$

2. Isogroup isoaxioms

$$\begin{gathered} \hat{U}(\hat{\theta}_1) \star \hat{U}(\hat{\theta}_2) = \hat{U}(\hat{\theta}_1 + \hat{\theta}_2), \\ \hat{U}(\hat{\theta}) \star \hat{U}(-\hat{\theta}) = \hat{U}(0) = \hat{I}, \quad k = 1, 2, 3; \end{gathered} \tag{70}$$

and

3. Isospecial isounitarity

$$IsoDet\hat{U}(\hat{\theta}) = \hat{I}, \ Det(U\hat{T}) = 1. \tag{71}$$

The latter condition essentially restricts the isogroup $\hat{SU}(2)$ to its isoconnected component $\hat{SU}^0(2)$, which is hereon tacitly assumed.

The above conditions imply the local isomorphism

$$\hat{SU}(2) \approx SU(2), \tag{72}$$

and the following explicit realization in terms of isoexponential (I-22)

$$\begin{gathered} \hat{U}(\hat{\theta}) = \Pi_k U_k(\theta_k)\hat{I} = \Pi_k \hat{e}^{\hat{i} \star \hat{J}_k \star \hat{\theta}_k} = \Pi_k (e^{iJ_k \hat{T} \theta_k})\hat{I}, \\ U_k(\theta_k) = e^{iJ_k \hat{T}\theta_k} \quad , k = 1, 2, 3, \end{gathered} \tag{73}$$

where $\hat{J}_k$ represents the isogenerators of the Lie-Santilli isoalgebra $\hat{su}(2)$ verifying the conditions

$$IsoTr\hat{J}_k = 0, \ \ Tr(\hat{J}_k\hat{T}) = 0, \tag{74}$$

and the isocommutation rules

$$\begin{aligned} [\hat{J}_i \hat{,} \hat{J}_j] = \hat{J}_i \star \hat{J}_j - \hat{J}_j \star \hat{J}_i = \\ = \hat{J}_i \hat{T} \hat{J}_j - \hat{J}_j \hat{T} \hat{J}_i = \epsilon_{ijk} \hat{J}_k. \end{aligned} \tag{75}$$

Note that, in accordance with Theorem I-2.7.2, the isorepresentations here considered are called *regular* because they can be constructed via non-unitary transformations of the conventional $su(2)$ algebra, resulting in the preservation of the conventional structure *constants* ϵ_{ijk}.

However, as we shall see in the next section, the isotopies of the $su(2)$ algebra imply realizations called *irregular* that cannot be constructed via non-unitary representations of $su(2)$ [58], in which case the structure constants ϵ_{ijk} are replaced by *structure functions* with an arbitrary (non-singular) functional dependence on local variables,

$$\hat{C}_{ijk} = C_{ijk}(z, \bar{z}, E, \nu, \alpha, \tau, \psi, \partial\psi, ...)\hat{I}. \tag{76}$$

As one can verify, $\hat{su}(2)$ admits the following iso-Casimir isoinvariant

$$\begin{aligned} \hat{J}^{\hat{2}} = \Sigma_k \hat{J}_k \star \hat{J}_k = \\ = \hat{J}_1 \hat{T} J_1 + \hat{J}_2 \hat{T} J_2 + \hat{J}_3 \hat{T} J_3. \end{aligned} \tag{77}$$

The maximal set of isocommuting isooperators is then given by $\hat{J}_3$ and $\hat{J}^{\hat{2}}$.

By again following Ref. [58], in order to compute the explicit form of the isorepresentations of $\hat{su}(2)$, we introduce the Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$[19] over the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$ [20] with d-dimensional isobasis $|\hat{b}^d_k>$ verifying isonormalization (I-75),

$$\begin{aligned} < \hat{b}^d_k| \star |\hat{b}^d_k > = < \hat{b}^d_k|\hat{T}|\hat{b}^d_k > = \hat{I}, \\ d = 1, 2, 3, ...N, , \quad k = 1, 2, , 3. \end{aligned} \tag{78}$$

From the local isomorphism $\hat{su}(2) \approx su(2)$ we know that the isoeigenvalue equations have the structure

$$\begin{aligned} \hat{J}_k \star |\hat{b}^d_k > = b^d_k|\hat{b}^d_k >, \\ \hat{J}^{\hat{2}} \star |\hat{b}^d_k > = \Sigma_k b^d_k(b^d_k + W)|\hat{b}^d_k > \ , \\ W = Det\hat{T} = 1/n_1^2 n_2^2, \end{aligned} \tag{79}$$

where $W = 1$ for regular isorepresentation, otherwise W is an arbitrary function of local quantities to be identified via subsidiary constraints from the medium in which extended particles are immersed.

The explicit form of the isorepresentations of $\hat{su}(2)$ is then given by the simple isotopy of the conventional case [72]

$$\begin{gathered}\hat{J}_{\pm} = \hat{J}_1 \pm \hat{J}_2, \\ (\hat{J}_1)_{ij} = i\tfrac{1}{2} < \hat{b}_k^d| \star (\hat{J}_- - \hat{J}_+) \star |\hat{b}_k^d >, \\ (\hat{J}_2)_{ij} = i\tfrac{1}{2} < \hat{b}_k^d| \star (\hat{J}_- + \hat{J}_+) \star |\hat{b}_k^d >, \\ (J_3)_{ij} =< \hat{b}_k^d| \star \hat{J}_3 \star |\hat{b}_k^d > .\end{gathered} \tag{80}$$

By continuing to follow Ref. [58], we now restrict our attention to the two-dimensional isofundamental (isoadjoint) isorepresentation of $\hat{su}(2)$ occurring for $d = 2$, in which case we assume

$$\hat{J}_k = \frac{1}{2}\hat{\sigma}_k, \quad k = 1, 2, 3, \tag{81}$$

and select the basic isounitary isotransform according to Sections I-2.8 and I-2.9

$$UU^\dagger = f(W) > 0, \;\; W = Det.\hat{I} = n_1^2 n_2^2, \tag{82}$$

where $f(W)$ is a smooth function such that $f(1) = 1$.

By using the above procedure, we have the following *regular Pauli-Santilli isomatrices* first introduced by Santilli in Ref. [58], Eqs. (3.2) (where the isometric elements are denoted $g_{kk} = n_k^{-2}, \; k - 1, 2,$

$$\begin{gathered}\hat{\Sigma}_k = \hat{\sigma}_k \hat{I}, \\ \hat{\sigma}_1 = (n_1 n_2) \begin{pmatrix} 0 & n_1^{-2} \\ n_2^{-2} & 0 \end{pmatrix}, \;\; \hat{\sigma}_2 = (n_1 n_2) \begin{pmatrix} 0 & -in_1^{-2} \\ in_2^{-2} & 0 \end{pmatrix}, \\ \hat{\sigma}_3 = (n_1 n_2) \begin{pmatrix} n_2^{-2} & 0 \\ 0 & -n_1^{-2}, \end{pmatrix}.\end{gathered} \tag{83}$$

with isocommutation rules

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = i2\epsilon_{ijk}\hat{\sigma}_k, \tag{84}$$

in which the 'regular' character of the isomatrices is established by the presence of the conventional (constant) structure constants.

We then have the isoeigenvalues isoequations

$$\hat{\sigma}_3 \star |\hat{b}^2_m >= \hat{\sigma}_3 \hat{T} |\hat{b}^2_m >= \pm \tfrac{1}{n_1 n_2} |\hat{b}^2_m >$$

$$\hat{\sigma}^{\hat{2}} = (\sigma_1 \hat{T} \hat{\sigma}_1 + \sigma_2 \hat{T} \hat{\sigma}_2^{+} \sigma_3 \hat{T} \hat{\sigma}_3) \hat{T} |\hat{b}^2_m >= \tag{85}$$

$$= 3 \tfrac{1}{n_1^2 n_2^2} |\hat{b}^2_m >,$$

showing that the regular Pauli-Santilli isomatrices preserve the conventional structure constants ϵ_{ijk} of Pauli matrices, but exhibit structure (84) with generalized isoeigenvalues containing two characteristic quantities n_1^2, n_2^2.

