RATIO MATHEMATICA 25(2013), 47-58 ISSN:1592-7415

The Divisors' Hyperoperations

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Abstract

In the set $\mathbb{N}$ of the Natural Numbers we define two hyperoperations based on the divisors of the addition and multiplication of two numbers. Then, the properties of these two hyperoperations are studied together with the resulting hyperstructures. Furthermore, from the coexistence of these two hyperoperations in $\mathbb{N}^*$, an H_v-ring is resulting which is dual.

Key words: Hyperstructures, H_v-structures

MSC2010: 20N20, 16Y99.

1 Introduction

In 1934, F. Marty introduced the definitions of the *hyperoperation* and of the *hypergroup* as a generalization of the operation and the group respectively.

Let H be a set and $\circ : H \times H \to P'(H)$ be a hyperoperation, [2], [3], [5], [6], [8]:
The hyperoperation $(\circ)$ in H is called ***associative***, if

$$(x \circ y) \circ z = x \circ (y \circ z), \forall x, y, z \in H.$$

The hyperoperation $(\circ)$ in H is called ***commutative***, if

$$x \circ y = y \circ x, \forall x, y \in H.$$

An algebraic hyperstructure $(H, \circ)$, i.e. a set H equipped with the hyperoperation $(\circ)$, is called ***hypergroupoid***. If this hyperoperation is associative, then the hyperstructure is called ***semihypergroup***. The semihypergroup $(H, \circ)$, is called ***hypergroup*** if it satisfies the ***reproduction*** axiom:

$$x \circ H = H \circ x, \forall x \in H.$$

One of the topics of great interest, in the last years, is the H_V*-structures*, which was introduced by T. Vougiouklis in 1990 [7]. The class of H_V-structures is the largest class of algebraic hyperstructures. These structures satisfy weak axioms, where the non-empty intersection replaces the equality, as bellow [8]:

i) The $(\circ)$ in H is called ***weak associative***, we write ***WASS***, if

$$(x \circ y) \circ z \cap x \circ (y \circ z) \neq \emptyset, \forall x, y, z \in H.$$

ii) The $(\circ)$ is called ***weak commutative***, we write ***COW***, if

$$(x \circ y) \cap (y \circ x) \neq \emptyset, \forall x, y \in H.$$

iii) If H is equipped with two hyperoperations $(\circ)$ and $(*)$, then $(*)$ is called ***weak distributive*** with respect to $(\circ)$, if

$$[x * (y \circ z)] \cap [(x * y) \circ (x * z)] \neq \emptyset, \forall x, y, z \in H.$$

The hyperstructure $(H, \circ)$ is called ***H_v-semigroup*** if it is *WASS* and it is called ***H_v-group*** if it is a reproductive (i.e. $x \circ H = H \circ x = H, \forall x \in H$) H_v-semigroup. It is called ***commutative H_v-group*** if $(\circ)$ is commutative and it is called ***H_v- commutative group*** if $(\circ)$ is weak commutative. The hyperstructure $(H, \circ, *)$ is called ***H_v-ring*** if both hyperstructures $(\circ)$ and $(*)$ are *WASS*, the reproduction axiom is valid for $(\circ)$, and $(*)$ is weak distributive with respect to $(\circ)$.

It is denoted [4] by E_* the set of the unit elements with respect to $(*)$ and by $I_*(x, e)$ the set of the inverse elements of x associated with the unit e, with respect to $(*)$.

An H_v-ring $(R, +, \cdot)$ is called ***Dual H_v*-ring**, if $(R, \cdot, +)$ is an H_v-ring, too [4].

Let $(H, \cdot)$ be a hypergroupoid. An element $e \in H$ is called *right unit* element if $a \in a \cdot e, \forall a \in H$ and is called *left unit element* if $a \in e \cdot a, \forall a \in H$. The element $e \in H$ is called *unit* element if $a \in a \cdot e \cap e \cdot a, \forall a \in H$.

