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On the planarity of line Mycielskian graph of a graph

Keerthi G. Mirajkar*
Anuradha V. Deshpande†

Abstract

The line Mycielskian graph of a graph G, denoted by $L_\mu(G)$ is defined as the graph obtained from $L(G)$ by adding $q+1$ new vertices $E^{'} = e_i^{'} : 1 \leq i \leq q$ and e, then for $1 \leq i \leq q$, joining $e_i^{'}$ to the neighbours of e_i and to e. The vertex e is called the root of $L_\mu(G)$. This research paper deals with the characterization of planarity of line Mycielskian Graph $L_\mu(G)$ of a graph. Further, we also obtain the characterization on outerplanar, maximal planar, maximal outerplanar, minimally nonouterplanar and 1-planar of $L_\mu(G)$.

Keywords: Planar graph, Outerplanar, Maximal planar, Maximal outerplanar, Minimally nonouterplanar and 1-planar.
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*Department of Mathematics, Karnatak University's Karnatak Arts College, Dharwad - 580001, Karnataka, INDIA; keerthi.mirajkar@gmail.com
†Department of Mathematics, Karnatak University's Karnatak Arts College, Dharwad - 580001, Karnataka, INDIA; anudesh08@gmail.com.
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1 Introduction

All graphs considered in this paper are finite, undirected and without loops. The phraseologies are referred from [Harary, 1969].

The graph G is said to be embedded in a surface S when its vertices are represented by points in S and each edge by a curve joining corresponding points in S, in such a way that no curve intersects itself and two curves intersect each other only at a common vertex.

A graph G is said to be planar if it can be embedded in the plane. A plane representation of a planar graph divides the plane into number of plane areas called regions or faces. The regions enclosed by the planar graph are called interior faces of the graph. The region surrounding the planar graph is called the exterior face of the graph. A planar graph G is called maximal planar if the addition of any edge to G creates a nonplanar graph. A maximal planar graph is a planar graph in which every face (including the exterior face) is bounded by a triangle [Dillencourt, 1991]. A planar graph is called outerplanar if it can be embedded in the plane so that all its points lie on the same face. An outerplanar graph is called maximal outerplanar if no line can be added without losing outerplanarity.

The inner vertex number $i(G)$ of a planar graph G is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be k - minimally nonouterplanar if $i(G) = k, k \geq 1$. An 1-minimally nonouterplnar graph is called minimally nonouterplanar [Kulli and Basavanagoud, 2004].

Two graphs are said to be homeomorphic if one graph can be obtained from the other by insertion of vertex of degree two into its edges or by the merger of adjacent edges, where the incident vertex is of degree two.

Crossing number $cr(G)$ of a graph G is the minimum number of crossings (of its edges) among the drawings of G in the plane.

[Ringel, 1965] introduced the concept of 1-planarity. A graph G is called 1-planar if it can be drawn in the plane so that all or any edge is crossed by at most one other edge.

The line Mycielskian graph of a graph G, denoted by $L_\mu(G)$ is defined as the graph obtained from $L(G)$ by adding $q + 1$ new vertices $E^{'} = e_i^{'} : 1 \leq i \leq q$ and e, then for $1 \leq i \leq q$, joining $e_i^{'}$ to the neighbours of e_i and to e. The vertex e is called the root of $L_\mu(G)$ [Mirajkar and Mathad, 2019].