It is evident that, under isounimodularity condition (71),

$$Det\hat{T} = 1, \;\; n_1 = 1/n_2, \tag{86}$$

isomatrices (83) reduce to

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1^{-2} \\ n_2^{-2} & 0 \end{pmatrix}, \;\; \hat{\sigma}_2 = \begin{pmatrix} 0 & -i n_1^{-2} \\ i n_2^{-2} & 0 \end{pmatrix},$$

$$\hat{\sigma}_3 = \begin{pmatrix} n_2^{-2} & 0 \\ 0 & -n_1^{-2} \end{pmatrix}, \tag{87}$$

by verifying conventional commutation rules (84) and conventional eigenvalues

$$\hat{\sigma}_3 \star |\hat{b}^2_m >= \pm |\hat{b}^2_m >$$

$$\hat{\sigma}^{\hat{2}} \star |\hat{b}^2_m >= 3 |\hat{b}^2_m > . \tag{88}$$

In order to search for additional realizations of regular Pauli-Santilli isomatrices, we now assume the following non-unitary transform

$$U = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} = U^{\dagger}, \tag{89}$$

under which we have the following second realization of regular Pauli-Santilli isomatrices

$$\hat{\sigma}_k = U \sigma_k U^{\dagger},$$

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1 n_2 \\ n_1 n_2 & 0 \end{pmatrix}, \;\; \hat{\sigma}_2 = \begin{pmatrix} 0 & -i n_1 n_2 \\ i n_1 n_2 & 0 \end{pmatrix}, \tag{90}$$

$$\hat{\sigma}_3 = \begin{pmatrix} n_1^2 & 0 \\ 0 & -n_2^2 \end{pmatrix}.$$

It is an instructive exercise for the interested reader to verify that the above isomatrices verify the isocommutation rules (84) and conventional isoeigenvalue (99), namely, *the second realization of the Pauli-Santilli isomatrices, Eqs. (83), also admit conventional structure constants and eigenvalues despite the degrees of freedom permitted by the two characteristic quantities* $n_1^2,\ n_2^2$.

We now assume the following non-diagonal realization of the non-unitary transform

$$U = \begin{pmatrix} 0 & n_1 \\ n_2 & 0 \end{pmatrix}, U^\dagger = \begin{pmatrix} 0 & n_2 \\ n_1 & 0 \end{pmatrix}, \tag{91}$$

$$UU^\dagger = \hat{I} = 1/\hat{T} > 0,$$

which characterizes the following third realization of the regular Pauli-Santilli isomatrices

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1 n_2 \\ n_1 n_2 & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -i n_1 n_2 \\ i n_1 n_2 & 0 \end{pmatrix},$$

$$\hat{\sigma}_3 = \begin{pmatrix} -n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}. \tag{92}$$

It is easy to see that the above third realization of the regular Pauli-Santilli isomatrices also verify conventional commutation rules (84) and eigenvalues (88).

Note that, while Pauli's matrices are invariant under unitary transforms, there exist a number of Pauli-Santilli isomatrices each of which is invariant under isounitary isotransforms (Section I-3.9).

3.4. Irregular Pauli-Santilli isomatrices.

3.4.1. Historical notes. One of the most fundamental, yet unresolved processes in nature is the synthesis of the neutron from the hydrogen in the core of stars, which is a pre-requisite for the production of light and, therefore, for the existence of life itself.

In this section, we would like to outline the main historical aspects on the synthesis of the neutron and identify the open problems because truly fundamental for the construction of the new mathematics needed for their solution.

Recall that stars initiate their life as an aggregate of hydrogen and grow via the accretion of hydrogen existing in interstellar spaces.

In 1910, H. Rutherford [73] conjectured that, when the pressure and temperature at the core of the star reaches certain values, the hydrogen atom is "compressed" into a neutral particle n which is called the *neutron* according to the reaction (see Section I-4)

$$e^- + p^+ \rightarrow n. \tag{93}$$

Rutherford hypothesis was experimentally confirmed by J. Chadwick in 1932 [74], and the neutron became part of scientific history.

Following the experimental verification that the neutron has the same spin $1/2$ of the electron and of the proton, in an attempt at maintaining the conservation of the angular momentum, E. Fermi [75] suggested that the synthesis of the neutron occurs with the emission of a hypothetical, massless and chargeless particle ν with spin $1/2$ which he called the *neutrino* (meaning "little neutron" in Italian), according to the reaction widely accepted by the scientific communioty for about one century

$$e^- + p^+ \rightarrow n + \nu. \tag{94}$$

After joining Harvard University in September 1977 under DOE support, R. M. Santilli [15]-[17] noted that, despite the salvaging of space-time symmetries and related conservation laws, reaction (93) is not compatible with quantum mechanical laws because the rest energy of the neutron E_n is $0.782\ MeV$ *bigger* than the sum of the rest energies of the proton E_p and of the electron E_e,

$$\begin{gathered} E_p = 938.272\ MeV, \ \ E_e = 0.511\ MeV, \ \ E_n = 939.565\ MeV, \\ E_n - (E_p + E_e) = 0.782\ MeV > 0. \end{gathered} \tag{95}$$

Therefore, Santilli presented a number of arguments according to which the synthesis of the neutron is clear evidence of Einstein's view on the lack of "completion" of quantum mechanics (see Einstein's name in the title of the 1981 paper [17] released from the Department of Mathematics of Harvard University). Subsequently, Santilli achieved in Ref. [76] (see the independent review [**?**] the non-relativistic representation of *all* characteristics of the neutron in its synthesis from the hydrogen representation, with a relativistic representation subsequently achieved in Ref. [62] (see the independent review in ref. [45]).

The technically most difficult problem of the above representation was the identification of the spin-orbit coupling for the electron when totally immersed inside the proton which was first solved non-relativistically in Ref. [76] and relativistically in Ref. [62] (see the review in Ref. [30], Chapter 6 in particular).

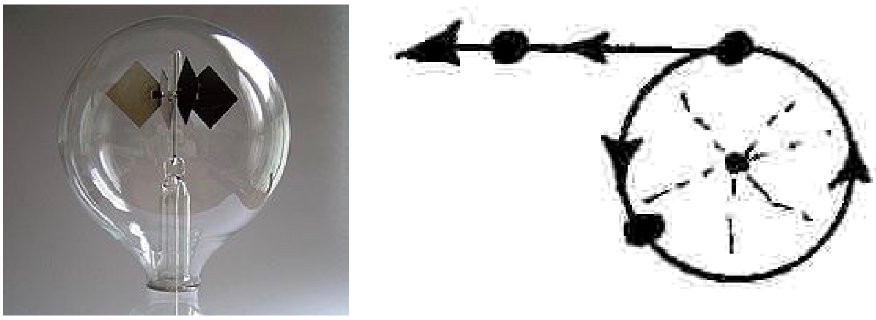

Figure 9: *To illustrate some of isosymmetries for interior systems, in the left we show the conversion of linear momentum into angular momentum for the case of photons causing the rotation of a small propeller in a vacuum chamber. In the right, we show the opposite conversion of angular momentum into linear momentum, as it is the case for the sling shot. The left view illustrates that the neutrino hypothesis is not necessary for the synthesis of the neutron when particles are represented as being extended. The right view illustrates that the emission of an antineutrino is not necessary in the neutron decay because the internal angular momentum of the electron can be converted into its external linear momentum without any violation of physical laws.*

Recall that quantum mechanics has an excellent consistency for bound states with *negative potentials* causing a *mass defect.* Santilli's first argument is that a representation of experimental data (95) via quantum mechanics is impossible because it would require a *positive potential* capable of producing a *mass excess,* which features imply the loss of physical consistency of Schrödinger equation *for bound states* (and not for free particles with positive kinetic energy) because the indicial equation of Schrödinger equation admits no consistent solutions for positive potential energies, (see Section I-4).

The inability of Dirac's and other quantum mechanical equations for the representation of experimental data (95) then followed.

Various conjectures, aimed at maintaining for the neutron structure the theory so effective for the hydrogen atom, were proved not to be consistent. For instance, the hypothesis that the missing energy of $0.782\ MeV$ is provided by the star via a relative energy between the electron and the proton had to be dismissed because the cross section $e-p$ at $0.782\ MeV$ is essentially null, thus preventing any fusion between the electron and the proton.

Similarly, the hypothesis that the missing energy is provided by the antineutrino $\bar{\nu}$ via reactions of the type

$$e^- + \bar{\nu} + p^+ \ \rightarrow \ n, \tag{96}$$

had to be equally abandoned because the cross section between neutrinos or antineutrinos and individual particles is identically null.

As it is well known, the neutrino hypothesis is necessary for the quantum mechanical treatment of synthesis (94), namely, for the point-like characterization of the proton, the electron and the neutron. Ref. [17] indicated that the neutrino hypothesis cannot any longer be consistently applied for the neutron synthesis whenever particles are represented as being extended.

Independently from that, there exist no known conventional possibility of identifying the *energy* needed for the creation of the neutrino since synthesis (94) already misses $0.782\ MeV$ for the synthesis of the neutron.

Another argument of Ref. [17] is that the conservation of the angular momentum is necessary for the synthesis of bound states with a Keplerian center under the validity of conventional space-time symmetries, such as for the synthesis of the hydrogen atom form an electron and a proton.

However, said conservation is no longer necessary for bound states at short distances without a Keplerian center, since in that case we have the validity of space-time isosymmetries for which the angular momentum can be transformed into linear momentum and vice-versa without any violation of physical laws (see Section 2.2 and Figure 9).