Let $(H, \cdot)$ be a hypergroupoid endowed with at least one unit element. An element $a' \in H$ is called an *inverse* element of the element $a \in$H, if there exists a unit element $e \in H$ such that $e \in a' \cdot a \cap a \cdot a'$.

Moreover, let us define here: If $x \in x \cdot y (resp. x \in y \cdot x) \forall y \in H$, then, x is called *left absorbing-like* element (resp. *right absorbing-like* element). An element $a \in H$ is called *idempotent* element if $a^2 = a$. The n^{th} power of an element h, denoted h^s, is defined to be the union of all expressions of n times of h, in which the parentheses are put in all possible ways. An H_v-group (H,·) is called *cyclic* with finite period with respect to $h \in H$, if there exists a positive integer s, such that $H = h^1 \cup h^2 \quad \ldots \cup h^s$. The minimum such s is called *period of the generator h.* If all generators have the same period, then H is *cyclic with period.* If there exists $h \in H$ and s positive integer, the minimum one, such that $H = h^s$, then H is called *single-power cyclic* and h is a generator with *single-power period s.* The cyclicity in the infinite case is defined similarly. Thus, for example, the H_v-group $(H, \cdot)$ is called *single-power cyclic with infinite period* with generator h if every element of H belongs to a power of h and there exists $s_0 \geq 1$, such that $\forall s \geq s_0$ we have:

$$h^1 \cup h^2 \cup \cdots \cup h^{s-1} \subset h^s.$$

2 The divisors' hyperoperation due to addition in N

Let $\mathbb{N}$ be the set of the Natural Numbers. Let us define the hyperoperation $(\oslash)$ in $\mathbb{N}$ as follows:

Definition 2.1. For every $x, y \in \mathbb{N}$

$$\oslash : \mathbb{N} \times \mathbb{N} \to \mathcal{P}(\mathbb{N}) - \{\emptyset\} : (x, y) \mapsto x \oslash y \subset \mathbb{N}$$

such that

$$x \oslash y = \{z \in \mathbb{N} : x + y = z \cdot \lambda, \lambda \in \mathbb{N}\}$$

where $(+)$ and $(\cdot)$ are the usual operations of the addition and multiplication in $\mathbb{N}$, respectively.
We call the above hyperoperation, *divisors' hyperoperation due to addition.*

Some properties of the divisors' hyperoperation due to addition

1. $x \oslash y = y \oslash x, \forall x, y \in \mathbb{N}$
2. $0 \oslash 0 = \mathbb{N}$
3. $0 \oslash 1 = 1 \oslash 0 = 1$
4. $\{1, x + y\} \subset x \oslash y, \forall x, y \in \mathbb{N}$
5. If $x + y = \kappa \cdot \nu \Rightarrow \{1, \kappa, \nu, \kappa \cdot \nu\} \subset x \oslash y, x, y, \kappa, \nu \in \mathbb{N}$.

Remark 2.2. If $x + y = p$, where $p \in \mathbb{N}$ is a prime number, then $x \oslash y = \{1, p\}, x, y \in \mathbb{N}$.

Proposition 2.3. The number 0 is a unit element of the divisors' hyperoperation due to addition.

Proof. Indeed, for $x \in \mathbb{N}$

$$x \oslash 0 = \{z \in \mathbb{N} : x + 0 = z \cdot \lambda, \lambda \in \mathbb{N}\} = \{z \in \mathbb{N} : x = z \cdot \lambda, \lambda \in \mathbb{N}\} \ni x.$$

Then,

$$x \in (x \oslash 0) \cap (0 \oslash x), \forall x \in \mathbb{N}. \quad \square$$

Remark 2.4. Since, there is no $x' \in \mathbb{N}$ such that $0 \in (x \oslash x') \cap (x' \oslash x)$ when $x \neq 0$, the number 0 is the only one in $\mathbb{N}$ having an inverse element (and that is 0) associated with the unique unit element 0 of the divisors' hyperoperation due to addition, i.e. $0 \in 0 \oslash 0$.

Proposition 2.5. The number 1 is an absorbing-like element of the divisors' hyperoperation due to addition.