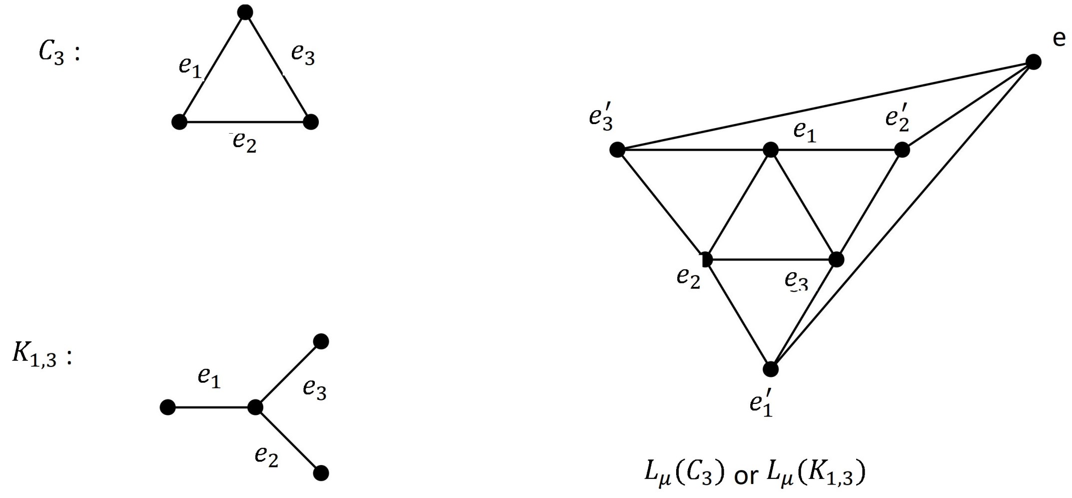

Figure 1. C_3, $K_{1,3}$ and their line Mycielskian graphs $L_\mu(C_{1,3})$ and $L_\mu(K_{1,3})$

Motivated by the the research work [Mirajkar and Mathad, 2019], the present problem is initiated. Further it is extended with the objective of obtaining the characterization results on planarity, outerplanar, maximal planar, maximal outerplanar, minimal nonouterplanar and 1-planar of $L_\mu(G)$.

2 Prelimnaries

The following important theorems and remark are used for proving further results.

Theorem 2.1. *[Kuratowski, 1930] A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.*

Theorem 2.2. *[Harary, 1969] A graph is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$ except $K_4 - x$.*

Theorem 2.3. *[Czap and Hudák, 2012] The complete graph K_{α_1}, $\alpha_1 \leq 6$ is 1-planar.*

Theorem 2.4. *[Mirajkar et al., 2019] The line Mycielskian graph $L_\mu(G)$ of a graph G is disconnected iff $G = K_2$.*

Remark 2.1. *[Mirajkar and Mathad, 2019] $L(G)$ is subgraph of $L_\mu(G)$.*

3 Results

Theorem 3.1. *The line Mycielskian graph $L_\mu(G)$ is planar if and only if the graph G is C_n, $n = 3$ or C_3.*

Proof. Suppose $L_\mu(G)$ is planar and $G = C_n$.

We consider the following cases.

Case 1. Suppose $n = 5$, then $G = C_5$. $L(C_5)$ is C_5 and by remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. The construction of $L_\mu(G)$ with five newly introduced vertices $e_1^{'}, e_2^{'}, e_3^{'}$, $e_4^{'}$ and $e_5^{'}$ corresponding to vertices of $L(C_5)$ and root vertex e produces five mutually adjacent vertices with degree four which is a subgraph homeomorphic to K_5. By theorm 2.1, a contradiction.

Case 2. Suppose $n = 4$, then $G = C_4$. $L(C_4)$ is C_4 and by remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$.

Here $L_\mu(G)$ is constructed by introducing new vertices $e_1^{'}, e_2^{'}, e_3^{'}$, and $e_4^{'}$ corresponding to the vertices e_1, e_2, e_3, and e_4 of $L(C_4)$ and root vertex e. Newly introduced vertices are connected with the vertices e_1, e_2, e_3 and e_4 of C_4 in such a way that the, $e_1^{'}$ and $e_3^{'}$ are adjacent to the two opposite vertices e_2 and e_4 and $e_2^{'}$ and $e_4^{'}$ are adjacent to e_1 and e_3. Root vertex e is connected to $e_1^{'}, e_2^{'}, e_3^{'}, e_4^{'}$. This construction produces the five mutually adjacent vertices with degree four and thus contains a subgraph homeomorphic to K_5, a contradiction (From theorem 2.1).