But *the neutron has no Keplerian center,* with the consequential lack of applicability of the Lorentz-Poincaré symmetry, and the ensuing lack of necessary conservation of the angular momentum in favor of alternative hypotheses.

In view of the above (and other) insufficiencies of the neutrino hypothesis, Santilli suggested in Ref. [78] the introduction in the *l.h.s.* (rather than the r.h.s.) of the synthesis of the mass-less, charge-less and spin-less particle called the *etherino* and denoted with the symbol a (from the Latin *aether*)

$$p^+ + a + e^- \ \rightarrow \ n, \tag{97}$$

whose scope is to represent the delivery of the missing $0.782\ MeV$ to the neutron.

An intriguing aspect is that *the etherino hypothesis can be shown to be compatible with the experimental data of the so-called "neutrino experiments," of course, under the condition of abandoning point-like abstractions of hadrons and representing them as they are in the physical reality, i.e., extended, deformable and hyperdense.*

By recalling that there is no known consistent way of accounting for the missing $0.782\ MeV$ as originating from the star, Ref. [**?**] submitted the hypothesis that the missing energy may originate from the ether as a universal medium of extremely high energy density for the characterization and propagation of particles and electromagnetic waves.

However, it should be stressed that *the etherino is not intended to be a particle, but to be an "impulse" representing the mechanism of supplying the missing* $0.782\ MeV$ *for the neutron, the origin of the missing energy from the ether being only one among other possibilities.*

In view of the above insufficiencies of quantum mechanics for the synthesis of the neutron, Santilli initiated in Refs. [15] - [17] the search for a "completion" of quantum mechanics into hadronic mechanics with particular reference to the "completion"of Lie's theory at large, and the $SU(2)$-spin symmetry in particular, for the characterization of the spin of the electron when "compressed" inside the hyperdense proton.

It should also be recalled that, during the same period, Santilli conducted a post Ph. D. Seminar Course at the Lyman Laboratory of Physics of Harvard University with a technical treatment of the insufficiency of quantum mechanics for the neutron synthesis via the *conditions of variational self-adjointness* for the existence of a Lagrangian or a Hamiltonian. This Seminar Course was eventually published by Springer-Verlag in monographs [25] [26] whose primary aim is the first known presentation of the axiom-preserving Lie-Santilli isotheory and the axiom-inducing Lie-Santilli genotheory.

In fact, possible representations of experimental data (95) for the neutron synthesis violate the conditions of variational self-adjointness, thus mandating the search for a covering theory.

The subsequent 1995 papers [55] [56] [58] achieved the regular isotopies $\hat{SU}(2)$ of the spin symmetry (reviewed in the preceding section).

However, *regular isotopies of the* $SU(2)$ *spin symmetry are insufficient for the neutron synthesis because it requires alterations (called mutations) of conventional eigenvalues that can be solely represented via irregular isorepresentations.*

The irregular isorepresentations of the $SU(2)$-spin symmetry were identified, apparently for the first time, by Santilli in the 1990 paper [78] and used to achieve the non-relativistic representation of *all* characteristics of the neutron in its synthesis from the hydrogen.

In the 1995 paper [62], Santilli presented a relativistic study of $\hat{SU}(2)$ as an isosubalgebra of the irregular isospinorial covering of the Lorentz-Poincaré symmetry (Section 2.5.11) and used the results to achieve a relativistic representation of the neutron synthesis.

The indicated irregular representations of the $SU(2)$-spin symmetry were then instrumental for the apparent confirmations of the EPR argument in Refs.[10] [11] reviewed in this section.

3.4.2. Non-relativistic formulation. The first irregular isotopies of Pauli's matrices, today known as *irregular Pauli-Santilli isomatrices* [45], have been introduced in Eqs. (2.32) of Ref. [78] via the use of the isorepresentations of $\hat{SU}(2)$ worked out in the preceding papers [55] [56], and are given by

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & -n_1 \\ n_2 & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -in_1 \\ in_2 & 0 \end{pmatrix}, \\ \hat{\sigma}_3 = \tfrac{1}{n)1n_2} \begin{pmatrix} n_1^2 & 0 \\ 0 & -n_2^2 \end{pmatrix}, \tag{98}$$

with irregular isocommutation rules

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = 2i \frac{1}{n_1 n_2} \epsilon_{ijk} \hat{\sigma}_k, \ \ i, j, k, = 1, 2, 3 \tag{99}$$

and isoeigenvalues

$$\hat{\sigma}_3 \star |\hat{u}> = \pm \tfrac{1}{n_1 n_2} |\hat{u}>, \\ \hat{\sigma}^{\hat{2}} \star |\hat{u}> = \tfrac{1}{n_1 n_2} (\tfrac{1}{n_1 n_2} + 2) |\hat{u}> . \tag{100}$$

It is easy to see that, when the hyperdense medium surrounding the immersed particle is homogeneous and isotropic, the characteristics quantities can be normalized to the values $n_1 = n_2 = n_3 = 1$, in which case isoeigenvalues (100) are conventional. We therefore have the following

LEMMA 3.1: Irregular isorepresentations of the Lie-Santilli isosymmetry $\hat{su}(2)$ represent the inhomogeneity and anisotropy of media in which extended particles are immersed.

Among a number of additional irregular Pauli-Santilli isomatrices with isotopic element $\hat{T}$ in Eq. (64) we quote Eqs. (3.2) of Ref. [10]

$$\hat{\sigma}_1 = n_1 n_2 \begin{pmatrix} 0 & n_1^{-2} \\ n_2^{-2} & 0 \end{pmatrix}, \ \hat{\sigma}_2 = n_1 n_2 \begin{pmatrix} 0 & -in_1^{-2} \\ in_2^{-2} & 0 \end{pmatrix}, \\ \hat{\sigma}_3 = n_1 n_2 \begin{pmatrix} n_2^{-2} & 0 \\ 0 & -n_1^{-2} \end{pmatrix}, \tag{101}$$

with irregular isocommutation rules

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = 2i \frac{1}{n_1 n_2} \hat{\sigma}_k, \tag{102}$$

and isoeigenvalues

$$\begin{aligned} \hat{\sigma}_3 \star |\hat{u}> &= \pm \tfrac{1}{n_1 n_2} |\hat{u}>, \\ \hat{\sigma}^{\hat{2}} \star |\hat{u}> &= 3 \tfrac{1}{n_1^2 n_2^2} |\hat{u}>, \end{aligned} \tag{103}$$

The above isorepresentation appears to be significant when the medium causes a proportional alteration/mutation of both the third component as well as the total value of the spin of a particle having the value $1/2$ in vacuum.

Another example of irregular Pauli-Santilli isomatrices is given by Eqs. (3.7) of Ref. [10]

$$\begin{gathered} \hat{\sigma}_1 = \begin{pmatrix} 0 & n_2 \\ n_1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -in_2 \\ in_1 & 0 \end{pmatrix}, \\ \hat{\sigma}_3 = \begin{pmatrix} n_1^2 & \\ 0 & -n_2^2 \end{pmatrix}, \end{gathered} \tag{104}$$

with irregular isocommutation rules

$$\begin{gathered} [\hat{\sigma}_1 \hat{,} \hat{\sigma}_2] = 2i \tfrac{1}{n_1^2 n_2^2} \hat{\sigma}_3, \quad [\hat{\sigma}_2 \hat{,} \hat{\sigma}_3] = 2i \hat{\sigma}_1, \\ [\hat{\sigma}_3 \hat{,} \hat{\sigma}_1] = 2i \hat{\sigma}_2, \end{gathered} \tag{105}$$

and mutated isoeigenvalues

$$\begin{aligned} \hat{\sigma}_3 \star |\hat{u}> &= \pm |\hat{u}>, \\ \hat{\sigma}^{\hat{2}} \star |\hat{u}> &= \tfrac{2}{n_1^2 n_2^2} |\hat{u}> . \end{aligned} \tag{106}$$

The above isorepresentation may be useful when the anisotropy and inhomogeneity of the medium maintain the spin value $1/2$ along the third axis, yet they are such to deform the remaining components.

Additional example of irregular Pauli-Santilli isomatrices are available from Refs. [10] and [58], and can be readily constructed by interested readers.

3.4.3. Relativistic formulation. Consider the iso-Minkowskian isospace $\hat{M}(\hat{x}, \bar{\Omega}, \hat{I})$ with isometric

$$\hat{\Omega} = \hat{\eta}\hat{I}, \quad \bar{\eta} = \hat{T}\eta,$$

$$\hat{T} = Diag.(\tfrac{1}{m_1^2}, \tfrac{1}{m_2^2}, \tfrac{1}{m_3^2}, \tfrac{1}{m_4^2}), \tag{107}$$

$$m_\mu = m_\mu(r, p, E, \nu, \alpha, \tau, \pi, \psi, ...) > 0. \ \ \mu = 1, 2, 3, 4,$$

where the new characteristic quantities m_μ have been introduced to avoid confusion with the previously used symbols n_μ.