Proof. Indeed,

$$1 \in x \oslash y, \forall x, y \in \mathbb{N} \Rightarrow 1 \in 1 \oslash y, \forall y \in \mathbb{N} \Rightarrow 1 \in (1 \oslash y) \cap (y \oslash 1), \forall y \in \mathbb{N}. \quad \square$$

Proposition 2.6. If $y = n \cdot x$, $x, n \in \mathbb{N}$ then $\{1, x, 1 + n, x(1 + n)\} \subset x \oslash y$.

Proof. Let $y = n \cdot x$, $x, n \in \mathbb{N}$ then

$$\begin{aligned} x \oslash y &= \{z \in \mathbb{N} : x + y = z \cdot \lambda, \lambda \in \mathbb{N}\} = \{z \in \mathbb{N} : x + nx = z \cdot \lambda, \lambda \in \mathbb{N}\} \\ &= \{z \in \mathbb{N} : x(1 + n) = z \cdot \lambda, \lambda \in \mathbb{N}\} \supset \{1, x, 1 + n, x(1 + n)\}. \end{aligned} \quad \square$$

Proposition 2.7. If $x \in \mathbb{N}$ is a prime number then $x^2 = \{1, 2, x, 2x\}$.

Proof. Let $x \in \mathbb{N}$, be a prime number then

$$x^2 = x \oslash x = \{z \in \mathbb{N} : x + x = z \cdot \lambda, \lambda \in \mathbb{N}\} = \{z \in \mathbb{N} : 2x = z \cdot \lambda, \lambda \in \mathbb{N}\}.$$

According to property 5, $\{1, 2, x, 2x\} \subset x^2$, but since x is prime, $x^2 = \{1, 2, x, 2x\}$. $\square$

Proposition 2.8. $x \oslash \mathbb{N}^* = \mathbb{N}^* \oslash x = \mathbb{N}^*, \forall x \in \mathbb{N}^*$.

Proof. Let $x \in \mathbb{N}^*$, then

$$x \oslash \mathbb{N}^* \supset x \oslash (nx) \ni n + 1,\ n \in \mathbb{N}^*, \text{ according to Proposition 2.6.}$$

So, we proved that $n+1 \in x \oslash \mathbb{N}^*, \forall x, n \in \mathbb{N}^*$ and since $1 \in x \oslash \mathbb{N}^*, \forall x \in \mathbb{N}^*$, we get

$$x \oslash \mathbb{N}^* = \mathbb{N}^* \oslash x = \mathbb{N}^*, \forall x \in \mathbb{N}^*. \quad \square$$

Remark 2.9. Notice that, for $x \in \mathbb{N}^*$

$$x \oslash \mathbb{N} = \bigcup_{n \in IN} (x \oslash n) = \bigcup_{n \in I\mathbb{N}} \{z \in \mathbb{N} : x + n = z \cdot \lambda, \lambda \in \mathbb{N}^*\} \supseteq$$
$$\supseteq \bigcup_{n \in I\mathbb{N}} \{z \in \mathbb{N} : x + nx = z \cdot \lambda, \lambda \in \mathbb{N}^*\} = \bigcup_{n \in IN} (x \oslash nx).$$

But from Proposition 2.6,

$$\bigcup_{n \in I\mathbb{N}} (x \oslash nx) \supset \bigcup_{n \in IN} \{1, x, n+1, x(n+1)\} \supset \bigcup_{n \in IN} \{n+1\} = \mathbb{N}^*.$$

So,

$$x \oslash N = N \oslash x = N^*, \forall x \in N^*.$$

Proposition 2.10. The divisors' hyperoperation due to addition is a weak associative one in $\mathbb{N}^*$.