Case 3. Suppose $n = 3$, then $G = C_3$. $L(C_3)$ is C_3 by remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. To construct line Mycielskian graph $L_\mu(G)$, three new vertices $e_1^{'}, e_2^{'}$ and $e_3^{'}$ corresponding to the edges of G and root vertex e are introduced. The edges between the vertices are drawn in such a way that the newly introduced vertices $e_1^{'}, e_2^{'}$ and $e_3^{'}$ are adjacent to the corresponding adjacent edges of e_1, e_2 and e_3 of G respectively. The root vertex e joins the vertices $e_1^{'}, e_2^{'}$ and $e_3^{'}$ such that no crossing of edges occur as shown in Figure 1. This implies that line graph $L_\mu(G)$ is planar for $G = C_3$.

From the above cases, it is observed that $L_\mu(G)$, for all $G = C_n$, $n \geq 4$, contains a subgraph homeomorphic to K_5, a contradiction. Thus $L_\mu(G)$ is planar only if G is C_n, $n = 3$.

Conversely, Suppose $G = C_3$, then the construction of $L_\mu(G)$ as discussed in above case 3 which results into planar graph.

Theorem 3.2. *The line Mycielskian graph $L_\mu(G)$ is planar if and only if the graph G is a path graph P_n, $n \geq 3$.*

Proof. Suppose $L_\mu(G)$ is planar and $G = P_n$.

We consider the following cases.

Case 1. Suppose $n = 2$, then $G = P_2$, then from theorem 2.4, $L_\mu(G)$ is disconnected graph, a contradiction.

Case 2. Suppose $G = P_n, n \geq 3$. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. $L(P_n)$ is P_{n-1}.

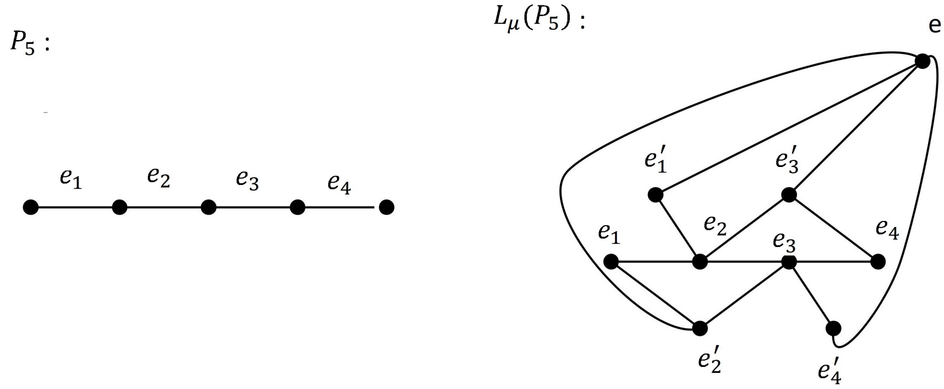

Figure 2. P_5 and its line Mycielskian graph $L_\mu(P_5)$

For the construction of $L_\mu(G)$, new vertices $e_1', e_2',.., e_{n-1}'$ corresponding to edges of G and root vertex e are introduced. In $L_\mu(G)$, the edges between the vertices are connected in such a way that, the newly introduced vertices e_1' and e_{n-1}' are adjacent to e_2 and e_{n-2} of $L(G)$ respectively and the other vertices e_2', e_3',.. , e_{n-2}' are adjacent to two vertices e_{i-1} and e_{i+1}, $i = 2, 3, .., (n-2)$. The root vertex e is adjacent to the vertices $e_1', e_2',.., e_{n-1}'$. All the faces are polygons without any crossings as shown in Figure 2. i.e., $L_\mu(G)$ is planar.
It is clear from the above two cases that $L_\mu(G)$ is planar only for $G = P_n, n \geq 3$.

Conversely, Suppose $G = P_n$, $n \geq 3$. Then the construction of $L_\mu(G)$ results into planar graph (as discussed above).

Theorem 3.3. *The line Mycielskian graph $L_\mu(G)$ of a graph G is planar if and only if one of the follwing conditions hold*
(i) $\Delta(G) = 2$, except for C_n, $n \geq 4$
(ii) $G = K_{1,3}$

Proof. Suppose $L_\mu(G)$ of G is planar.

We discuss the following cases based on the maximum degree of G.