The relativistic formulation of the irregular isorepresentation of $\hat{SU}(2)$ were derived, apparently for the first time, in Eqs. (6.4c)-(6.4d) of Ref. [62], and can be written

$$J_k = \frac{1}{2}\epsilon_{kij}\hat{\gamma}_i \star \gamma_j, \tag{108}$$

where $\hat{\gamma}$ are the regular Dirac-Santilli isomatrices (I-89), i.e.,

$$\hat{\gamma}_k = \tfrac{1}{m_k}\begin{pmatrix} 0 & \hat{\sigma}_k \\ -\hat{\sigma}_k & 0 \end{pmatrix},$$

$$\hat{\gamma}_4 = \tfrac{i}{m_4}\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \tag{109}$$

and $\hat{\sigma}_k$ are the *regular Pauli-Santilli isomatrices.*

The irregular character of isorepresentation (108) is established by the presence of structure functions in the isocommutation rules, Eqs, (6.4c) of Ref. [62],

$$[J_i \hat{,} J_j] = \epsilon_{ijk}\frac{1}{m_k^2}J_k, \tag{110}$$

and in the irregular isoeigenvalues

$$J_3 \star |\hat{\psi}>= \pm\tfrac{1}{2}\tfrac{1}{m_1 m_2}|\hat{\psi}>,$$

$$\hat{J^2} \star |\hat{\psi}>= \tfrac{1}{4}(\tfrac{1}{m_1 m_2} + \tfrac{1}{m_2 m_3} + \tfrac{1}{m_3 m_1})|\hat{\psi}>, \tag{111}$$

that, as shown in Ref. [62], permit a relativistic representation of the spin of the neutron in its synthesis from the hydrogen.

Again one should note that, when the medium is homogeneous and isotropic, isoeigenvalues (101) are conventional.

Note that the assumption of mutated spin for an extended particle within a hyperdense medium implies the *inapplicability (rather than the violation) of the Fermi-Dirac statistics, Pauli's exclusion principle and other quantum mechanical laws* with the understanding that said mutations are internal, thus solely testable under *external* strong interactions, as indicated beginning with the *title* of Harvard's 1978 paper [15].

3.5. Isotopies of hadronic spin and angular momentum.

3.5.1. Historical notes. An electron orbiting in vacuum around the proton in the hydrogen atom experiences no resistive forces, thus verifying known symmetries and conservation laws.

When the same electron has been "compressed" inside the proton according to Rutherford [73], Santilli [78] argued that the sole possible angular moment is that permitted by *constraints* exercised on the electron by the internal medium.

Since the electron is about $2,000$ times lighter than the proton, *the most stable configuration is that for which the electron is "constrained" to orbit with a value of the angular momentum equal to the proton spin,* since any different configuration would imply big resistive forces (Figure 9).

Needless to say, fractional angular momenta are anathema for the quantum mechanical description of point-particles in vacuum.

However, the angular momentum of extended particles immersed within hyperdense hadronic media can acquire values other than integers, depending on the local physical conditions of the medium surrounding the particle, such as pressure, density, anisotropy, inhomogeneity, etc.

The first known quantitative study of *constrained angular momenta* of extended particles within hyperdense hadronic media was done at the non-relativistic level by Santilli in Ref. [78] of 1990 following the preceding isotopies of the rotational symmetry, Refs. [55] [56]. The study was then extended to the relativistic level in Ref. [62] of 1990.

These studies are crucial for quantitative representations of the synthesis of hadrons providing apparent verifications of the EPR argument, and can be summarized as follows.

3.5.2. Non-relativistic representation. Recall the central assumption of isosymmetries according to which conventional generators are preserved (because representing conventional total conservation laws), and only their product is lifted into the isotopic form (1) (to represent the extended character of the particles and their non-Hamiltonian interactions).

Hence, the definition of the *isoangular isomomentum,* also called *hadronic angular momentum,* on an iso-Euclidean isospace is the same as that of quantum mechanics

$$L_k = \epsilon_{ijk}\hat{r}_i \star \hat{p}_j = \epsilon_{ijk} r_i p_j, \tag{112}$$

although it is defined on a Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ with isostates $|\hat{\psi}>$ on an isocomplex isofield $\hat{\mathcal{C}}$, with *isolinear isomomentum* Eqs, (I-79), and isocommutation rules are then given by Eqs. (I-81).

It is then easy to verify the following isocommutation rules for the hadronic angular isomomentum, Eqs. (2.22b) [78]

$$[L_i \hat{,} L_j] = i\hat{I}\epsilon_{ijk}L_k, \tag{113}$$

where, as one can see, the characteristics of the medium, represented by the isounit $\hat{I}$, enter directly in the isocommutation rules.

The use of the *isosperical isoharmonic isofunctions* (see page 240 of Ref. [30] for details)

$$\begin{gathered} \hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}) = UY(\theta, \phi)U^\dagger = \hat{T}^{-1}Y_{\ell m}(\theta, \phi), \\ UU^\dagger = \hat{I} = 1/\hat{T} \neq I, \end{gathered} \tag{114}$$

where $Y_{\ell m}(\theta, \phi$ are the conventional spherical harmonic functions, yields the following isoeigenvalues, Eqs. (2.25), Ref. [78],

$$\begin{gathered} L_3 \star \hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}) = \hat{I}m\hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}), \\ L^{\hat{2}} \star \hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}) = \hat{I}\ell(\hat{I}\ell + 1)\hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}), \\ m = \ell, \ell - 1, ..., -\ell, \;\; m = 1, 2, 3, ... \end{gathered} \tag{115}$$

where one can see again the mutation of the eigenvalues caused by the surrounding medium.

Applications to particle physics then require specific realizations of the isounit $\hat{I}$, such as the simple assumption of expressions (4) used in Ref. [78]

$$\rho = |\hat{I}| =\approx |e^\gamma|, \tag{116}$$

where ρ is a function of all possible or otherwise needed local variables of the medium.

3.5.3. Isotopies of non-relativistic spin-orbit coupling. As one can see, isoeigenvalues (115) do not allow a representation of the constrained hadronic angular momentum of the electron when compressed inside the proton (Figure 9).

In view of this insufficiency, Santilli conducted in Ref. [76] (see Also Ref. [30], Chapter 6) a study of the eigenvalues of the combined spin and angular momentum of the electron in the indicated interior conditions.

We consider then the *total hadronic momentum*

$$J_{tot} = L_\ell \hat{\otimes} J_s, \tag{117}$$

with corresponding basis $|\hat{Y}\hat{\otimes}\hat{u} >$ and isoexpectation values, Eqs. (2.34), Ref. [78],

$$\begin{gathered}
J_{3,tot}|\hat{Y}\hat{\otimes}\hat{u} >= (\rho_m(\ell) \pm \tfrac{m(s)}{n_1 n_2})|\hat{Y}\hat{\otimes}\hat{u} > \\
J^{\hat{2}}_{tot} \star |\hat{Y}\hat{\otimes}\hat{u} >= (\rho\ell \pm \tfrac{s}{n_1 n_2})(\rho\ell \pm \tfrac{s}{n_1 n_2} + 1)|\hat{Y}\hat{\otimes}\hat{u} > \\
\ell = 0, 1, 2, 3, ... \;\; s = 0, \tfrac{1}{2}1, \tfrac{3}{2}, ..., \\
m(\ell) = \ell, \ell - 1, ..., -\ell, \;\; m(s) = s, s - 1, ..., -s.
\end{gathered} \tag{118}$$

Following a laborious journey initiated in 1977, isoeigenvalues (118) finally permitted Santilli to achieve the desired solution for $\ell = 1$ and $s = \frac{1}{2}$, Eq. (2.36), Ref. [78],

$$\rho = \frac{1}{2}\frac{1}{n_1 n_2}, \tag{119}$$

for which *the total hadronic angular momentum of the electron in the synthesis of the neutron is identically null,* $J_{tot} = 0$, *and the spin of the neutron coincides with that of the proton.*

More detailed studies pertaining to electric and magnetic dipoles excluded the alternative $J = 1$ of eigenvalues (118), as well as total hadronic angular momenta of the electron other than zero.

The preceding studies permitted a quantitative non-relativistic representation of the spin of the neutron in its synthesis from the hydrogen atom. A representation of the remaining characteristics of the neutron (mass, radius, charge, dipole moments, etc.) is reviewed in Section 4.5.