Proof. For $x, y, z \in \mathbb{N}^*$

$$(x \oslash y) \oslash z = \{w \in \mathbb{N}^* : x + y = w \cdot \lambda, \lambda \in \mathbb{N}^*\} \oslash z = \bigcup_{w \in IN*} (w \oslash z) =$$
$$= \bigcup_{w \in IN*} \{w' \in \mathbb{N}^* : w + z = w' \cdot \lambda', \lambda' \in \mathbb{N}^*\}$$
$$\supset \{w' \in \mathbb{N}^* : x + y + z = w' \cdot \lambda', \lambda' \in \mathbb{N}^*\} \qquad (I)$$

On the other hand

$$x \oslash (y \oslash z) = x \oslash \{v \in \mathbb{N}^* : y + z = v \cdot \rho, \rho \in \mathbb{N}^*\} = \bigcup_{v \in IN*} (x \oslash v) =$$
$$= \bigcup_{v \in IN*} \{v' \in \mathbb{N}^* : x + v = v' \cdot \rho', \rho' \in \mathbb{N}^*\}$$
$$\supset \{v' \in \mathbb{N}^* : x + y + z = v' \cdot \rho', \rho' \in \mathbb{N}^*\} \qquad (II)$$

From (I) and (II) we get:

$$(x \oslash y) \oslash z \cap x \oslash (y \oslash z) = \{n \in N^* : x+y+z = n \cdot \mu, \mu \in N^*\} \neq \emptyset, \forall x, y, z \in \mathbb{N}^*. \quad \square$$

Since the divisors' hyperoperation due to addition is commutative, according to Propositions 2.8 and 2.10, we get the following:

Proposition 2.11. The hyperstructure $(\mathbb{N}^*, \oslash)$ is a commutative H_v-group.

Proposition 2.12. For $(x, y, z) \in N^* \times N^* \times N^*$, if $x = z$, then the divisors' hyperoperation due to addition is strong associative.

Proof. Let $(x, y, z) \in \mathbb{N}^* \times \mathbb{N}^* \times N^*$ such that $x = z$, then due to commutativity we get:

$$(x \oslash y) \oslash z = (x \oslash y) \oslash x = x \oslash (x \oslash y) = x \oslash (y \oslash x) = x \oslash (y \oslash z). \quad \square$$

Proposition 2.13. The $(\mathbb{N}^*, \oslash)$, is a single-power cyclic H_v-group with infinite period where every $x \in \mathbb{N}^*$ is a generator.

Proof. For $x \in \mathbb{N}^*$, notice that

$$\begin{aligned}
&x^1 = \{x\} \\
&x^2 = x \oslash x = \{z \in \mathbb{N}^* : 2x = z \cdot \lambda, \lambda \in \mathbb{N}^*\} \supset \{1, 2\} \\
&x^3 = x^2 \oslash x = \{z \in \mathbb{N}^* : 2x = z \cdot \lambda, \lambda \in \mathbb{N}^*\} \oslash x = \bigcup_{z \in IN*} (z \oslash x) \\
&\quad = \bigcup_{z \in IN*} \{w \in \mathbb{N}^* : z + x = w \cdot \rho, \rho \in \mathbb{N}^*\} \supset \\
&\quad \supset \{w' \in \mathbb{N}^* : 2x + x = w' \cdot \rho', \rho' \in \mathbb{N}^*\} \cup \\
&\quad \cup \{w'' \in \mathbb{N}^* : x + x = w'' \cdot \rho'', \rho'' \in \mathbb{N}^*\} \supset \\
&\quad \supset \{1, 3\} \cup \{1, 2\} = \{1, 2, 3\}.
\end{aligned}$$

We shall prove that $x^n \supset \{1, 2, 3, \ldots, n\}$, $\forall x \in \mathbb{N}^*$, $n \in \mathbb{N}^*, n \geq 2$, by induction.