Case 1. Suppose G is a graph with $\Delta(G) = 5$. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. Then the line Mycielskian graph $L_\mu(G)$ contains K_5 as its subgraph. By Theorem 2.1, $L_\mu(G)$ is nonplanar, a contradiction.

Case 2. Next suppose G is a graph with $\Delta(G) = 4$. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$ and contains K_4 as its subgraph. $L_\mu(G)$ is obtained by introducing new vertices $e_1^{'}, e_2^{'}, e_3^{'}$ and $e_4^{'}$ corresponding to the edges of subgraph K_4 of $L(G)$ and root vertex e by the definition. Vertices of K_4 and newly introduced vertices are connected. The edge from root vertex e is drawn in such a way that it is adjacent to the vertices $e_1^{'}, e_2^{'}, e_3^{'}$ and $e_4^{'}$. This construction produces subgraph K_4 with each vertex degree ≥ 4. One of the newly introduced vertices of is $L_\mu(G)$ is adjacent to the three vertices of K_4 with path length 1. It is also adjacent to the remaining one vertex of K_4 with path length 2. This produces the subgraph homeomorphic to K_5, a contradiction.

Case 3. Suppose $\Delta(G) = 3$, G contains $K_{1,3}$ as its subgraph. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. Since $L(K_{1,3})$ is C_3, the construction of $L_\mu(K_{1,3})$ is same as $L_\mu(C_3)$ as shown in Figure 1. From theorem 3.1, $L_\mu(C_3)$ is planar. Thus $L_\mu(K_{1,3})$ is also planar.

Further in the construction of $L_\mu(K_{1,3})$, presence of additional edge in $K_{1,3}$ to any vertex leads to increase in the number of vertices and edges (2 vertices and 3 edges) in $L_\mu(G)$ and thus contains K_4. On constructing $L_\mu(G)$, newly introduced vertices, root vertex and vertices of K_4 produces five mutually adjacent vertices with degree 4 which is a subgraph homeomorphic to K_5. By theorem 2.1, $L_\mu(G)$ is nonplanar, a contradiction.

Case 4. Suppose $\Delta(G) = 2$, Obiviously G is either path graph P_n or cycle C_n. By theorem 3.1 and theorem 3.2, $L_\mu(C_n)$, $n = 3$ and $L_\mu(P_n)$, $n \geq 3$ are planar.

From all the above cases, it is noted that, $L_\mu(G)$ contains subgraph homeomorphic to K_5 for $\Delta(G) \geq 3$, a contradiction and is planar only for $\Delta(G) = 2$.

Converse is obivious.

Theorem 3.4. *The line Mycielskian graph $L_\mu(G)$ is outerplanar if and only if G is P_3.*

Proof. Suppose $L_\mu(G)$ is outerplanar. Then $L_\mu(G)$ is planar. By theorems 3.1, 3.2 and 3.3, $L_\mu(G)$ is planar only for the graphs C_3, $K_{1,3}$ and P_n, $n \geq 3$.
Suppose G=C_3 or $K_{1,3}$. From theorem 3.1 and Figure 1, $L_\mu(G)$ contains a subgraph homeomorphic to $K_{2,3}$. From theorem 2.2, $L_\mu(G)$ is not outerplanar, a contradiction.

Assume $G = P_4$. line graph of P_4 is P_3. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. On constructing $L_\mu(G)$ with newly introduced vertices and root vertex (as explained in case 2 of theroem 3.2) forms a subgraph homeomorphic to $K_{2,3}$. By theorem 2.2 $L_\mu(P_4)$ is not outerplanar. Which is contradiction. i.e., G cannot be P_4.

Similarly, for higher values of n i.e., for $G = P_4$, $n \geq 5$, the same construction (case 2 of theorem 3.2) repeates and generates a subgraph with every four vertices of $L_\mu(G)$ which is homeomorphic to $K_{2,3}$, a contradiction.

Next assume $G = P_3$. Line graph of P_3 is P_2 and by remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$. $L_\mu(G)$ is constructed by introducing root vertex e and new vertices $e_1^{'}$ and $e_2^{'}$ corresponding to the edges of e_1 and e_2 of G respectively. The edges from new vertices are drawn in such a way that $e_1^{'}$ is adjacent to e_2 and $e_2^{'}$ to e_1 which forms C_5, as shown in Figure 3, which is outerplanar. Therefore, $G = P_3$.