3.5.4. Isotopies of relativistic spin-orbit couplings. The hadronic spin $\hat{S} = S\hat{I}$ is a realization of the $\hat{SU}(2)$ isosubalgebra of $\hat{\mathcal{P}}(3.1)$ with generators (57), while the hadronic angular momentum $\hat{L} = L\hat{I}$ is a realization of the isorotational $\hat{SO}(3)$ isosubalgebra. Their relativistic formulation on iso-Minkowskian isospace (107) has been first derived in Eqs. (6.4a) (6.4b), Ref. [62] and are given by

$$\begin{gathered}
S_k = 2\epsilon_{kij}\hat{\gamma}_i \star \hat{\gamma}_j, \\
L_k = \epsilon_{kij} r_i \star p_j,
\end{gathered} \tag{120}$$

where $\hat{\gamma}_k$ are the Dirac-Santilli isomatrices.

We then have the irregularisocommutation rules

$$[S_i \hat{,} S_j] = \epsilon_{kij} m_k^2 \hat{S}_k,$$
$$[L_i \hat{,} L_j] = \epsilon_{ijk} m_k^2 L_k, \tag{121}$$

and isoeigenvalues, Eqs, (6.4d) Ref. [62]

$$\hat{S}_3 \star |\hat{\psi}>= \pm \tfrac{1}{m_1 m_2} |\hat{\psi}>,$$
$$\hat{S}^{\hat{2}} \star |\hat{\psi}>= (m_1^{-2} m_2^{-2} + m_2^{-2} m_3^{-2} + m_3^{-2} m_1^{-2}) |\hat{\psi}>,$$
$$\hat{L}_3 \star |\hat{\psi}>= \pm m_1 m_2 |\hat{\psi}>, \tag{122}$$
$$\hat{L}^{\hat{2}} \star |\hat{\psi}>= (m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2) |\hat{\psi}> .$$

The most salient difference between relativistic isoeigenvalues (122) and their non-relativistic counterparts (155) is that *the former admit fractional hadronic angular momenta while the latter do not.*

In fact, for the following values admitted by a homogeneous and isotropic medium [62]

$$m_1 = m_2 = m_3 = \frac{1}{\sqrt{2}}, \tag{123}$$

isoeigenvalues (122) become

$$\hat{S}_3 \star |\hat{\psi}>= \pm \tfrac{1}{2} |\hat{\psi}>,$$
$$\hat{S}^{\hat{2}} \star |\hat{\psi}>= \tfrac{3}{4} |\hat{\psi}>,$$
$$\hat{L}_3 \star |\hat{\psi}>= \pm \tfrac{1}{2} |\hat{\psi}>, \tag{124}$$
$$\hat{L}^{\hat{2}} \star |\hat{\psi}>= \tfrac{3}{4} |\hat{\psi}> .$$

Consequently, *isoeigenvalues (122) permit a quantitative representation of the hadronic angular momentum of the electron as being constrained to be equal to the proton spin* [61] [62] (Figure zzzz).

In this case too, the total hadronic angular momentum of the electron is null because the only stable hadronic spin-orbit coupling is in singlet, and the spin of the electron can be assumed in first good approximation not to be mutated since the electron is about $2,000$-times lighter than the proton.

Hadronic spins, hadronic angular momenta and hadronic spin-orbit couplings were studied in detail Chapter 6 of Ref. [30] resulting in Lemma 6.12.1 here reproduced without proof:

LEMMA 3.2: When immersed within hadrons or nuclei with spin $1/2$, an elementary particle having spin $1/2$ in vacuum can only have a null total hadronic angular momentum.

As we shall see in Section 4.6, the above configuration of the synthesis of the neutron from the hydrogen is an apparent verification of the EPR argument.

3.6. Realization of hidden variables.

As recalled in Section 1.1, the conventional quantum mechanical realization of the Lie symmetry $SU(2)$ does not allow a consistent representation of hidden variables λ [3] [4].

It is easy to see that, despite the local isomorphism $\hat{SU}(2) \approx SU(2)$, the Lie-Santilli isosymmetry $\hat{SU}(2)$ does indeed allow explicit and concrete realizations of hidden variables thanks to the degree of freedom permitted by the isotopic element (1) in the structure of the Lie-Santilli isoproduct (2) with realizations of the isotopic element of type (3).

In this section, we review the explicit and concrete realization of *regular hidden variables,* namely, realizations that can be derived via non-unitary transforms of the Lie algebra $su(2)$, and then review *irregular hidden variables,* namely, realizations that do not admit such a simple derivation.

Regular and irregular realizations of hidden variables have been first identified by Santilli in Ref. [58] of 1993, and then used for the proof of the EPR argument [10] reviewed in Section 3.7.

Realizations of regular hidden variables are easily provided by Pauli-Santilli isomatrices (83) with the identifications

$$n_1^2 = \lambda_1, \;\; n_2^2 = \lambda_2, \tag{125}$$

yielding the desired realization, Eqs. (3.9), Ref. [58],

$$\hat{\sigma}_1 = (\lambda_1 \lambda_2) \begin{pmatrix} 0 & \lambda_1^{-1} \\ \lambda_2^{-1} & 0 \end{pmatrix}, \;\; \hat{\sigma}_2 = (\lambda_1 \lambda_2) \begin{pmatrix} 0 & -i\lambda_1^{-1} \\ i\lambda_2^{-1} & 0 \end{pmatrix},$$
$$\hat{\sigma}_3 = (\lambda_1 \lambda_2) \begin{pmatrix} \lambda_2^{-1} & 0 \\ 0 & -\lambda_1^{-1} \end{pmatrix} \tag{126}$$

verifying isocommutation rules

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = i\epsilon_{ijk} \hat{\sigma}_k, \tag{127}$$

and isoeigenvalue isoequations

$$\hat{\sigma}_3 \star |\hat{b} >= \pm(\lambda_1 \lambda_2)|\hat{b} >$$
$$\hat{\sigma}^{\hat{2}} = 3(\lambda_1 \lambda_2)^2 |\hat{b} > . \tag{128}$$

We consider now the particular case of Eq. (3), i.e.,

$$Det.\hat{T} = 1, \;\; n_1^2 = 1/n_2^2 = \lambda, \tag{129}$$

derivable via the basic non-unitary transformation

$$\hat{T} = (UU^\dagger)^{-1} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}. \tag{130}$$

In this case, isomatrices (83) become (Eqs. (3.9) of [58])

$$\hat{\sigma}_1(\lambda) = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix}, \;\; \hat{\sigma}_2(\lambda) = \begin{pmatrix} 0 & -i\lambda^{-1} \\ i\lambda & 0 \end{pmatrix},$$
$$\hat{\sigma}_3(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix}. \tag{131}$$

It is an instructive exercise for the interested reader to verify that the above realization of the regular Pauli-Santilli isomatrices verifies isocommutation rules with the same stricture constants of the $SU(2)$ algebra

$$[\hat{\sigma}_i(\lambda)\hat{,}\hat{\sigma}_j(\lambda)] = i2\epsilon_{ijk}\hat{\sigma}_k(\lambda), \tag{132}$$

and admit conventional eigenvalues

$$\hat{\sigma}_3(\lambda) \star |\hat{b} >= \pm|\hat{b} >$$
$$\hat{\sigma}(\lambda)^{\hat{2}} = 3|\hat{b} > . \tag{133}$$

Consequently, we have the following property [58]:

LEMMA 3.3. Regular Pauli-Santilli isomatrices provide an explicit and concrete realization of regular hidden variables directly in the spin $1/2$ algebra.

Note that, besides being positive-definite, hidden variables have an unrestricted functional dependence on all needed local variables, Eqs. (66).

An example of irregular hidden variables is provided by the $\hat{SU}(2)$ component of the spinorial covering of the Lorentz-Poincaré-Santilli isosymmetry.

To illustrate this realization, introduce *three* additional hidden variables for the characterization of isospace (107)

$$m_\mu = \lambda_\mu, \;\; \mu = 1, 2, 3, 4. \tag{134}$$

Realization (108) then implies the following irregular Dirac-Santilli isomatrices

$$\begin{gathered}\hat{\gamma}_k(\lambda) = \tfrac{1}{\gamma_k}\begin{pmatrix} 0 & \hat{\sigma}_k(\lambda) \\ -\hat{\sigma}_k & 0 \end{pmatrix}, \\ \hat{\gamma}_4 = \tfrac{i}{m_4}\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix},\end{gathered} \tag{135}$$

where $\hat{\sigma}_k$ are the *regular or irregular Pauli-Santilli isomatrices,* with isocommutation rules

$$[S_i(\lambda)\hat{,}S_j(\lambda)] = \epsilon_{ijk}\frac{1}{\lambda_k}S_k, \tag{136}$$

and isoeigenvalues

$$\begin{gathered}S_3 \star |\hat{\psi}>= \pm\tfrac{1}{2}\tfrac{1}{\sqrt{\lambda_1\lambda_2}}|\hat{\psi}>, \\ S^{\hat{2}} \star |\hat{\psi}>= \tfrac{1}{4}(\tfrac{1}{\sqrt{\lambda_1\lambda_2}} + \tfrac{1}{\sqrt{\lambda_2\lambda_3}} + \tfrac{1}{\sqrt{\lambda_3\lambda_1}})|\hat{\psi}> .\end{gathered} \tag{137}$$

Consequently, we have the following property [62]

LEMMA 3.4: The axioms of Dirac's equation admit up to five generally different, regular or irregular hidden variables.