Suppose that for $n = k, k \in \mathbb{N}^*, k \geq 2$:

$$x^k \supset \{1, 2, 3, \ldots, k\}$$

We shall prove that the above is valid for $n = k + 1$, i.e.

$$x^{k+1} \supset \{1, 2, 3, \ldots, k, k + 1\}.$$

Indeed,

$$x^{k+1} = (x^k \oslash x) \cup (x^{k-1} \oslash x^2) \cup \ldots \cup (x \oslash x^k).$$

Then

$$\begin{aligned}
x^{k+1} &\supset (x^{k-1} \oslash x^2) \supset \{1, 2, 3, \ldots, k - 1\} \oslash \{1, 2\} \supset \\
&\subset \{1, 2, 3, \ldots, k\} \cup \{k + 1\} = \{1, 2, 3, \ldots, k, k + 1\}.
\end{aligned}$$

Therefore every element of $\mathbb{N}^*$ belongs to a special power of x, thus, is a generator of the single-power cyclic H_v-group. $\square$

3 The divisors' hyperoperation due to multiplication in N

Now, let us define the hyperoperation $(\otimes)$ in $\mathbb{N}$ as follows:

Definition 3.1. For every $x, y \in IN$

$$\otimes \quad : \mathbb{N} \times \mathbb{N} \to \quad P(\mathbb{N}) - \{\emptyset\} : (x, y) \mapsto x \otimes y \subset \mathbb{N}$$

such that

$$x \odot y = \{z \in \mathbb{N} : x \cdot y = z \cdot \lambda, \lambda \in \mathbb{N}\}$$

where $(\cdot)$ is the usual operation of the multiplication in $\mathbb{N}$.

We call the above hyperoperation, *divisors' hyperoperation due to multiplication.*

Some properties of the divisors' hyperoperation due to multiplication

1. $x \otimes y = y \otimes x, \forall x, y \in \mathbb{N}$
2. $0 \otimes x = x \otimes 0 = \mathbb{N}, \forall x \in \mathbb{N}$
3. $1 \otimes 1 = 1$, i.e 1 is an idempotent element
4. $\{1, x, y, xy\} \subset x \otimes y, \forall x, y \in \mathbb{N}$

Remark 3.2. If x is a prime number, then $1 \otimes x = x \otimes 1 = \{1, x\}$.

Proposition 3.3. $E_{\otimes} = \mathbb{N}$.

Proof. For $x, e \in \mathbb{N}, x \otimes e = \{z \in \mathbb{N} : x \cdot e = z \cdot \lambda, \lambda \in \mathbb{N}\} \ni x$. So, according to property 1, we get

$$x \in (x \otimes e) \cap (e \otimes x), \forall x, e \in \mathbb{N}$$

That means that the set of the unit elements with respect to $(\otimes)$ is the set $\mathbb{N}$, i.e. $E_{\otimes} = \mathbb{N}$. □

Proposition 3.4. i) $I_{\otimes}(x, 0) = \{0\}, y \in \mathbb{N}$ ii) $I_{\otimes}(x, 1) = \mathbb{N}$.

Proof. i) Straightforward from property 2.

ii) Take the unit element 1, then from property 4, we get

$$1 \in (x \otimes y) \cap (y \otimes x), \forall x, y \in \mathbb{N}$$

which means that $I_{\otimes}(x, 1) = \mathbb{N}$. □

Proposition 3.5. If a unit element p is a prime number, then

$$I_{\otimes}(x,p) = \begin{cases} \mathbb{N}, & x = np, n \in \mathbb{N} \\ pN, & x \neq np, n \in \mathbb{N}. \end{cases}$$

Proof. Let $p \in \mathbb{N}$ be a unit element and $p =$ prime number. Then p has no other divisors than 1 and itself. So, let $x = np$, $n \in \mathbb{N}$, then for $x' \in \mathbb{N}$

$$\begin{aligned} x \otimes x' &= \{z \in \mathbb{N} : x \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} = \\ &= \{z \in \mathbb{N} : (np) \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \ni p, \forall x' \in \mathbb{N}. \end{aligned}$$

That means that $I_{\otimes}(x,p) = \mathbb{N}$. Let $x \neq np, n \in \mathbb{N}$, then $p \in \{z \in \mathbb{N} : x \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \Leftrightarrow x' = pn, n \in \mathbb{N} \Leftrightarrow I_{\otimes}(x,p) = pN$. □

Seems to be particularly interesting, one to study cases where the unit element is not a prime number. The following two examples study the cases where the unit element is 6 and 9.