By observing the above cases , it can be stated that $L_\mu(G)$ for $G = P_n$, $n \geq 4$ contains subgraph homeomorphic to $K_{2,3}$, a contradiction and hence is outerplanar only for $G = P_n$, $n = 3$.

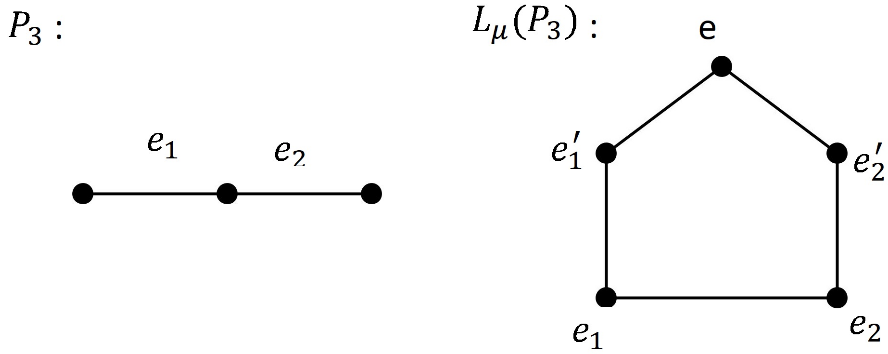

Figure 3. P_3 and its line Mycielskian graph $L_\mu(P_3)$

Conversely, suppose $G = P_3$. From theorem 3.2, $L_\mu(G)$ is planar for G=P_n, $n \geq 3$. i.e., $L_\mu(P_3)$ produces C_5 and is planar (as discussed above). Clearly which is outerplanar.

Theorem 3.5. *The line Mycielskian graph $L_\mu(G)$ is maximal planar if and only if G is C_3 or $K_{1,3}$ or P_n, $n \geq 4$.*

Proof. Suppose $L_\mu(G)$ is maximal planar, then it is planar. From theorem 3.1, theorem 3.2 and theorem 3.3, $L_\mu(G)$ is planar if G is C_3 or $K_{1,3}$ or P_n, $n \geq 4$.

Suppose G is C_3 or $K_{1,3}$. On the construction of $L_\mu(C_3)$ or $L_\mu(K_{1,3})$ as shown in Figure 1, it can be observed that, three of its interior faces are triangles and by the definition of homeomorphic graph, the other two faces can be triangularised by merging adjacent edges if the incident vertex is of degree two. i.e., $L_\mu(G)$ has triangulation plane. $L_\mu(G)$ is maximal planar.

Suppose G is P_n, $n \geq 3$. From theorem 3.2, $L_\mu(G)$ is planar.

We consider the following cases.

Suppose $G = P_3$. From theorem 3.4, the construction of $L_\mu(P_3)$ forms C_5 and does not contain any triangle faces as shown in Figure 3. Thus $L_\mu(P_3)$ is not maximal planar, a contradiction.

Next, suppose $G = P_n$, $n \geq 4$. The construction of $L_\mu(G)$ contains some of its interior faces as C_4's and some as C_5's. By definition of homeomorphic graph, all the faces can be triangularised by merging the adjacent edges. Thus $L_\mu(G)$ has triangulation plane and is maximal planar for $G = P_n$, $n \geq 4$.

Conversely, suppose G = C_3 or $K_{1,3}$ or P_n, $n \geq 4$. From theorem 3.1, theorem 3.3 and theorem 3.2, $L_\mu(G)$ is planar.

Let us first consider G = C_3 or $K_{1,3}$.

Three interior faces of $L_\mu(G)$ are triangles and two faces are C_4's and by the definition of homeomorphic graph, they can be triangularized by merging adjacent edges. i.e., $L_\mu(G)$ has triangulation plane. Hence, $L_\mu(G)$ is maximal planar.

Next suppose $G = P_n$, $n \geq 4$. From Figure 2, all the faces of $L_\mu(G)$ can be triangularized by merging adjacent edges and thus $L_\mu(G)$ has triangulation plane and it is maximal planar.