Additional realizations of irregular hidden variables can be found in Eqs. (3.11) of Ref. [58] or can be easily derived from the preceding realization of the Pauli-Santilli isomatrices.

3.7. Apparent admission of classical counterparts.

As it is well known, Bell's inequality [3] [4], von Neumann's theorem [5], and the theory of local realism at large (see review [6] with a comprehensive literature) are generally assumed to be evidence of the impossibility of "completing" quantum mechanics into a broader theory, with ensuing rejection of the EPR argument [1].

Following decades of preparatory works reviewed in Paper I [9] and in the preceding sections of this paper, Santilli proved in Ref. [10] of 1998 (see also the detailed study in Ref. [30], particularly Chapter 4 and Appendix 4C, page 166) that:

1) Bell's inequality, von Neumann's theorem and related studies are indeed valid, but under the *tacit* assumption of representing particles as being point-like, with ensuing sole admission of linear, local and potential interactions (exterior dynamical problems).

2) Bell's inequality, von Neumann's theorem and related studies are *inapplicable* (rather than being violated) for extended particles within physical media, due to the presence of additional non-liner, non-local and non-potential interactions (interior dynamical systems).

3) The latter systems represented with the axiom-preserving "completion" of 20th century applied mathematics into isomathematics and the ensuing "completion" of quantum mechanics into hadronic mechanics [29]-[31] verify Statement 2 and admit well defined classical counterparts.

To review the preceding advances, consider two quantum mechanical particles with spin $1/2$ denoted 1 and 2 which verify the $SU(2)$ spin symmetry.

Assume that, as a result of some interaction, the two particles have antiparallel spins represented in the Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$. The total state in $ca\hat{l}H$ is then given by

$$|S_{1-2}>=\frac{1}{\sqrt{2}}(|S_{1\uparrow}>\times|S_{2\downarrow}>-|S_{1\downarrow}>\times|S_{2\uparrow}>), \tag{138}$$

with conventional l normalization

$$<S_{1-2}|S_{1-2}>=1, \tag{139}$$

where $\times$ is the conventional associative product.

Let a_1, b_1 and a_2, b_2 be unit vectors along the z-axis of a conventional Euclidean space $E(r, \delta, I)$ for particle 1 and 2, respectively. Introduce the quantum mechanical probability

$$P(a_1, b_1)=<S_{1-2}|(\sigma_1\otimes a_1)\times(\sigma_2\otimes b_1)|S_{1-2}>=-a_1\otimes b_1, \tag{140}$$

where $\otimes$ is the conventional scalar product.

Then, Bell's inequality can be written [4] (see Ref. [6] for numerous equivalent formulations)

$$D^{QM}_{Bell}=Max|P(a_1, b_1)-P(a_1, b_2)+P(a_2, b_1)+P(a_2, b_2)|\leq 2, \tag{141}$$

and implies the following property:

LEMMA 3.5: Particles in vacuum verifying the Lie symmetry $SU(2)$ admit no classical counterparts.

PROOF: The classical counterpart of Bell's inequality is given by

$$D_{Max}^{Classical} = Max|a_1 \otimes b_1 - a_1 \otimes b_2| + |a_2 \otimes b_1 + a_2 \otimes b_2| = 2\sqrt{2}. \qquad (142)$$

But the quantum mechanical value of D_{Bell}^{QM} is *always* smaller than its classical counterpart $D_{Max}^{Classical}$,

$$D_{Bell}^{QM} < D_{Max}^{Classical}, \qquad (143)$$

by therefore establishing the impossibility for an $SU(2)$-invariant system to admit identical classical images. Q. E. D.

Santilli [10] has shown that inequality (141) is inapplicable for the same particles when they are in interior dynamical conditions, e.g., when they are in the core of a star, or at the limit, when they are in the interior of a gravitational collapse.

Considers two extended particles also denoted 1 and 2. Suppose that said particles verify the regular $\hat{SU}(2)$ isosymmetry with spin $1/2$ (Section 3.3), thus implying the elaboration via isomathematics (Section I-3) and the verification of the isotopic branch of hadronic mechanics (Section I-4).

Suppose that the two extended particles with spin $1/2$ are characterized by the following isotopic elements:

$$\begin{aligned} &Particle\ 1: \ \hat{T}_1 = Diag(\lambda_1, 1/\lambda_1), \\ &Particle\ 2: \ \hat{T}_2 = Diag(\lambda_2, 1/\lambda_2), \end{aligned} \qquad (144)$$

with realization (83) of the Pauli-Santilli isomatrices.

Suppose that, due to preceding interactions, the two extended particles are in single overlapping/entanglement thus having opposite spins.

Let $\hat{I}_1$ and $\hat{I}_2$ be the isounits for particles 1 and 2, respectively. The systems of the assumed two isoparticles is then characterized by the total isounit

$$\hat{I}_{tot} = \hat{I}_1 \times \hat{I}_2 = \frac{1}{\hat{T}_{tot}} = \frac{1}{\hat{T}_1 \times \hat{T}_2}. \qquad (145)$$

In this case, the total isostate on the Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ [19] over the isofield of isocomplex isonumbers $\hat{calC}$ [20] is given by

$$|\hat{S}_{1-2}>= \frac{1}{\sqrt{2}}(|\hat{S}_{1\uparrow} > \hat{\times}|\hat{S}_{2\downarrow} > -|\hat{S}_{1\downarrow} > \times|\hat{S}_{2\uparrow} >). \qquad (146)$$

The lack of validity of inequality (141) for irregular isorepresentations of $\hat{SU}(2)$ is evident (e.g., because of the anomalous spin isoeigenvalues) and, as such, it is ignored.

A significant aspect of Ref. [10] is the proof of the inapplicability of inequality (141), not only for regular isorepresentation of $\hat{SU}(2)$, but also when such isorepresentations are isounimodular, Eqs. (144).

Let a_1, b_1, a_2, b_2 be unit vectors along the z-axis of an iso-Euclidean isospace. Introduce the isoprobability (Eq. (32.39), page 99, Ref. [30])

$$\begin{aligned}\hat{P}(a,b) &=< \hat{S}_{1-2}| \star (\hat{\Sigma}_1 \hat{\otimes}_1 a) \times (\hat{\Sigma}_2 \hat{\otimes}_2 b) |\hat{S}_{1-2} > \hat{I}_{tot} = \\ &=< \hat{S}_{1-2}| \star (\hat{\sigma}_1 \otimes a) \times (\hat{\sigma}_2 \otimes b) |\hat{S}_{1-2} > \hat{I}_{tot},\end{aligned} \tag{147}$$

with isonormalization (here referred to individual diagonal elements of isotopic elements and isounits)

$$< \hat{S}_{1-2}| \star |\hat{S}_{1-2} >=< \hat{S}_{1-2}|\hat{T}_{tot}|\hat{S}_{1-2} >= \hat{I}_{tot} \tag{148}$$

where: $\star$ is the total isoproduct; $\hat{\otimes}_k,\ k = 1, 2$, is the isoscalar isoproduct; and we have used simplifications of the type

$$\hat{\Sigma}_1 \hat{\otimes}_1 a = (\hat{\sigma}_1 \hat{I}_1)(\hat{T}_1 \otimes) a = \hat{\sigma}_1 \otimes a. \tag{149}$$

An isotopy of the conventional case yields the following isobasis, Eq. (6.5) of Ref. [10],

$$|S_{1-2} >= \frac{1}{2} \left\{ \begin{pmatrix} \lambda_1^{-1/2} \\ 0 \end{pmatrix} \begin{pmatrix} o \\ \lambda_2^{1/2} \end{pmatrix} - \begin{pmatrix} 0 \\ \lambda_2^{1/2} \end{pmatrix} \begin{pmatrix} \lambda_1^{-1/2} \\ 0 \end{pmatrix} \right\}. \tag{150}$$

The appropriate use of products and isoproducts then yield expression (5.6) Ref. [10], i.e.,

$$\begin{aligned}&< \hat{S}_{1-2}|\hat{T}_{tot} (\hat{\sigma}_1 \otimes_1 a) \times (\hat{\sigma}_2 \otimes_2 b) \hat{T}_{tot} |\hat{S}_{1-2} >= \\ &= -a_x b_x - a_y b_y - \tfrac{1}{2}(\lambda_1 \lambda_2^{-1} + \lambda_1^{-1} \lambda_2) a_z b_z.\end{aligned} \tag{151}$$

The continuation of the isotopy of the conventional case, yields the main result, Eq. (5.8) of Ref. [10], which provides the following *isotopic "completion" of Bell's inequality,*

$$\begin{aligned}&\hat{D}^{HM}_{Max} = D^{HM}_{Max} \hat{I}_{tot} = \\ &Max|\hat{P}(a_1, b_1) - \hat{P}(a_1, b_2) + \hat{P}(a_2, b_1) + \hat{P}(a_2, b_2)| = \\ &= [\tfrac{1}{2}(\lambda_1 \lambda_2^{-1} + \lambda_1^{-1} \lambda_2) D^{QM}_{Bell} \, \hat{I}_{tot},\end{aligned} \tag{152}$$

with consequential:

LEMMA 3.6. Extended particles within physical media that are invariant under the Lie-Santilli isosymmetry $\hat{SU}(2)$ admit identical classical counterparts.