Example 3.6. Let 6 be the unit element. Assume that $x = 6n$, $n \in \mathbb{N}$, then

$$x \otimes x' = \{z \in \mathbb{N} : (6n) \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \ni 6, \forall x' \in \mathbb{N}.$$

Then, $I_{\otimes}(x,6) = \mathbb{N}$.
Assume that $x = 3m \neq 6n$, $n, m \in \mathbb{N}$, then

$$\begin{aligned} 6 \in x \otimes x' &= \{z \in \mathbb{N} : (3m) \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \Leftrightarrow x' = 2n, n \in \mathbb{N} \\ &\Leftrightarrow I_{\otimes}(x,6) = 2\mathbb{N}. \end{aligned}$$

Assume that $x = 2m \neq 6n$, $n, m \in \mathbb{N}$, then

$$\begin{aligned} 6 \in x \otimes x' &= \{z \in \mathbb{N} : (2m) \cdot x' = z \cdot \lambda, \lambda \in N\} \Leftrightarrow x' = 3n, n \in \mathbb{N} \\ &\Leftrightarrow I_{\otimes}(x,6) = 3N. \end{aligned}$$

Assume that $x = 2m + 1 \neq 3n$, $n, m \in \mathbb{N}$, then

$$\begin{aligned} 6 \in x \otimes x' &= \{z \in \mathbb{N} : (2m+1) \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \\ &\Leftrightarrow x' = 6n, n \in \mathbb{N} \Leftrightarrow I_{\otimes}(x,6) = 6\mathbb{N}. \end{aligned}$$

Example 3.7. Let 9 be the unit element. Assume that $x = 9n$, $n \in \mathbb{N}$, then

$$x \otimes x' = \{z \in \mathbb{N} : (9n) \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \ni 9, \forall x' \in \mathbb{N}.$$

Then, $I_{\otimes}(x,9) = \mathbb{N}$. Assume that $x = 3m \neq 9n$, $n, m \in \mathbb{N}$, then

$$\begin{aligned} 9 \in x \otimes x' &= \{z \in \mathbb{N} : (3m) \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \\ &\Leftrightarrow x' = 3n, n \in \mathbb{N} \Leftrightarrow I_{\otimes}(x,9) = 3N. \end{aligned}$$

Assume that $x \neq 3m$, $m \in \mathbb{N}$, then

$$9 \in x \otimes x' = \{z \in \mathbb{N} : x \cdot x' = z \cdot \lambda, \lambda \in \mathbb{N}\} \Leftrightarrow x' = 9n, n \in \mathbb{N} \Leftrightarrow I_{\otimes}(x,9) = 9N.$$

Proposition 3.8. Every element $x \in \mathbb{N}$ is an absorbing-like element of the divisors' hyperoperation due to multiplication.

Proof. According to property 4, $x \in x \otimes y$, $\forall x, y \in \mathbb{N}$, which means, that for every x$\in$ $\mathbb{N}$, $x \in x \otimes y$, $\forall y \in \mathbb{N}$ and due to property 1, $\forall x \in \mathbb{N}, x \in (x \otimes y) \cap (y \otimes x), \forall y \in \mathbb{N}$. Then, every natural number is an absorbing-like element of the divisors' hyperoperation due to multiplication. $\square$

Proposition 3.9. The divisors' hyperoperation due to multiplication is a strong associative one in $\mathbb{N}$.

Proof. For $x, y, z \in \mathbb{N}$

$$\begin{aligned}
(x \otimes y) \otimes z = \{w \in \mathbb{N} : x \cdot y = w \cdot \lambda, \lambda \in \mathbb{N}\} \otimes z = \bigcup_{w \in IN} (w \otimes z) = \\
= \bigcup_{w \in IN} \{w' \in \mathbb{N} : w \cdot z = w' \cdot \lambda', \lambda' \in \mathbb{N}\} = \\
= \bigcup_{\lambda \in IN} \{w' \in \mathbb{N} : \big[\frac{1}{\lambda}(xy)\big] z = w' \cdot \lambda', \lambda' \in \mathbb{N}\} = \\
= \bigcup_{\lambda \in IN} \{w' \in \mathbb{N} : x \big[\frac{1}{\lambda}(yz)\big] = w' \cdot \lambda', \lambda' \in \mathbb{N}\} = \\
= \bigcup_{v \in IN} \{w' \in \mathbb{N} : x \cdot v = w' \cdot \lambda', \lambda' \in N\} = \\
= \bigcup_{v \in IN} (x \otimes v) = x \otimes \{v \in \mathbb{N} : y \cdot z = v \cdot \lambda, \lambda \in \mathbb{N}\} = x \otimes (y \otimes z).
\end{aligned}$$