□

Theorem 3.6. *For any graph G, The line Mycielskian graph $L_\mu(G)$ of a graph G is not maximal outerplanar.*

Proof. Suppose $L_\mu(G)$ is maximal outerplanar, then it is outerplanar. From theorem 3.4, $L_\mu(G)$ is outerplanar only for $G = P_3$. From Figure 3, it is obvious that addition of an edge between any two non-adjacent vertices does not violate the property of planarity, a contradiction.

□

Theorem 3.7. *For any graph G, the line Mycielskian graph $L_\mu(G)$ of a graph G is not minimally nonouterplanar.*

Proof. Suppose $L_\mu(G)$ is minimally nonouterplanar, then $L_\mu(G)$ is planar. From theorem 3.1, theorem 3.2 and theorem 3.3, $L_\mu(G)$ is planar only if G is C_n, $n = 3$ or P_n, $n \geq 3$ or $K_{1,3}$.

We consider the following three cases.

Case 1. Suppose $G = C_n$, $n = 3$ or $G = K_{1,3}$. From Figure 1, $L_\mu(G)$ contains at least three points belong to interior region, a contradiction.

Case 2. Suppose $G = P_n$, $n \geq 3$.

Subcase 2.1 Suppose $n = 3$ then $G = P_3$. From Figure 3, in $L_\mu(G)$ all the points belong to exterior region. No point belong to interior region, a contradiction.

Subcase 2.2. Suppose $G = P_n$, $n \geq 4$, then $L_\mu(G)$ is constructed with newly introduced vertices $e_1^{'}, e_2^{'},.., e_{n-1}^{'}$ corresponding to edges of G and root vertex e (construction explained in case 2 of theorem 3.2).This construction generates subgraph with every four vertices of $L_\mu(G)$ which is homeomorphic to $K_{2,3}$. This process continues for the successive value of n in G and thus $L_\mu(G)$ contains more than one subgraph homeomorphic to $K_{2,3}$, a contradiction.

In either cases, $L_\mu(G)$ is not minimally nonouterplanar.

□

Theorem 3.8. *If the graph G is a cycle C_n, $n \geq 4$, then line Mycielskian graph $L_\mu(G)$ is 1-planar with n crossings .*

Proof. Let G be C_n, $n \geq 4$. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$.

From theorem 3.1, $L_\mu(C_n)$, $n \geq 4$ is nonplanar.

Here $L_\mu(G)$ is constructed by introducing new vertices $e_1^{'}, e_2^{'}, e_3^{'}, e_4^{'},.., e_n^{'}$ and root vertex e. Root vertex, newly introduced vertices and vertices of $L(C_n)$ are connected as explained in the theorem 3.1. This construction produces the five mutually adjacent vertices with degree ≥ 4 and thus contains a subgraph homeomorphic to K_5. From theorem 2.3, $L_\mu(G)$ is 1-planar.

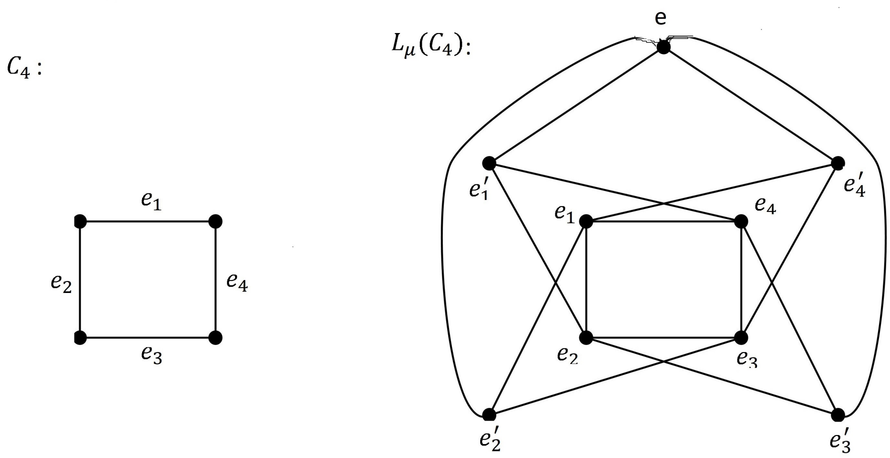

Figure 4. C_4 and its line Mycielskian graph $L_\mu(C_4)$

In $L_\mu(G)$, every crossing arises from the edges drawn between the vertices $L(C_n)$ and newly introduced vertices. Each newly introduced vertex is adjacent with two vertices of $L(C_n)$ and each crosing occurs from the two adjacent vertices of $L(C_n)$. As there are n vertices in $L(C_n)$, the number of crossings are n.