PROOF: Isoinequality (141) establishes the lack of universal validity of Bell's inequality (128) because the factor $\frac{1}{2}(\lambda_1\lambda_2^{-1} + \lambda_1^{-1}\lambda_2)$ can have values *bigger* than one, thus implying

$$D^{HM}_{Max} \ \geq \ D^{QM}_{Bll}. \tag{153}$$

Consider then a classical iso-Euclidean isospace $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ representing motion of classical extended particles 1 and 2 within physical media [30] with isometric elements

$$\hat{\delta}_{11} = 1, \ \ \hat{\delta}_{22} = 1, \ \ \hat{\delta}_{33} = \frac{1}{2}(\lambda_1\lambda_2^{-1} + \lambda_1^{-1}\lambda_2) = 2, \tag{154}$$

in which case

$$D^{HM}_{Max} \equiv htD^{Classical}_{Max}, \tag{155}$$

by therefore establishing that systems of extended particles within physical media verifying the $\hat{SU}(2)$ isosymmetry admits an identical classical counterpart along the EPR argument. Q.E.D.

It is an instructive exercise for the interested reader to prove that the above lemma also holds for different isorenormalizations, e.g., Eqs. (171) of next section, with the understanding that different isorenormalizations imply different isobasis and different hidden variable terms in Eqs. (151).

Note the crucial role of hidden variables for the proof of Lemma 3.6. It is an instructive exercise for interested readers to prove that Lemma 3.6 holds for any other regular, isounimodular isorepresentation of the isotopic $\hat{SU}(2)$ symmetry in terms of hidden variables presented in Section 3.3.

The proof of the lack of applicability of von Neumann's theorem [5] for extended particles in interior conditions is elementary. Recall that von Neumann's theorem is based on the uniqueness of the eigenvalues E of a Hermitean operator H, $H|\psi >= E|\psi >$ under unitary transformation on $\mathcal{H}$,

$$UH|\psi > U^\dagger = UE|\psi > U^\dagger = EU|\psi > U^\dagger, \ \ UU^\dagger = U^\dagger U = I, \tag{156}$$

under the tacit assumption of point particles in vacuum.

By contrast, when the same particles is in interior conditions, it is subjected to an infinite number of different physical different interactions with the medium represented by the isotopic element $\hat{T}$ with ensuing isoeigenvalue equation (Section I-4), [9],

$$abel1H \star |\hat{\psi}_{\hat{T}} r >= HT|\hat{\psi}_{\hat{T}} >= E_{\hat{T}}|\hat{p}si_{\hat{T}} >, \tag{157}$$

thus establishing that a given quantum mechanical operator H representing the energy of an extended particle in interior conditions has an infinite number of *generally different* isoeigenvalues $E_{\hat{T}}$ depending on the infinite number of different interior conditions.

Note that, for each given $\hat{T}$ the isoeigenvalue $E_{\hat{T}}$ is invariant under isounitary isotransformations (Section I-3-9).

3.8. Apparent admission of classical determinism.

Consider a point-like particle in empty space represented in the 3-dimensional Euclidean space $E(r, \delta, I)$, where r represents coordinates, $\delta = Diag.$-$(1, 1, 1)$ represents the Euclidean metric and $I = Diag.(1, 1, 1,)$ represents the space unit.

Let the operator representation of said point-like particle be done in a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$ with states $\psi(r)$ and familiar normalization

$$< \psi(r)|\ |\psi(r) >= \int_{-\infty}^{+\infty} \psi(r)^\dagger \psi(r) dr = 1. \tag{158}$$

As it is well known, the primary objections against the EPR argument [2]- [6] were based on Heisenberg uncertainty principle according to which *the position r and the momentum p of said particle cannot both be measured exactly at the same time.*

By introducing the *standard deviations* Δr and Δp, the uncertainty principle is generally written in the form

$$\Delta r \Delta p\ \geq\ \frac{1}{2}\hbar, \tag{159}$$

which is easily derivable via the vacuum expectation value of the canonical commutation rule

$$\Delta r \Delta p\ \geq\ |\frac{1}{2i} < \psi|\ [r, p]\ |\psi >\ | = \frac{1}{2}\hbar. \tag{160}$$

Standard deviations have the known form (see, e.g., Ref. [79]) with $\hbar = 1$

$$\Delta r = \sqrt{< \psi(r)|[\ r - (< \psi(r)|\ r\ |\psi(r) >)]^2|\psi(r) >},$$
$$\Delta p = \sqrt{< \psi(p)|\ [p - (< \psi(p)|\ p\ |\psi(p) >)]^2|\psi(p) >}, \tag{161}$$

where $\psi(r)$ and $\psi(p)$ are the wavefunctions in coordinate and momentum spaces, respectively.

We consider now an extended particle, this time, in interior conditions, e.g., in the core of a star, classically represented by the *iso-Euclidean isospace* $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ with isounit $\hat{I} = 1/\hat{T} > 0$, isocoordinates $\hat{r} = r\hat{I}$, isometric

$$\hat{\delta} = \hat{T}\delta, \tag{162}$$

and isotopic element (4)) under conditions (5).

For simplicity, we assume that the extended particle has no Hamiltonian interactions due to the dominance of the latter interactions over the former.

Consequently, we can represent the extended particle in the isospace $\hat{\mathcal{H}}$ over the isofield $\hat{\mathcal{C}}$ and introduce the time independent *isoplanewave* [18]

$$\begin{aligned} \hat{\psi}(\hat{r}) = \tilde{\psi}(\hat{r})\hat{I} = \\ = \hat{N} \star (\hat{e}^{\hat{i} \star \hat{k} \star \hat{r}})\hat{I} = N(e^{ik\hat{T}\hat{r}})\hat{I}, \end{aligned} \tag{163}$$

where $\hat{N} = N\hat{I}$ is an *isonormalization isoscalar,* $\hat{k} = k\hat{I}$ is the *isowavenumber,* and the isoexponentiation is given by Eq. (I-22) [26].

The corresponding representation in isomomentum isospace is given by

$$\tilde{\psi}(\hat{p}) = \hat{M} \star \hat{e}^{\hat{i} \star \hat{n} \star \hat{p}}, \tag{164}$$

where $\hat{M} = M\hat{I}$ is an isonormalization isoscalar and $\hat{n} = n\hat{I}$ is the isowavenumber in isomomentum isospace.

The *isopropability isofunction* is then given by (Ref. [30] page 99)

$$\hat{\mathcal{P}} = \hat{<} | \star | \hat{>} =< \hat{\psi}(\hat{r}) | \, T \, | \hat{\psi}(\hat{r}) > I, \tag{165}$$

that, written in terms of isointegrals (Ref. [29] page 354), becomes

$$\begin{aligned} \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \star \hat{\psi}(\hat{r}) \star \hat{d}\hat{r} = \\ = \int_{-\infty}^{+\infty} \tilde{\psi}(\hat{r})^\dagger \tilde{\psi}(\hat{r})(dr + r\hat{T}d\hat{I}), \end{aligned} \tag{166}$$

where one should keep in mind that the isodifferential $\hat{d}\hat{r}$ given by Eqs. (I-29).

The *isoexpectation isovalues* of a Hermitean operator $\hat{Q}$ are then given by [30]

$$\hat{<}| \star \hat{Q} \star |\hat{>} =< \hat{\psi}(\hat{r})| \star \hat{Q} \star |\hat{\psi}(\hat{r}) >=$$

$$= \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger} \star \hat{Q} \star \hat{\psi}(\hat{r}) \hat{d}\hat{r} = \tag{167}$$

$$= \int_{-\infty}^{+\infty} \tilde{\psi}(\hat{r})^{\dagger} \hat{Q} \tilde{\psi}(\hat{r}) \hat{d}\hat{r},$$

with corresponding expressions for the isoexpectation isovalues in isomomentum isospace.