So, $(x \otimes y) \otimes z = x \otimes (y \otimes z), \forall x, y, z \in \mathbb{N}$. $\square$

Proposition 3.10. The hyperstructure $(\mathbb{N}, \otimes)$ is a commutative hypergroup.

Proof. Indeed, for $x \in \mathbb{N}$,

$$x \otimes \mathbb{N} = (x \otimes 0) \cup [\bigcup_{n \in IN*} (x \otimes n)] = \mathbb{N} \cup \Big[\bigcup_{n \in I\mathbb{N}*} (x \otimes n)\Big] = \mathbb{N}.$$

So, $x \otimes \mathbb{N} = \mathbb{N} \otimes x = \mathbb{N}$, $\forall x \in \mathbb{N}$.

Also, according to property 1 and Proposition 3.9 we get that $(\mathbb{N}, \otimes)$ is a commutative hypergroup. $\square$

Remark 3.11. For $x \in \mathbb{N}^*$,

$$x \otimes \mathbb{N}^* = \bigcup_{n \in IN*} (x \otimes n) = \bigcup_{n \in IN*} \{z \in \mathbb{N} : x \cdot n = z \cdot \lambda, \lambda \in \mathbb{N}^*\} \supset \bigcup_{n \in IN*} \{n\} = \mathbb{N}^*.$$

So, $x \otimes \mathbb{N}^* = \mathbb{N}^* \otimes x = \mathbb{N}^*, \forall x \in \mathbb{N}^*$.

Proposition 3.12. For every $x \in \mathbb{N}, x^{n-1} \subseteq x^n, n \in \mathbb{N}, n \geq 2$.

Proof. For $x \in \mathbb{N}$ and $n \in \mathbb{N}, n \geq 2$

$$x^n = (x^{n-1} \otimes x) \cup (x^{n-2} \otimes x^2) \cup \ldots \cup (x^{n-p} \otimes x^p)$$

where $p = \left[\frac{n}{2}\right]$ the integer part of $\frac{n}{2}$, [1]. Then,

$$x^n \supseteq x^{n-1} \otimes x \supseteq x^{n-1} \otimes 1 \supseteq x^{n-1}. \qquad \square$$

4 On a dual H_v-ring in $\mathbb{N}^*$

Proposition 4.1. $(x \otimes y) \oslash (x \otimes z) \supset x \otimes (y \oslash z), \forall x, y, z \in \mathbb{N}^*$.

Proof. For $x, y, z \in \mathbb{N}^*$, we get

$$\begin{aligned} x \otimes (y \oslash z) &= x \otimes \{w \in \mathbb{N}^* : y + z = w \cdot \lambda, \lambda \in \mathbb{N}^*\} = \bigcup_{w \in IN*} (x \otimes w) = \\ &= \bigcup_{w \in IN*} \{w' \in \mathbb{N}^* : x \cdot w = w' \cdot \lambda', \lambda' \in \mathbb{N}^*\} = \\ &= \bigcup_{\lambda \in IN*} \{w' \in \mathbb{N}^* : x \cdot \frac{y+z}{\lambda} = w' \cdot \lambda', \lambda' \in \mathbb{N}^*\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(x \otimes y) \oslash (x \otimes z) = \\ &= \{v \in \mathbb{N}^* \colon x \cdot y = v \cdot \rho, \rho \in \mathbb{N}^*\} \oslash \{v' \in \mathbb{N}^* \colon x \cdot z = v' \cdot \rho', \rho' \in \mathbb{N}^*\} \\ &= \bigcup_{v, v' \in IN*} (v \oslash v') = \bigcup_{v, v' \in IN*} \{k \in \mathbb{N}^* \colon v + v' = k \cdot \mu, \mu \in \mathbb{N}^*\} = \\ &= \bigcup_{\rho, \rho' \in IN*} \{\kappa \in \mathbb{N}^* \colon \frac{xy}{\rho} + \frac{xz}{\rho'} = \kappa \cdot \mu, \mu \in \mathbb{N}^*\} \supset \\ &\supset \bigcup_{\mu' \in IN*} \{\kappa' \in \mathbb{N}^* \colon x \cdot \frac{y+z}{\mu'} = \kappa' \cdot \tau', \tau' \in \mathbb{N}^*\} = x \otimes (y \oslash z). \end{aligned}$$