□

Theorem 3.9. *If G is $K_{1,3} \bullet K_2$, then the line Mycielskian graph $L_\mu(G)$ is 1-planar with crossing number* 1.

Proof . Let G be $K_{1,3} \bullet K_2$. By remark 2.1, $L(G)$ is subgraph of $L_\mu(G)$.

From theorem 3.3, $L_\mu(K_{1,3})$ is planar. But inclusion of K_2 to $K_{1,3}$ increases number of vertices and edges in $L_\mu(G)$ for which the construction is explained in theorem 3.3 case 3. This construction produces five mutually adjacent vertices with degree 4. Thus $L_\mu(G)$ contains a subgraph homeomorphic to K_5. From theorem 2.3, $L_\mu(G)$ is 1-planar.

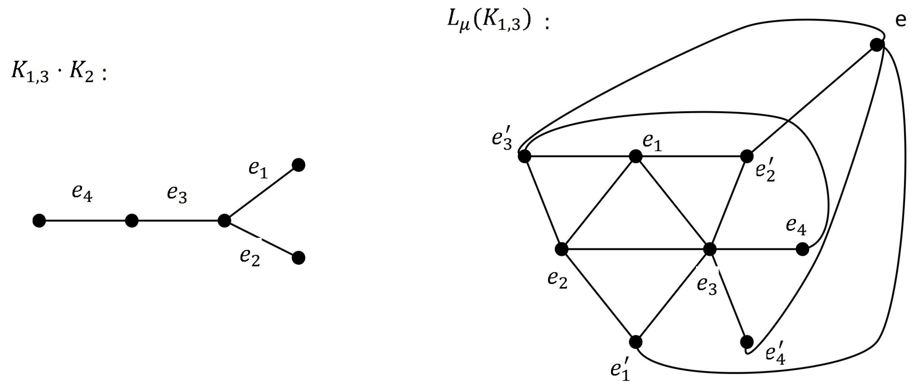

Figure 5. $K_{1,3} \bullet K_2$ and its line Mycielskian graph $L_\mu((K_{1,3} \bullet K_2)$

In graph $G = K_{1,3} \bullet K_2)$ shown in Figure 5, the number of edges are four. Since only one vertex e_3 of G is adjacent to three edges, in $L_\mu(G)$ the degree of newly introduced vertex corresponding to this edge is four as shown in Figure 5, where as the degree of other newly introduced vertices e_1', e_2', and e_4' is ≤ 3. Thus, this results the graph $L_\mu(G)$ to have only one crossing from the edges drawn between the root vertex e and newly introduced vertices e_2', e_3' and e_4'.

4 Conclusion

In this research paper we obtained the characterization results on planarity, outerplanar, maximal planar, maximal outerplanar, minimal nonouterplanar and 1-planar of line Mycielskian graph $L_\mu(G)$. The results revealed that $L_\mu(G)$ is planar only for the graphs C_3, P_n,$n \geq 3$, $K_{1,3}$ and maximal planar if G is C_3, P_n,$n \geq 4$ and $K_{1,3}$. It is outerplanar only if G= P_3.There is no existance of any graph whose $L_\mu(G)$ is maximal outerplanar and not minimally nonouterplanar. Further we also obtained $L_\mu(G)$ for 1-planar when G= C_n, $n \geq 4$ and $(K_{1,3} \bullet K_2)$ with n crossings and 1 crossing respectively.

Future line of this research can be extended to study some more properties of planarity such as crossing numbers, genus, thickness and coarseness of line Mycielskian graph of a graph.

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