Santilli then introduced apparently for the first time in Ref. [11] the *isotopic operator*

$$\hat{\mathcal{T}} = \hat{T}\hat{I} = I, \tag{168}$$

that, despite its seemingly irrelevant value, is indeed the correct operator formulation of the isotopic element for the "completion" of the isoproduct from its scalar form (1) to the isoscalar form

$$\hat{n}^{\hat{2}} = \hat{n} \star \hat{n} = \hat{n} \star \hat{\mathcal{T}} \star \hat{n} = \ n^2 I. \tag{169}$$

In Sections 3.6, 3.7, we have shown that the Lie-Santilli isosymmetry $\hat{SU}(2)$ admits an explicit and concrete realization of hidden variables that allowed the construction of identical classical counterparts for interior dynamical systems.

Ref. [11] introduced the isoexpectation isovalue of the isotopic operator

$$\hat{<}| \star \hat{\mathcal{T}} \star |\hat{>} =< \hat{\psi}(\hat{r})| \star \hat{\mathcal{T}} \star |\hat{\psi}(\hat{r}) > \hat{I} =$$

$$= \int_{-\infty}^{+\infty} \tilde{\psi}(\hat{r})^{\dagger} \hat{T} \tilde{\psi}(\hat{r}) \hat{d}\hat{r}, \tag{170}$$

and assumed the isonormalization (again, intended for diagonal matrix elements)

$$\hat{<}| \star \hat{\mathcal{T}} \star |\hat{>} =$$

$$= \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger} \hat{T} \hat{\psi}(\hat{r}) \hat{d}\hat{r} = \hat{T}. \tag{171}$$

Consider then the *isostandard isodeviation* for isocoordinates $\Delta\hat{r} = \Delta r\hat{I}$ and isomomenta $\Delta\hat{p} = \Delta p\hat{I}$, where Δr and Δp are the standard deviations in our space.

By using isocanonical isocommutation rules (I-81), we obtain the expression

$$\Delta\hat{r} \star \Delta\hat{p} = \Delta r \Delta p \hat{I} \ \approx \ \tfrac{1}{2}| < \hat{\psi}(\hat{r})| \star [\hat{r} \hat{,} \hat{p}] \star \hat{\psi}(\hat{r}) > |\hat{I} =$$

$$= \tfrac{1}{2}| < \hat{\psi}(\hat{r})|\hat{T}\, [\hat{r} \hat{;} \hat{p}]\, \hat{T}|\hat{\psi}(\hat{r}) > \hat{I}, \tag{172}$$

One should note the replacement of the symbol $\geq$ in Eq. (160) with the symbol $\approx$ in Eq. (172). This is due to the fact that the historical arguments applying for a point-like particle in vacuum no longer apply for an interior system because the pressure exercised by the medium on the particle (Figure 4) reduce the lower limit of Eq. (160) to the approximate value of Eq. (172).

Under the above assumptions, by eliminating the common isounit $\hat{I}$, Ref. [11] achieved the desired result here called *isodeterministic isoprinciple*

$$\begin{aligned} \Delta r \Delta p \approx \tfrac{1}{2} | < \hat{\psi}(\hat{r})| \star [\hat{r}\hat{,}\hat{p}] \star |\hat{\psi}(\hat{r}) >= \\ = \tfrac{1}{2} | < \hat{\psi}(\hat{r})| \hat{T}\, [\hat{r}\hat{,}\hat{p}]\, \hat{T} |\hat{\psi}(\hat{r}) >= \\ \tfrac{1}{2} \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger} \hat{T} \hat{\psi}(\hat{r}) \hat{d}\hat{r} = \tfrac{1}{2} T \ \ll \ 1 \end{aligned} \tag{173}$$

where the property $\Delta r \Delta p \ll \ 1$ follows from the fact that the isotopic element $\hat{T}$ has values smaller than 1 in the fitting of all experimental data dealing with hadronic media such as hadrons, nuclei and stars, and null value for gravitational collapse [31].

In the event Eq. (35), page 14 of Ref. [11] should be compatible with Eq. (173) above, it is sufficient to turn into a comma the sign $=$ in the right of the central expression of Eq. (35), or absorb the factor $1/2$ of Eq. (173) into the isorenormalization.

In this way, thanks to a laborious scientific journey initiated at Harvard University in late 1977, and thanks to contributions by numerous mathematicians, theoreticians and experimentalists, Santilli reached the following verification of the EPR argument [11]:

LEMMA 3.7 (ISODETERMINISTIC PRINCIPLE): The isostandard isodeviations for isocoordinates $\Delta \hat{r}$ and isomomenta $\Delta \hat{p}$, as well as their product, progressively approach classical determinism for extended particles in the interior of hadrons, nuclei, and stars, and achieve classical determinism at the extreme densities in the interior of gravitational collapse.

PROOF: Define the isostandard isodeviations via the following isotopy of quantum mechanical expressions (161) (where we ignore the common multiplication by the isounit)

$$\begin{aligned} \Delta r = \sqrt{< \hat{\psi}(\hat{r}) | [\, \hat{r} - < \hat{\psi}(\hat{r}) | \star \hat{r} \star | \hat{\psi}(\hat{r}) >]^{\hat{2}} \hat{|} \psi(\hat{r}) >}, \\ \Delta p = \sqrt{< \hat{\psi}(\hat{p}) |\, [\hat{p} - < \hat{\psi}(\hat{p}) | \star \hat{p} \star | \hat{\psi}(\hat{p}) >]^{\hat{2}} | \hat{\psi}(\hat{p}) >}, \end{aligned} \tag{174}$$

where the differentiation between the isotopic elements for isocoordinates and isomomenta is ignored for simplicity. But the isotopic element represents the interactions of the particle with the physical medium and tends toward null values for gravitational collapse, Eqs. (I-91) (I-92). Therefore, isosquare in expression (171) implies the expressions

$$\begin{aligned}\Delta r &= \sqrt{\hat{T} < \hat{\psi}(\hat{r})|[\,\hat{r}- < \hat{\psi}(\hat{r})| \star \hat{r} \star |\hat{\psi}(\hat{r}) >]^2|\hat{\psi}(\hat{r}) >},\\ \Delta p &= \sqrt{\hat{T} < \hat{\psi}(\hat{p})|\,[\hat{p}- < \hat{\psi}(\hat{p})| \star \hat{p} \star |\hat{\psi}(\hat{p}) >]^2|\hat{\psi}(\hat{p}) >},\end{aligned} \tag{175}$$

that approach indeed null value under the indicated limit conditions of gravitational collapse

$$\begin{aligned}Lim_{\hat{T}=0}\Delta r &= 0,\\ Lim_{\hat{T}=0}\Delta p &= 0,\end{aligned} \tag{176}$$

Q.E.D.

3.9. Apparent removal of quantum divergencies.

Recall from Section I-4.13 that, under condition (I-96), corresponding to condition (173), there is a rapid convergence of isoseries (I-97), as well as the removal of the singularity of Dirac's delta distribution, Eq. (I-98) (Figure I-14).

The above properties can be now formalized according to the following:

COROLLARY 3.7.1. Einstein's determinism according to Lemma 3.7 implies the removal of quantum mechanical divergencies.

PROOF. Lemma 3.7 is based on values of the isotopic element $\hat{T}$ being smaller than 1, which values imply in turn the rapid convergence of perturbative series without divergencies (Section I-4-13).
Q.E.D.

The removal of quantum divergencies, that have been cause of controversies for about one century, illustrates the far reaching implications of Einstein's determinism for interior dynamical systems.

3. CONCLUDING REMARKS.

Following the study of basic methods in Paper I, in this paper we have provided an apparent confirmation of proofs [10] [11] of the EPR argument [1] for extended, thus deformable particles within hyperdense media with ensuing linear and non-linear, local and non-local and potential as well as non-potential/non-Hamiltonian interactions.

This study has been conducted via the use of isomathematics and isomechanics admitting a conventional Hamiltonian H or Lagrangian L for the invariant representation of linear, local and potential interactions, plus the isotopic element $\hat{T}$ of isoproducts $A \star B = A\hat{T}B$, Eq. (1), for the invariant representation of non-linear, non-local and non-Hamiltonian interactions.

Following the outline and upgrade of isosymmetries for time-reversal invariant interior systems (Section 3), we have apparently confirmed the proof of Ref. [10] according to which extended particles in interior dynamical conditions admit identical classical counterpart.

We have then apparently confirmed the proof of Ref. [11] according to which extended particles progressively approach classical determinism when in the interior of hadrons, nuclei and stars, and achieve full determinism at the limit of gravitational collapse, essentially as predicted by A. Einstein, B. Podolsky and N. Rosen [1].

To illustrate the far reaching implications of what appears to be Einstein's most important legacy, we have shown for the first time that the recovering of Einstein's determinism for interior conditions appears to imply the removal of quantum divergencies due to the rapid convergence of the isoseries of hadronic mechanics, the removal of the singularity in Dirac's delta distribution and other features.

A number of illustrations and novel applications in mathematics, physics and chemistry are presented in the forthcoming Paper III.

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