So, $(x \otimes y) \oslash (x \otimes z) \supset x \otimes (y \oslash z)$ and then,

$$x \otimes (y \oslash z) \cap (x \otimes y) \oslash (x \otimes z) \neq \emptyset, \forall x, y, z \in \mathbb{N}^*. \qquad \square$$

Proposition 4.2. The divisors' hyperoperation due to addition is weak distributive with respect to the divisors' hyperoperation due to multiplication in $\mathbb{N}^*$.

Proof. For $x, y, z \in \mathbb{N}^*$, we get

$$\begin{aligned} x \oslash (y \otimes z) &= x \oslash \{w \in \mathbb{N}^* : y \cdot z = w \cdot \lambda, \lambda \in \mathbb{N}^*\} = \bigcup_{w \in IN*} (x \oslash w) = \\ &= \bigcup_{w \in IN*} \{w' \in \mathbb{N}^* : x + w = w' \cdot \lambda', \lambda' \in \mathbb{N}^*\} \supset \\ &\supset \{w'' \in \mathbb{N}^* : x + y = w'' \cdot \lambda'', \lambda'' \in \mathbb{N}^*\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(x \oslash y) \otimes (x \oslash z) = \\ &\quad = \{v \in \mathbb{N}^* : x + y = v \cdot \rho, \rho \in \mathbb{N}^*\} \otimes \{v' \in \mathbb{N}^* : x + z = v' \cdot \rho', \rho' \in \mathbb{N}^*\} = \\ &\quad = \bigcup_{v, v' \in IN*} (v \otimes v') = \bigcup_{v, v' \in IN*} \{v'' \in \mathbb{N}^* : v \cdot v' = v'' \cdot \rho'', \rho'' \in \mathbb{N}^*\} \supset \\ &\quad \supset \{\kappa \in N^* : (x + y) \cdot (x + z) = \kappa \cdot \mu, \mu \in \mathbb{N}^*\} \supset \\ &\quad \supset \{\kappa' \in \mathbb{N}^* : x + y = \kappa' \cdot \mu', \mu' \in \mathbb{N}^*\}. \end{aligned}$$

So, $x \oslash (y \otimes z) \cap (x \oslash y) \otimes (x \oslash z) \supset \{\tau \in \mathbb{N}^* : x + y = \tau \cdot \sigma, \sigma \in \mathbb{N}^*\}$ and then

$$x \oslash (y \otimes z) \cap (x \oslash y) \otimes (x \oslash z) \neq \emptyset, \forall x, y, z \in \mathbb{N}^*.$$ □

Proposition 4.3. The hyperstructure $(\mathbb{N}^*, \oslash, \otimes)$ is a commutative dual H_v-ring.

Proof. Indeed, according to Propositions 2.11 and 3.10 the hyperstructures $(\mathbb{N}^*, \oslash)$ and $(\mathbb{N}^*, \otimes)$ are commutative H_v-group and commutative hypergroup respectively. From Propositions 4.1 and 4.2 we get that $(\otimes)$ is weak distributive with respect to $(\oslash)$ and $(\oslash)$ is weak distributive with respect to $(\otimes)$, respectively. □

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